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Frontiers in Stochastic Analysis – BSDEs, SPDEs and their Applications

Edinburgh, July 2017
Selected, Revised and Extended Contributions

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International Workshop on
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Preface

It was our pleasure to be among the organizers of the *International workshop on BSDEs, SPDEs, and applications*, held at the University of Edinburgh in July 2017. The workshop brought together more than 200 active researchers in probability theory, for over 150 research talks, in addition to poster presentations and networking events. The meeting also included the *8th World BSDE symposium*.

The papers in this volume give a taste of those areas presented at the meeting, covering a range of actively researched areas. We hope that they act as a stimulus for further research in this exciting subfield of probability theory. We now summarize the key themes of each of the papers in the volume:

The first paper, by Dirk Becherer, Martin Büttner, and Klebert Kentia, considers the monotone stability approach to BSDEs with jumps. This is an approach to studying basic questions of existence and uniqueness of solutions to backward SDEs, by leveraging the result of the "comparison theorem" for these equations. This is made more difficult than in the standard case due to the presence of jumps, which imply that additional requirements on the generator of the BSDE must be imposed. This paper uses this result to provide existence results without a standard Lipschitz continuity condition and then further explores how these equations appear in some applied problems in mathematical finance.

The second paper, by Mireille Bossy and Jean-François Jabir, studies McKean stochastic differential equations, in particular, a framework where the dynamics of a process Y depend on the (conditional) distribution of Y given a related process X. The well-posedness of this equation is proven, under appropriate continuity and regularity assumptions.

The third paper, by Philippe Briand and Adrien Richou, studies the uniqueness of solutions to BSDEs with drivers which may grow quadratically, without an assumption of convexity. If the driver and terminal value are assumed to be bounded, the uniqueness of solutions to these equations is well known; however in the unbounded case, the study of these equations is significantly more difficult. This paper studies the case where the terminal value is unbounded and is determined by the path of a forward SDE.

The fourth paper, by Antonella Calzolari and Barbara Torti, studies the question of martingale representation, when a filtration is enlarged by additional information. In particular, a model is studied in which information arrives from two sources—a Brownian motion and the occurrence of a random time. In this setting, they show that while the Brownian motion and the martingale associated with the random time have the predictable representation property in each of their filtrations, the combination of these two sources can introduce the necessity for a third martingale in a representing set (alternatively, the *multiplicity* of the joint filtration may be three).

The fifth paper, by Samuel N. Cohen and Martin Tegnér, considers the pricing of European options in a setting with estimation uncertainty. The paper considers estimating the parameters in a Heston stochastic volatility model for stock prices, along with their statistical uncertainties. It then explores, if the future dynamics of the price are only constrained to lie within the estimated bounds on the parameters, how to find the range of possible prices for a financial option. This is done by means of numerical solutions of BSDEs.

The sixth paper, by Gonçalo dos Reis and Greig Smith, studies a class of transport PDEs which have a representation from a stochastic perspective, in terms of branching processes with regime switching. This is then used to study the convergence of Monte Carlo approximations to these equations, and a comparison with alternative Laplacian–perturbation methods is given.

The seventh paper, by Nicole El Karoui, Caroline Hillairet, and Mohamed Mrad, gives a method of constructing an aggregate consistent utility from a collection of heterogeneous agents. Working in a setting of a financial market and assuming no arbitrage, they consider the marginal utilities of each agent and their corresponding investment preferences, and from these, construct a utility function which gives the same aggregate preferences. This is then applied to studying the yield curve in bond markets.

The eighth paper, by Monique Jeanblanc and Dongli Wu, returns to the theme of enlargement of filtrations, in this case studying how BSDEs vary when additional information is given. This question is then extended to the related question of how an optimal control (in particular the choice of an optimal investment in a financial market) would change under an increase in the information available.

The final paper, by Mauro Rosestolato, focusses on path-dependent stochastic differential equations in Hilbert spaces. This paper in particular focuses on the continuity and (Gâteaux) differentiability of the solution to such an equation with respect to the initial value given and with respect to perturbations of the other coefficients.

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Contents

On the Monotone Stability Approach to BSDEs with Jumps: Extensions, Concrete Criteria and Examples

Dirk Becherer, Martin Büttner and Klebert Kentia

Abstract We show a concise extension of the monotone stability approach to backward stochastic differential equations (BSDEs) that are jointly driven by a Brownian motion and a random measure of jumps, which could be of infinite activity with a non-deterministic and time-inhomogeneous compensator. The BSDE generator function can be non-convex and needs not satisfy global Lipschitz conditions in the jump integrand. We contribute concrete sufficient criteria, that are easy to verify, for results on existence and uniqueness of bounded solutions to BSDEs with jumps, and on comparison and a-priori L^∞-bounds. Several examples and counter examples are discussed to shed light on the scope and applicability of different assumptions, and we provide an overview of major applications in finance and optimal control.

Keywords Backward stochastic differential equations · Random measures · Monotone stability · Lévy processes · Step processes · Utility maximization · Entropic risk measure · Good deal valuation bounds

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1 Introduction

We study bounded solutions (Y, Z, U) to backward stochastic differential equations with jumps

$$Y_t = \xi + \int_t^T f_s(Y_{s-}, Z_s, U_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s - \int_t^T\int_E U_s(e)\,\widetilde{\mu}(\mathrm{d}s, \mathrm{d}e)\,,$$

which are jointly driven by a Brownian motion B and a compensated random measure $\widetilde{\mu} = \mu - \nu^{\mathbb{P}}$ of some integer-valued random measure μ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This is an extension of the classical BSDE theory on Wiener space towards BSDEs which involve jumps (JBSDEs), that are driven by the compensated random measure $\widetilde{\mu}$, and do evolve on non-Brownian filtrations. Such JBSDEs do involve an additional stochastic integral with respect to the compensated jump measure $\widetilde{\mu}$ whose integrand U, differently from Z, typically takes values in an infinite dimensional function space instead of an Euclidean space.

Comparison theorems for BSDEs with jumps require more delicate technical conditions than in the Brownian case, see [4, 15, 54]. The starting point for our article will be a slight generalization of the seminal $(\mathbf{A}_\gamma)$-condition for comparison due to [54]. Our first contribution are extensions of comparison, existence and uniqueness results for bounded solutions of JBSDEs to the case of infinite jump activity for a family (2.6) of generators, that do not need to be Lipschitz in the U-argument. This shows how the monotone stability approach to BSDEs with jumps, pioneered by [44] for one particular generator, permits for a concise proof in a setting, that may be of particular appeal in a pure jump case without a Brownian motion, see Corollary 4.12. While the strong approximation step for this approach is usually laborious, we present a compact proof with a $\mathcal{S}^1$-closedness argument and more generality of the generator in the U-argument for infinite activity of jumps. To be useful towards applications, our second contribution are sufficient concrete criteria for comparison and wellposedness that are comparably easy to verify in actual examples, because they are formulated in terms of concrete properties for generator functions f from a given family (2.6) w.r.t. to basically Euclidean arguments, instead of assuming inequalities to hold for rather abstract random processes or fields. This is the main thrust for the sufficient conditions of the comparison results in Sect. 3 (see Theorem 3.9 and Proposition 3.11, compared to Proposition 3.1 or the result by [54] and respective enhancements [38, 52, 57]) and of the wellposedness Theorem 4.13 (in comparison to Theorem 4.11, whose conditions are more general but more abstract). A third contribution are the many examples and applications which illustrate the scope and applicability of our results and of the, often technical, assumptions that are needed for JBSDE results in the literature. Indeed, the range of the imposed combinations of several technical assumptions is often not immediately clear. We believe that more discussion of examples and counter examples may help to shed light on the scope and the differences of some assumptions prevailing in the literature, and might also caution against possible pitfalls.

The approach in this paper can be described in more detail as follows: The comparison results will provide basic a-priori estimates on the L^∞-norm for the Y-component of the JBSDE solution. This step enables a quick intermediate result on existence and uniqueness for JBSDEs with finite jump activity. To advance from here to infinite activity, we approximate the generator f by a monotone sequence of generators for which solutions do exist, extending the monotone stability approach from [37] and (for a particular JBSDE) [44]. For the present paper, the compensator $\nu(\omega, \mathrm{d}t, \mathrm{d}e)$ of $\mu(\omega, \mathrm{d}t, \mathrm{d}e)$ can be stochastic and does not need to be a product measure like $\lambda(\mathrm{d}e) \otimes \mathrm{d}t$, as it would be natural e.g. in a Lévy-process setting, but it is allowed to be inhomogeneous in that it can vary predictably with (ω, t). In this sense, ν is only assumed to be absolutely continuous to some reference product measure $\lambda \otimes \mathrm{d}t$ with λ being σ-finite, see Eq. (2.1). Such appears useful, but requires some care in the specification of generator properties in Sect. 2. For the filtration we assume that $\widetilde{\mu}$ jointly with B (or alone) satisfies the property of weak predictable representation for martingales, see (2.2). As explained in Example 2.1, such setup permits for a range of stochastic dependencies between B and $\widetilde{\mu}$, which appear useful for modeling of applications, and encompasses many interesting driving noises for jumps in BSDEs; This includes Lévy processes, Poisson random measures, marked point processes, (semi-)Markov chains or much more general step processes, connecting to a wide range of literature, e.g. [3, 14, 15, 17, 25–27].

The literature on BSDE started with the classical study [50] of square integrable solutions to BSDEs driven solely by Brownian motion B under global Lipschitz assumptions. One important extension concerns generators f which are non-Lipschitz but have quadratic growth in Z, for which [37] derived bounded solutions by pioneering a monotone stability approach, and [56] by a fixed point approach. Square integrable solutions under global Lipschitz conditions for BSDEs with jumps from a Poisson random measures are first studied by [4, 55]. There is a lot of development in JBSDE theory recently. See for instance [2, 22, 38, 39, 49] for results under global Lipschitz conditions on the generator with respect to on (Z, U). In the context of non-Lipschitz generators that are quadratic (also in Z, with exponential growth in U), JBSDEs have been studied to our knowledge at first by [44] using a monotone stability approach for a specific generator that is related to exponential utility, by [23] using a quadratic-exponential semimartingale approach from [6], and by [40] or [35] with again different approaches, relying on duality methods or, respectively, the fixed-point idea of [56] for quadratic BSDEs. For extensive surveys of the active literature with more references, let us refer to [38, 57], who contribute results on L^p-solutions for generators, being monotone in the Y-component, that are very general in many aspects. Their assumptions on the filtrations or generator's dependence on (Y, Z) are for instance more general than ours. But the present paper still contributes on other aspects, noted above. For instance, [57] assumes finite activity of jumps and a Lipschitz continuity in U. More relations to some other related literature are being explained in many examples throughout the paper, see e.g. in Sect. 5. Moreover, it is fair to say that results in the JBSDE literature often involve combinations of many technical assumptions; To understand the scope, applicability and differences of those assumptions, it appears helpful to discuss concrete examples and applications.

The paper is organized as follows. Section 2 introduces the setting and mathematical background. In Sects. 3, 4, we prove comparison results and show existence as well as uniqueness for bounded solutions to JBSDEs, both for finite and infinite activity of jumps. Last but not least, Sect. 5 surveys key applications of JBSDEs in finance. We discuss several examples to shed light on the scope of the results and of the underlying technical assumptions, and discuss connections to the literature.

2 Preliminaries

This section presents the technical framework, sets notations and discusses key conditions. First we recall essential facts on stochastic integration w.r.t. random measures and on bounded solutions for Backward SDEs which are driven jointly by Brownian motions and a compensated random measure. For notions from stochastic analysis not explained here we refer to [28, 31].

Inequalities between measurable functions are understood almost everywhere w.r.t. an appropriate reference measure, typically $\mathbb{P}$ or $\mathbb{P} \otimes \mathrm{d}t$. Let $T < \infty$ be a finite time horizon and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ a filtered probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ satisfying the usual conditions of right continuity and completeness, assuming $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ being trivial (under $\mathbb{P}$); Thus we can and do take all semimartingales to have right continuous paths with left limits, so-called càdlàg paths. Expectations (under $\mathbb{P}$) are denoted by $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$. We will denote by $\mathbf{A}^T$ the transpose of a matrix $\mathbf{A}$ and simply write $\boldsymbol{x}\boldsymbol{y} := \boldsymbol{x}^T\boldsymbol{y}$ for the scalar product for two vectors $\boldsymbol{x}, \boldsymbol{y}$ of same dimensionality. Let H be a separable Hilbert space and denote by $\mathcal{B}(E)$ the Borel σ-field of $E := H\backslash\{0\}$, e.g. $H = \mathbb{R}^l, l \in \mathbb{N}$, or $H = \ell^2 \subset \mathbb{R}^{\mathbb{N}}$. Then $(E, \mathcal{B}(E))$ is a standard Borel space. In addition, let B be a d-dimensional Brownian motion. Stochastic integrals of a vector valued predictable process Z w.r.t. a semimartingale X, e.g. $X = B$, of the same dimensionality are scalar valued semimartingales starting at zero and denoted by $\int_{(0,t]} Z\mathrm{d}X = \int_0^t Z\mathrm{d}X = Z \bullet X_t$ for $t \in [0, T]$. The *predictable* σ-field on $\Omega \times [0, T]$ (w.r.t. $(\mathcal{F}_t)_{0\le t\le T}$) is denoted by $\mathcal{P}$ and $\widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(E)$ is the respective σ-field on $\widetilde{\Omega} := \Omega \times [0, T] \times E$.

Let μ be an integer-valued random measure with compensator $\nu = \nu^{\mathbb{P}}$ (under $\mathbb{P}$) which is taken to be absolutely continuous to $\lambda \otimes \mathrm{d}t$ for a σ-finite measure λ on $(E, \mathcal{B}(E))$ satisfying $\int_E 1 \wedge |e|^2 \lambda(\mathrm{d}e) < \infty$ with some $\widetilde{\mathcal{P}}$-measurable, bounded and non-negative density ζ, such that

$$\nu(\mathrm{d}t, \mathrm{d}e) = \zeta(t, e)\, \lambda(\mathrm{d}e)\, \mathrm{d}t = \zeta_t\, \mathrm{d}\lambda\, \mathrm{d}t, \tag{2.1}$$

with $0 \le \zeta(t, e) \le c_\nu$ $\mathbb{P} \otimes \lambda \otimes \mathrm{d}t$-a.e. for some constant $c_\nu > 0$. Note that $L^2(\lambda)$ and $L^2(\zeta_t \mathrm{d}\lambda)$ are separable Hilbert spaces since λ (and $\lambda_t := \zeta_t\, \mathrm{d}\lambda$) is σ-finite and $\mathcal{B}(E)$ is finitely generated. Since the density ζ can vary with (ω, t), the compensator

ν can be time-inhomogeneous and stochastic. Such permits for a richer dependence structure for $(B, \widetilde{\mu})$; For instance, the intensity and distribution of jump heights could vary according to some diffusion process. Yet, it also brings a few technical complications, e.g. function-valued integrand processes U from $\mathcal{L}^2(\widetilde{\mu})$ (as defined below) for the JBSDE need not take values in one given L^2-space (for a.e. (ω, t)), like e.g. $L^2(\lambda)$ if $\zeta \equiv 1$, and the specifications of the domain and of the measurability for the generator functions should take account of such.

For stochastic integration w.r.t. $\widetilde{\mu}$ and B we define sets of $\mathbb{R}$-valued processes

$$\mathcal{S}^p := \mathcal{S}^p(\mathbb{P}) := \Big\{ Y \text{ càdlàg} \,:\, |Y|_p := \Big\| \sup_{0\le t\le T} |Y_t| \Big\|_{L^p(\mathbb{P})} < \infty \Big\} \quad \text{for } p \in [1, \infty]\,,$$

$$\mathcal{L}^2(\widetilde{\mu}) := \Big\{ U \;\widetilde{\mathcal{P}}\text{-measurable} \,:\, \|U\|^2_{\mathcal{L}^2(\widetilde{\mu})} := \mathbb{E}\Big(\int_0^T\!\!\int_E |U_s(e)|^2\, \nu(\mathrm{d}s, \mathrm{d}e) \Big) < \infty \Big\}\,,$$

and the set of $\mathbb{R}^d$-valued processes

$$\mathcal{L}^2(B) := \Big\{ \theta \;\mathcal{P}\text{-measurable} \,:\, \|\theta\|^2_{\mathcal{L}^2(B)} := \mathbb{E}\Big(\int_0^T \|\theta_s\|^2\, \mathrm{d}s \Big) < \infty \Big\},$$

where $\widetilde{\mu} = \widetilde{\mu}^{\mathbb{P}} = \mu - \nu$ denotes the compensated measure of μ (under $\mathbb{P}$). Recall that for any predictable function U, $\mathbb{E}(|U| * \mu_T) = \mathbb{E}(|U| * \nu_T)$ by the definition of a compensator. If $(|U|^2 * \mu)^{1/2}$ is locally integrable, then U is integrable w.r.t. $\widetilde{\mu}$, and $U * \widetilde{\mu}$ is defined as the purely discontinuous local martingale with jump process $\big(\int_E U_t(e)\, \mu(\{t\}, \mathrm{d}e)\big)_t$ by [31, Definition II.1.27] noting that ν is absolutely continuous to $\lambda \otimes \mathrm{d}t$. For $Z \in \mathcal{L}^2(B)$ and $U \in \mathcal{L}^2(\widetilde{\mu})$ we recall that $Z \bullet B$ and $U * \widetilde{\mu} = (U * \widetilde{\mu}_t)_{0\le t\le T}$ with $U * \widetilde{\mu}_t = \int_0^t \int_E U_s(e)\, \widetilde{\mu}(\mathrm{d}s, \mathrm{d}e)$ are square integrable martingales by [31, Theorem II.1.33]. For $Z, Z' \in \mathcal{L}^2(B)$ and $U, U' \in \mathcal{L}^2(\widetilde{\mu})$ we have for the predictable quadratic covariations that $\big\langle U * \widetilde{\mu}, U' * \widetilde{\mu}\big\rangle_t = \int_0^t\!\int_E U_s(e) U'_s(e)\, \nu(\mathrm{d}s, \mathrm{d}e)$ by [31, Theorem II.1.33], $\big\langle \int Z\, \mathrm{d}B, \int Z'\, \mathrm{d}B \big\rangle_t = \int_0^t Z_s^T Z'_s\, \mathrm{d}s$ and $\big\langle \int Z\, \mathrm{d}B, U * \widetilde{\mu}\big\rangle_t = 0$ by [31, Theorem I.4.2].

We denote the space of square integrable martingales by $\mathcal{M}^2$ and its norm by $\|\cdot\|_{\mathcal{M}^2}$ with $\|M\|_{\mathcal{M}^2} = \mathbb{E}(M_T^2)^{1/2}$. We recall [28, Theorem 10.9.4] that the subspace of BMO($\mathbb{P}$)-martingales BMO($\mathbb{P}$) contains any square integrable martingale M with uniformly bounded jumps and bounded conditional expectations for increments of the quadratic variation process:

$$\sup_{0\le t\le T} \Big\| \mathbb{E}\big((M_T - M_t)^2 \,|\, \mathcal{F}_t \big) \Big\|_{L^\infty(\mathbb{P})} = \sup_{0\le t\le T} \big\| \mathbb{E}\big(\langle M\rangle_T - \langle M\rangle_t \,|\, \mathcal{F}_t \big) \big\|_{L^\infty(\mathbb{P})} \le \text{const} < \infty.$$

We will assume that the continuous martingale B and the compensated measure $\widetilde{\mu}$ of an integer-valued random measure μ (or $\widetilde{\mu}$ alone, see Example 2.1 and Corollary 4.12 with trivial $B = 0$) jointly have the weak predictable representation property (weak PRP) w.r.t. the filtration $(\mathcal{F}_t)_{0\le t\le T}$, in the sense that every square integrable martingale M has a (unique) representation, i.e.

$$\text{for all } M \in \mathcal{M}^2 \text{ there exists } Z, U \text{ such that } \; M = M_0 + \int Z \,\mathrm{d}B + U * \widetilde{\mu}\,, \tag{2.2}$$

with (unique) $Z \in \mathcal{L}^2(B)$ and $U \in \mathcal{L}^2(\widetilde{\mu})$. Let us note that in the literature [31, III.§4c] or [28, XIII.§2] the weak representation property is defined as a decomposition like (2.2) for any local martingale M with integrands Z, U being integrable in the sense of local martingales. Such clearly implies our formulation above. Indeed, for a (locally) square integrable martingale M in such a decomposition both integrands must be at least locally square integrable and $\langle M\rangle = \int |Z|^2\,\mathrm{d}t + |U|^2 * \nu$ by strong orthogonality of the stochastic integrals. Then $E[\langle M\rangle_T] < \infty$ implies that Z, U are in the respective $\mathcal{L}^2$-spaces. We exemplify how (2.2) connects with a wide literature.

Example 2.1 The weak predictable representation property (2.2) holds in the cases below. Cases 1.–4. are well known from classical theory [28], see [7, Example 2.1] for details.

1. Let X be a Lévy process with $X_0 = 0$ and predictable characteristics (α, β, ν) (under $\mathbb{P}$). Then the continuous martingale part X^c (rescaled to a Brownian motion if $\beta \neq 0$, or being trivial if $\beta = 0$) and the compensated jump measure $\widetilde{\mu}^X = \mu^X - \nu$ of X have the weak PRP w.r.t. the usual filtration $\mathbb{F}^X$ generated by X. An example for a Lévy process of infinite activity is the Gamma process. One can add that weak PRP even holds in the sense of Theorem III.4.34 from [31] for the more general class of PII-processes with independent increments. This class encompasses the more familiar Lévy processes without requiring time-homogeneity or stochastic continuity.
2. Assume that B and $\widetilde{\mu}$ satisfy (2.2) under $\mathbb{P}$. Let $\mathbb{P}'$ be an equivalent probability measure with density process Z. Then the Brownian motion $B' := B - \int (Z_-)^{-1}\,\mathrm{d}\langle Z, B\rangle$ and $\widetilde{\mu}' := \mu - \nu^{\mathbb{P}'}$ have the weak PRP (2.2) also w.r.t. $\mathbb{P}'$ under the same filtration. This offers plenty of scope to construct examples where W and $\widetilde{\mu}$ are not independent, based on other examples.
3. Let B be a Brownian motion independent of a step process X (in the sense of [28, Chap. 11]). Then B and $\widetilde{\mu}$, the compensated measure of the jump measure μ^X of X, have the weak PRP w.r.t. the usual filtration generated by X and B. An example for a step process is a multivariate (non-explosive) point process, as appearing in [17].
4. A (semi-)Markov chain X, possibly time-inhomogeneous, is a step process. Thus weak PRP (2.2) holds for a filtration generated by a Brownian motion and an independent Markov chain, relating later results to literature [3, 15, 16] on BSDEs driven by compensated random measures of the respective pure-jump (semi-)Markov processes. Markov chains X on countable state spaces can be chosen [15] to take values in the set of unit vectors $\{e_i : i \in \mathbb{N}\}$ of the sequence space $\ell^2 \subset \mathbb{R}^{\mathbb{N}}$, with jumps ΔX taking values $e_i - e_j$, $i, j \in \mathbb{N}$.

5. The pure jump martingale $U * \widetilde{\mu}$ (for $U \in \mathcal{L}^2(\widetilde{\mu})$) may be written as a series of mutually orthogonal martingales. More precisely, assume that the compensator coincides with the product measure $\lambda \otimes \mathrm{d}t$, i.e. $\zeta = 1$. Let $(u^n)_{n\in\mathbb{N}}$ be an orthonormal basis (ONB) of the separable Hilbert space $L^2(\lambda)$ with scalar product $\langle u, v\rangle := \int_E u(e)v(e)\,\lambda(\mathrm{d}e)$. Let $U_t = \sum_{n\in\mathbb{N}} \langle U_t, u^n\rangle u^n$ be the basis expansion of U_t for $U \in \mathcal{L}^2(\widetilde{\mu})$, $t \le T$. Then it holds (in $\mathcal{M}^2$)

$$U * \widetilde{\mu} = \sum_{n\in\mathbb{N}} \int_0^{\cdot} \langle U_t, u^n\rangle \int_E u^n(e)\,\widetilde{\mu}(\mathrm{d}t, \mathrm{d}e) =: \sum_{n\in\mathbb{N}} \int_0^{\cdot} \alpha^n_t\,\mathrm{d}L^n_t = \sum_{n\in\mathbb{N}} \alpha^n \bullet L^n, \tag{2.3}$$

for $\alpha^n_t := \langle U_t, u^n\rangle$ and $L^n := u^n * \widetilde{\mu}$. Indeed, setting $F^n_t := \sum_{k=1}^n \langle U_t, u^k\rangle u^k = \sum_{k=1}^n \alpha^k_t u^k$ one sees that $\| \sum_{k=1}^\infty |\alpha^k|^2 \|_{L^1(\mathbb{P}\otimes \mathrm{d}t)} \le \|U\|^2_{\mathcal{L}^2(\widetilde{\mu})} < \infty$. By dominated convergence one obtains as $n \to \infty$

$$\|F^n - U\|^2_{\mathcal{L}^2(\widetilde{\mu})} = \mathbb{E}\Big(\int_0^T\int_E |F^n_t(e) - U_t(e)|^2 \lambda(\mathrm{d}e)\,\mathrm{d}t\Big) = \mathbb{E}\Big(\int_0^T \sum_{k=n+1}^\infty |\alpha^k_t|^2\,\mathrm{d}t\Big) \to 0.$$

Isometry implies that the stochastic integrals $F^n * \widetilde{\mu}$ converge to $U * \widetilde{\mu}$ in $\mathcal{M}^2$, proving (2.3).
In particular, we see how the PRP (2.2) w.r.t. a random measure can be rewritten as series of ordinary stochastic integrals w.r.t. scalar-valued strongly orthogonal martingales L^n, which are in fact Lévy processes with deterministic characteristics $(0, 0, \int u^n(e)\,\lambda(\mathrm{d}e))$. In this sense, the general condition (2.2) links well with results on PRP and BSDEs for Lévy processes in [46, 47] who study a specific Teugels martingale basis consisting of compensated power jump processes for Lévy processes which satisfy exponential moment conditions. For a systematic analysis of related PRP results, comprising general Lévy processes, see [20, 21].
6. Previous arguments could extend to the general case with $\zeta \not\equiv 1$ in (2.1). To this end, suppose U^n to be in $\mathcal{L}^2(\widetilde{\mu})$ such that for all $t \le T$ the sequence $(U^n_t)_{n\in\mathbb{N}}$ is ONB of $L^2(\lambda_t)$ for $\mathrm{d}\lambda_t = \zeta_t \mathrm{d}\lambda$ with scalar product $\langle u, v\rangle_t := \int_E u(e)v(e)\,\zeta(t, e)\,\lambda(\mathrm{d}e)$. Analogously to case 5. above, with $\alpha^n_t := \langle U_t, U^n_t\rangle_t$ and $L^n := U^n * \widetilde{\mu}$ one gets equalities of martingales (in $\mathcal{M}^2$)

$$U * \widetilde{\mu} = \sum_{n\in\mathbb{N}} \int_0^{\cdot} \langle U_t, U^n_t\rangle_t \int_E U^n_t(e)\,\widetilde{\mu}(\mathrm{d}t, \mathrm{d}e) =: \sum_{n\in\mathbb{N}} \alpha^n \bullet L^n\,.$$

To proceed, we now define a solution of the Backward SDE with jumps to be a triple (Y, Z, U) of processes in the space $\mathcal{S}^p \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ for a suitable $p \in (1, \infty]$ that satisfies

$$Y_t = \xi + \int_t^T f_s(Y_{s-}, Z_s, U_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s - \int_t^T\int_E U_s(e)\,\widetilde{\mu}(\mathrm{d}s, \mathrm{d}e), \quad 0 \le t \le T, \tag{2.4}$$

for given data (ξ, f), consisting of a $\mathcal{F}_T$-measurable random variable ξ and a generator function $f_t(y, z, u) = f(\omega, t, y, z, u)$. The values p will be specified below in the respective results, although a particular focus will be on bounded BSDE solutions (i.e. $p = \infty$). Because we permit ν to be time-inhomogeneous with a bounded but possibly non-constant density ζ in (2.1), it does not hold in general that U_t takes values a.e. in one space $L^2(\lambda)$ for $U \in \mathcal{L}^2(\widetilde{\mu})$. This requires some extra consideration about the domain of definition and measurability of f, as the generator function f needs to be defined for u-arguments from a suitable domain, which cannot be some fixed L^2-space in general (and needs to be larger than $L^2(\lambda)$), as integrability of $u = U_t(\omega, \cdot)$ over $e \in E$ may vary with (ω, t). On suitable larger domains, one typically may have to admit for f to attain non-finite values. To this end, let us denote by $L^0(\mathcal{B}(E), \lambda)$ the space of all $\mathcal{B}(E)$-measurable functions with the topology of convergence in measure and define

$$|u - u'|_t := \Big(\int_E |u(e) - u'(e)|^2 \, \zeta(t, e) \, \lambda(\mathrm{d}e) \Big)^{\frac{1}{2}}, \tag{2.5}$$

for functions u, u' in $L^0(\mathcal{B}(E), \lambda)$. Terminal conditions ξ for BSDE considered in this paper will be taken to be square integrable $\xi \in L^2(\mathcal{F}_T)$ and often even as bounded $\xi \in L^\infty(\mathcal{F}_T)$. Generator functions $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times L^0(\mathcal{B}(E), \lambda) \to \overline{\mathbb{R}}$ are always taken to be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+1}) \otimes \mathcal{B}(L^0(\mathcal{B}(E), \lambda))$-measurable. Main Theorems 3.9 and 4.13 are derived for families of generators having the form

$$f_t(y, z, u) := \widehat{f}_t(y, z) + \int_A g_t(y, z, u(e), e)\zeta(t, e)\lambda(\mathrm{d}e) \quad \text{(where finitely defined)} \tag{2.6}$$

and $f_t(y, z, u) := \infty$ elsewhere, or more specially (for a g-component not depending on y, z)

$$f_t(y, z, u) := \widehat{f}_t(y, z) + \int_A g_t(u(e), e) \, \zeta(t, e) \, \lambda(\mathrm{d}e) \quad \text{(where finitely defined)} \tag{2.7}$$

and $f_t(y, z, u) := \infty$ elsewhere, for a $\mathcal{B}(E)$-measurable set A and component functions $\widehat{f}$, g where $\widehat{f} : \Omega \times [0, T] \times \mathbb{R}^{1+d} \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+1})$-measurable and $g : \Omega \times [0, T] \times \mathbb{R}^{1+d} \times \mathbb{R} \times E \to \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+2}) \otimes \mathcal{B}(E)$-measurable. Clearly statements for generators of the form (2.6) are also true for those of the (more particular) form (2.7). (In)finite activity relates to generators with $\lambda(A) < \infty$ (respectively $\lambda(A) = \infty$). A simple but useful technical Lemma clarifies how we can (and always will) choose a bounded representative for U in a BSDE solution (Y, Z, U) with bounded Y.

Lemma 2.2 *Let $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ be a solution of some JBSDE (2.4) with data (ξ, f). Then there exists a representative U' of U, bounded pointwise by $2|Y|_\infty$, such that $U' = U$ in $\mathcal{L}^2(\widetilde{\mu})$ and $\mathbb{P} \otimes \mathrm{d}t$-a-e., and (Y, Z, U') solves the BSDE (ξ, f).*

Proof We reproduce a brief argument sufficient to our general setting, similarly to e.g. [44, Corollary 1] or [7, proof of Theorem 3.5]. Use that $\mu(\omega, \mathrm{d}t, \mathrm{d}e) = \sum_{s\geq 0} \mathbb{1}_D(\omega, s)\, \delta_{(s,\beta_s(\omega))}(\mathrm{d}t, \mathrm{d}e)$ for an optional E-valued process β and a thin set D, since μ is an integer-valued random measure [31, II.§1b]. Clearly the jump $\Delta Y_t(\omega) = (Y_t - Y_{t-})(\omega) = \int_E U_t(\omega, e)\, \mu(\omega; \{t\}, \mathrm{d}e)$ is equal to $\mathbb{1}_D(\omega, t) U_t(\omega, \beta_t(\omega))$ and bounded by $2|Y|_\infty$. For $U'_t(\omega, e) := U_t(\omega, e)\mathbb{1}_D(\omega, t)\mathbb{1}_{\{\beta_t\}}(e)$, we have $U_t(\omega, \beta_t(\omega)) = U'_t(\omega, \beta_t(\omega))$ on D, and $\sum_{s\geq 0} \mathbb{1}_D(\omega, s)|U_s - U'_s|^2(\omega, \beta_s(\omega)) = 0$ implies $E[|U - U'|^2 * \nu_T] = E[|U - U'|^2 * \mu_T] = 0$. Since $U = U'$ in $\mathcal{L}^2(\widetilde{\mu})$ and $U_t = U'_t$ in $L^0(\mathcal{B}(E), \lambda)$, the BSDE is solved by (Y, Z, U'). □

Under these conditions, we can and will take U to be bounded by twice the norm of Y; Defining $|U|_\infty := \operatorname{ess\,sup}_{(\omega,t,e)} |U_t(e)|$ for $U \in \mathcal{L}^2(\widetilde{\mu})$ yields $|U|_\infty \leq 2|Y|_\infty$ for any bounded BSDE solution (Y, Z, U). The next lemma notes that the stochastic integrals of bounded JBSDE solutions are BMO-martingales when some truncated generator function is bounded from above (below) by $+(-)\langle M\rangle$ for a BMO-martingale M; Moreover, their BMO-norms depend only on $|Y|_\infty$, the BMO-norm of M and the horizon T. See [36, Lemma 1.3] for details of the proof, and note that BMO-properties of integrals of (bounded) BSDEs are of course a well-studied topic, cf. [42] and references therein.

Lemma 2.3 *Let $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ be a bounded solution to the BSDE (ξ, f). Assume there is $M \in \mathrm{BMO}(\mathbb{P})$ such that $\int_t^T f_s(Y_{s-}, Z_s, U_s)\,\mathrm{d}s \leq \langle M\rangle_T - \langle M\rangle_t$ or $-\int_t^T f_s(Y_{s-}, Z_s, U_s)\,\mathrm{d}s \leq \langle M\rangle_T - \langle M\rangle_t$. Then $\int Z\,\mathrm{d}B$ and $U * \widetilde{\mu}$ are BMO-martingales and their BMO-norms (resp. L^2-norms) are bounded by a constant depending on $|Y|_\infty$ and $\|M\|_{\mathrm{BMO}(\mathbb{P})}$ (resp. on $|Y|_\infty$, $\|M\|_{\mathcal{M}^2}$).*

3 Comparison Theorems and A-Priori-Estimates

The stage for the main comparison Theorem 3.9 and the a-priori-L^∞-estimate of Proposition 3.11 in this section is set by the next proposition. Its line of proof follows the seminal Theorem 2.5 by [54], with slight generalizations that are needed in the sequel. Just some details for the change of measure argument are elaborated a bit differently, measurable dependencies of the random field γ are specified in more detail, and less is assumed on the generators. Instead of imposing specific conditions on the generators which imply existence of solutions, we only insist that we have solutions and impose a generalized $(\mathbf{A}_\gamma)$-condition as explained in Example 3.8.

Proposition 3.1 *Let $(Y^i, Z^i, U^i) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ be solutions to the BSDE (2.4) for data (ξ_i, f_i), $i = 1, 2$. Assume that f_2 is Lipschitz continuous w.r.t. y and z. Let $\gamma : \Omega \times [0, T] \times \mathbb{R}^{d+3} \times E \to [-1, \infty)$ with $(\omega, t, y, z, u, u', e) \mapsto \gamma_t^{y,z,u,u'}(e)$*

be a $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^{d+3})\otimes\mathcal{B}(E)$-measurable function such that for $\overline{\gamma} := \gamma^{Y^2_-,Z^2,U^1,U^2}$ it holds

$$f_2(t,Y^2_{t-},Z^2_t,U^1_t) - f_2(t,Y^2_{t-},Z^2_t,U^2_t) \le \int_E \overline{\gamma}_t(e)\,(U^1_t(e)-U^2_t(e))\,\zeta(t,e)\,\lambda(\mathrm{d}e),\ \mathbb{P}\otimes\mathrm{d}t\text{-a.e.} \tag{3.1}$$

and the stochastic exponential $\mathcal{E}(\int\beta\,\mathrm{d}B + \overline{\gamma}*\widetilde{\mu})$ is a martingale for β from (3.2). Then a comparison result holds, that means that the inequalities $\xi_1\le\xi_2$ and $f_1(t,Y^1_{t-},Z^1_t,U^1_t)\le f_2(t,Y^1_{t-},Z^1_t,U^1_t)$, $\mathbb{P}\otimes\mathrm{d}t$-a.e., together imply $Y^1_t\le Y^2_t$ for all $t\le T$.

In results like the above, in [54] and further enhancements [38, 52, 57], the key assumption needed for comparison is the existence of an abstract random field γ such that inequalities are satisfied between processes. In contrast, the subsequent results of this section offer sufficient criteria for comparison that can be verified more easily by checking concrete dependencies w.r.t. to basically Euclidean arguments for generator functions f of the type (2.6). See also [24] for a simpler version in a setting with a jump measure of Lévy-type on $E=\mathbb{R}^1\setminus\{0\}$ and $\zeta\equiv 1$.

Proof We define $\widehat{\xi}:=\xi_1-\xi_2$, $\widehat{Y}:=Y^1-Y^2$, $\widehat{Z}:=Z^1-Z^2$ and $\widehat{U}:=U^1-U^2$. The processes

$$\begin{aligned}
\alpha_s &:= \mathbb{1}_{\{Y^1_{s-}\neq Y^2_{s-}\}}\frac{f_2(s,Y^1_{s-},Z^1_s,U^1_s)-f_2(s,Y^2_{s-},Z^1_s,U^1_s)}{(Y^1_{s-}-Y^2_{s-})},\\
\beta_s &:= \mathbb{1}_{\{Z^1_s\neq Z^2_s\}}\frac{f_2(s,Y^2_{s-},Z^1_s,U^1_s)-f_2(s,Y^2_{s-},Z^2_s,U^1_s)}{\|Z^1_s-Z^2_s\|^2}(Z^1_s-Z^2_s)
\end{aligned} \tag{3.2}$$

and $R_t:=\exp(\int_0^t\alpha_s\,\mathrm{d}s)$ are bounded due to the Lipschitz assumption on f_2. As in [54], applying Itô's formula to $R\widehat{Y}$ between $\tau\wedge t$ and $\tau\wedge T$ for some stopping times τ yields

$$\begin{aligned}
(R\widehat{Y})_{\tau\wedge t} &= (R\widehat{Y})_{\tau\wedge T} + \int_{\tau\wedge t}^{\tau\wedge T} R_s\big(f_1(s,Y^1_{s-},Z^1_s,U^1_s)-f_2(s,Y^2_{s-},Z^2_s,U^2_s)\big)\,\mathrm{d}s\\
&\quad - \int_{\tau\wedge t}^{\tau\wedge T} R_s\widehat{Z}_s\,\mathrm{d}B_s - \int_{\tau\wedge t}^{\tau\wedge T}\int_E R_s\widehat{U}_s(e)\,\widetilde{\mu}(\mathrm{d}s,\mathrm{d}e) - \int_{\tau\wedge t}^{\tau\wedge T} R_s\alpha_s\widehat{Y}_{s-}\,\mathrm{d}s.
\end{aligned}$$

Set $M:=\int R\widehat{Z}\,\mathrm{d}B+(R\widehat{U})*\widetilde{\mu}$ and $N:=\int\beta\,\mathrm{d}B+\overline{\gamma}*\widetilde{\mu}$. Then $\mathrm{d}\mathbb{Q}:=\mathcal{E}(N)_T\mathrm{d}\mathbb{P}$ defines an absolutely continuous probability by the martingale property of the stochastic exponential $\mathcal{E}(N)\ge 0$; cf. [28, Lemma 9.40]. By Girsanov $L:=M-\langle M,N\rangle$ is a local $\mathbb{Q}$-martingale, and the inequality

$$\begin{gathered}
f_1(s,Y^1_{s-},Z^1_s,U^1_s)-f_2(s,Y^2_{s-},Z^2_s,U^2_s)\le\alpha_s\widehat{Y}_{s-}+\beta_s\widehat{Z}_s+\int_E\overline{\gamma}_s(e)\widehat{U}_s(e)\,\zeta_s(e)\,\lambda(\mathrm{d}e)\ \mathbb{P}\otimes\mathrm{d}s\text{-a.e.}\\
\text{implies}\quad (R\widehat{Y})_{\tau\wedge t}\le(R\widehat{Y})_{\tau\wedge T}-(L^\tau_T-L^\tau_t).
\end{gathered} \tag{3.3}$$

Localizing L along a sequence of stopping times $\tau_n \uparrow \infty$ and taking conditional expectations, we obtain $\mathbb{E}_{\mathbb{Q}}\big((R\widehat{Y})_{t\wedge\tau^n} \,\big|\, \mathcal{F}_t\big) \le \mathbb{E}_{\mathbb{Q}}\big((R\widehat{Y})_{\tau^n\wedge T} \,\big|\, \mathcal{F}_t\big)$ for each $n \in \mathbb{N}$. Dominated convergence yields the estimate $R_t\widehat{Y}_t \le \mathbb{E}_{\mathbb{Q}}\big(R_T\widehat{\xi}\,\big|\,\mathcal{F}_t\big) \le 0$ and thus $Y_t^1 \le Y_t^2$. □

Remark 3.2 1. Switching roles of f_1 and f_2, one gets that if f_1 is Lipschitz in y,z and satisfies (3.1) instead of f_2, then $\xi_1 \le \xi_2$ and $f_1(t, Y_{t-}^2, Z_t^2, U_t^2) \le f_2(t, Y_{t-}^2, Z_t^2, U_t^2)$ imply $Y_t^1 \le Y_t^2$.

2. The result of Proposition 3.1 remains valid (with a similar proof) if one requires that the Y-components of JBSDE solutions to compare are in $\mathcal{S}^2$ instead of $\mathcal{S}^\infty$, and the stochastic exponential $\mathcal{E}(\beta \bullet B + \overline{\gamma} * \widetilde{\mu})$ is in $\mathcal{S}^2$. However, as it is stated, Proposition 3.1 is exactly what we will need to apply in the sequel to derive, e.g., Proposition 4.3 and Theorem 4.13.

Example 3.3 Sufficient conditions for $\mathcal{E}(\overline{\gamma} * \widetilde{\mu})$ to be a martingale are, for instance,

1. $\Delta(\overline{\gamma} * \widetilde{\mu}) > -1$ and $\mathbb{E}\big(\exp(\langle \overline{\gamma} * \widetilde{\mu}\rangle_T)\big) = \mathbb{E}\big(\exp\big(\int_0^T \int_E |\overline{\gamma}_s(e)|^2\, \nu(\mathrm{d}s, \mathrm{d}e)\big)\big) < \infty$; see [51, Theorem 9]. This holds i.p. if $\int_E |\overline{\gamma}_s(e)|^2\, \zeta(s,e)\, \lambda(\mathrm{d}e) < const. < \infty$ $\mathbb{P} \otimes \mathrm{d}s$-a.e. and $\overline{\gamma} > -1$.
2. $\Delta(\overline{\gamma} * \widetilde{\mu}) \ge -1 + \delta$ for $\delta > 0$ and $\overline{\gamma} * \widetilde{\mu}$ is a BMO($\mathbb{P}$)-martingale due to Kazamaki [33].
3. $\Delta(\overline{\gamma} * \widetilde{\mu}) \ge -1$ and $\overline{\gamma} * \widetilde{\mu}$ is a uniformly integrable martingale and $\mathbb{E}\big(\exp(\langle \overline{\gamma} * \widetilde{\mu}\rangle_T)\big) < \infty$; see [41, Theorem I.8]. Such a condition is satisfied when $\overline{\gamma}$ is bounded and $|\overline{\gamma}| \le \psi$, $\mathbb{P} \otimes \mathrm{d}t \otimes \lambda$-a.e. for a function $\psi \in L^2(\lambda)$ and $\zeta \equiv 1$. The latter is what is required for instance in the comparison Theorem 4.2 of [52].

Note that under above conditions, also the stochastic exponential $\mathcal{E}(\int \beta \mathrm{d}B + \overline{\gamma} * \widetilde{\mu})$ for β bounded and predictable is a martingale, as it is easily seen by Novikov's criterion.

Let us also refer to [14, Sects. 19 and A.9] for related so-called balance conditions on generators for JBSDE comparison by change of measure arguments.

In the statement of Proposition 3.1, the dependence of the process $\overline{\gamma}$ on the BSDE solutions is not needed for the proof as the same result holds if $\overline{\gamma}$ is just a predictable process such that the estimate on the generator f_2 and the martingale property (3.1) hold. The further functional dependence is needed for the sequel, as required in the following

Definition 3.4 We say that an $\overline{\mathbb{R}}$-valued generator function f satisfies condition ($\mathbf{A}_\gamma$) if there is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$-measurable function $\gamma : \Omega \times [0,T] \times \mathbb{R}^{d+3} \times E \to (-1, \infty)$ given by $(\omega, t, y, z, u, u', e) \mapsto \gamma_t^{y,z,u,u'}(e)$ such that for all $(Y, Z, U, U') \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times (\mathcal{L}^2(\widetilde{\mu}))^2$ with $|U|_\infty < \infty$, $|U'|_\infty < \infty$ it holds for $\overline{\gamma} := \gamma^{Y_-,Z,U,U'}$

$$f_t(Y_{t-}, Z_t, U_t) - f_t(Y_{t-}, Z_t, U_t') \le \int_E \overline{\gamma}_t(e)(U_t(e) - U_t'(e))\zeta(t,e)\lambda(\mathrm{d}e),\ \mathbb{P} \otimes \mathrm{d}t\text{-a.e.}$$

$$\text{and } \mathcal{E}\Big(\int \beta \mathrm{d}B + \overline{\gamma} * \widetilde{\mu}\Big) \text{ is a martingale for every bounded and predictable } \beta. \tag{3.4}$$

We will say that f satisfies condition ($\mathbf{A}'_\gamma$) if the above holds for all bounded U and U' with additionally $U * \widetilde{\mu}$ and $U' * \widetilde{\mu}$ in BMO($\mathbb{P}$).

Clearly, existence and applicability of a suitable comparison result for solutions to JBSDEs implies their uniqueness. In other words, if there exists a bounded solution for a generator being Lipschitz w.r.t. y and z which satisfies ($\mathbf{A}_\gamma$) or ($\mathbf{A}'_\gamma$), we obtain that such a solution is unique.

Example 3.5 The natural candidate for γ for generators f of the form (2.6) is given by

$$\gamma_s^{y,z,u,u'}(e) = \frac{g_s(y,z,u,e) - g_s(y,z,u',e)}{u-u'} \, \mathbb{1}_A(e) \, \mathbb{1}_{\{u \neq u'\}}, \tag{3.5}$$

which is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$-measurable since g is. Assuming absolute continuity of g in u, we can express $\gamma_s^{y,z,u,u'}(e) = \int_0^1 \frac{\partial}{\partial u} g_s(y,z,tu+(1-t)u',e)\,\mathrm{d}t \, \mathbb{1}_A(e)$, by noting that

$$(u-u')\int_0^1 \frac{\partial}{\partial u} g_s(y,z,tu+(1-t)u',e)\,\mathrm{d}t \, \mathbb{1}_A(e) = \int_0^1 \frac{\partial}{\partial t}\left[(g_s(y,z,tu+(1-t)u',e))\right]\mathrm{d}t \, \mathbb{1}_A(e).$$

For generators of type (2.7) the γ simply is

$$\gamma_s^{y,z,u,u'}(e) = \int_0^1 \frac{\partial}{\partial u} g_s(tu+(1-t)u',e)\mathrm{d}t \, \mathbb{1}_A(e).$$

Definition 3.6 We say that a generator f satisfies condition ($\mathbf{A_{fin}}$) or ($\mathbf{A_{infi}}$) (on a set D) if

1. ($\mathbf{A_{fin}}$): f is of the form (2.6) with $\lambda(A) < \infty$, is Lipschitz continuous w.r.t. y and z uniformly in (t,ω,u), and the map $u \mapsto g(t,y,z,u,e)$ is absolutely continuous (in u) for all (ω,t,y,z,e) (in $D \subseteq \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times E$), i.e. $g(t,y,z,u,e) = g(0) + \int_0^u g'(t,y,z,x,e)\mathrm{d}x$, with density function g' being strictly greater than -1 (on D) and locally bounded (in u) from above, uniformly in (ω,t,y,z,e).
2. ($\mathbf{A_{infi}}$): f is of the form (2.7), is Lipschitz continuous w.r.t. y and z uniformly in (t,ω,u), and the map $u \mapsto g_t(u,e)$ is absolutely continuous (in u) for all (ω,t,e) (in D), i.e. $g(t,u,e) = g(0) + \int_0^u g'(t,x,e)\mathrm{d}x$, with density function g' being such that for all $c \in (0,\infty)$ there exists $K(c) \in \mathbb{R}$ and $\delta(c) \in (0,1)$ with $-1+\delta(c) \le g'(x)$ and $|g'(x)| \le K(c)|x|$ for all x with $|x| \le c$.

Remark 3.7 Note that under condition ($\mathbf{A_{infi}}$) the density function g' is necessarily locally bounded, in particular with $|g'(x)| \le K(c)c =: \bar{K}(c) < \infty$ for all $x \in [-c,c]$. Observe that the conditions are not requiring the function g to be convex and moreover refrain from requiring it to be continuously differentiable in u. Both can be helpful in application examplres, see Sect. 5.1.2.

Example 3.8 Sufficient conditions for condition ($\mathbf{A}_\gamma$) and ($\mathbf{A}'_\gamma$) are

1. γ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^{d+3}) \otimes \mathcal{B}(E)$-measurable function satisfying the inequality in (3.4) and
$$C_1(1 \wedge |e|) \leq \gamma_t^{y,z,u,u'}(e) \leq C_2(1 \wedge |e|)$$
on $E = \mathbb{R}^l \setminus \{0\}$ ($l \in \mathbb{N}$), for some $C_1 \in (-1, 0]$ and $C_2 > 0$. In this case $\exp(\langle \int \beta \mathrm{d}B + \overline{\gamma} * \widetilde{\mu} \rangle_T)$ is clearly bounded and the jumps of $\int \beta \mathrm{d}B + \overline{\gamma} * \widetilde{\mu}$ are bigger than -1. Hence $\mathcal{E}\left(\int \beta \mathrm{d}B + \overline{\gamma} * \widetilde{\mu}\right)$ is a positive martingale [51, Theorem 9]. Thus Definition 3.4 generalizes the original ($\mathbf{A}_\gamma$)-condition introduced by [54] for Poisson random measures.
2. ($\mathbf{A_{fin}}$) is sufficient for ($\mathbf{A}_\gamma$). This follows from Example 3.3, (3.5) and $\lambda(A) < \infty$.
3. ($\mathbf{A_{infi}}$) is sufficient for ($\mathbf{A}'_\gamma$). To see this, let u, u' be bounded by c and γ be the natural candidate in Example 3.5. Then $|\gamma_s^{y,z,u,u'}(e)| \leq \int_{u'}^{u} |g'(x)| \mathrm{d}x/(u - u') \leq K(c)(|u| + |u'|)$. Hence $\int \beta \mathrm{d}B + \overline{\gamma} * \widetilde{\mu}$ is a BMO-martingale by the BMO-property of $U * \widetilde{\mu}$ and $U' * \widetilde{\mu}$ with some lower bound $-1 + \delta$ for its jumps. And $\mathcal{E}(\int \beta \mathrm{d}B + \overline{\gamma} * \widetilde{\mu})$ is a martingale by part 2 of Example 3.3.
4. Condition ($\mathbf{A_{fin}}$) above is satisfied if, e.g., f is of the form (2.6) with $\lambda(A) < \infty$, is Lipschitz continuous w.r.t. y and z, and the map $u \mapsto g(t, y, z, u, e)$ is continuously differentiable for all (ω, t, y, z, e) (in D) such that the derivative is strictly greater than -1 (on $D \subseteq \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times E$) and locally bounded (in u) from above, uniformly in (ω, t, y, z, e).
5. Condition ($\mathbf{A_{infi}}$) is valid if for instance f is of the form (2.7), is Lipschitz continuous w.r.t. y and z, and the map $u \mapsto g_t(u, e)$ is twice continuously differentiable for all (ω, t, e) with the derivatives being locally bounded uniformly in (ω, t, e), the first derivative being (locally) bounded away from -1 with a lower bound $-1 + \delta$ for some $\delta > 0$, and $\frac{\partial g}{\partial u}(t, 0, e) \equiv 0$.

As an application of the above, we can now provide simple conditions for comparison in terms of concrete properties of the generator function, which are easier to verify than the more general but abstract conditions on the existence of a suitable function γ as in Proposition 3.1 or the general conditions by [15]. Note that no convexity is required in the z or u argument of the generator. The result will be applied later to prove existence and uniqueness of JBSDE solutions.

Theorem 3.9 (Comparison Theorem) *A comparison result between bounded BSDE solutions in the sense of Proposition 3.1 holds true in each of the following cases:*

1. (finite activity) *f_2 satisfies ($\mathbf{A_{fin}}$).*
2. (infinite activity) *f_2 satisfies ($\mathbf{A_{infi}}$) and $U^1 * \widetilde{\mu}$ and $U^2 * \widetilde{\mu}$ are* BMO*($\mathbb{P}$)-martingales for the corresponding JBSDE solutions (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2).*

Proof This follows directly from Proposition 3.1 and Example 3.8, noting that representation (3.5) in connection with condition ($\mathbf{A_{fin}}$) resp. ($\mathbf{A_{infi}}$) meets the sufficient conditions in Example 3.3. □

Unlike classical a-priori estimates that offer some L^2-norm estimates for the BSDE solution in terms of the data, the next result gives a simple L^∞-estimate for the Y-component of the solution. Such will be useful for the derivation of BSDE solution bounds and for truncation arguments.

Proposition 3.10 *Let $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ be a solution to the BSDE (ξ, f) with $\xi \in L^\infty(\mathcal{F}_T)$, f be Lipschitz continuous w.r.t. (y, z) with Lipschitz constant $K_f^{y,z}$ and satisfying $(\boldsymbol{A}_\gamma)$ with $f_\cdot(0,0,0)$ bounded.*
Then $|Y_t| \le \exp\big(K_f^{y,z}(T-t)\big)\big(|\xi|_\infty + (T-t)|f_\cdot(0,0,0)|_\infty\big)$ for $t \le T$.

Proof Set $(Y^1, Z^1, U^1) = (Y, Z, U)$, $(\xi^1, f^1) = (\xi, f)$, $(Y^2, Z^2, U^2) = (0, 0, 0)$ and $(\xi^2, f^2) = (0, f)$. Then following the proof of Proposition 3.1, Eq. (3.3) becomes

$$(RY)_{\tau\wedge t} \le (RY)_{\tau\wedge T} + \int_{\tau\wedge t}^{\tau\wedge T} R_s f_s(0,0,0)\,\mathrm{d}s - (L_T^\tau - L_t^\tau), \quad t \in [0, T],$$

for all stopping times τ where $L := M - \langle M, N\rangle$ is in $\mathcal{M}_{\mathrm{loc}}(\mathbb{Q})$, $M := \int RZ\,\mathrm{d}B + (RU) * \widetilde{\mu}$ is in $\mathcal{M}^2$, $N := \int \beta\,\mathrm{d}B + \overline{\gamma} * \widetilde{\mu}$ with $\overline{\gamma} := \gamma^{0,0,U,0}$ and the probability measure $\mathbb{Q} \approx \mathbb{P}$ is given by $\mathrm{d}\mathbb{Q} := \mathcal{E}(N)_T\mathrm{d}\mathbb{P}$. Localizing L along some sequence $\tau^n \uparrow \infty$ of stopping times yields

$$\mathbb{E}_{\mathbb{Q}}\big((RY)_{\tau^n\wedge t} \,\big|\, \mathcal{F}_t\big) \le \mathbb{E}_{\mathbb{Q}}\Big((RY)_{\tau^n\wedge T} + \int_{\tau\wedge t}^{\tau\wedge T} R_s f_s(0,0,0)\,\mathrm{d}s \,\Big|\, \mathcal{F}_t\Big).$$

By dominated convergence, we conclude that $\mathbb{P}$-a.e

$$Y_t \le \mathbb{E}_{\mathbb{Q}}\Big(\frac{R_T}{R_t}\xi + \int_t^T \frac{R_s}{R_t} f_s(0,0,0)\,\mathrm{d}s \,\Big|\, \mathcal{F}_t\Big) \le \mathrm{e}^{K_f^{y,z}(T-t)}\big(|\xi|_\infty + (T-t)|f_\cdot(0,0,0)|_\infty\big).$$

Analogously, if we define $\overline{N} := \int \beta\,\mathrm{d}B + \overline{\overline{\gamma}} * \widetilde{\mu}$ with $\overline{\overline{\gamma}} := \gamma^{0,0,0,U}$, and $\overline{\mathbb{Q}}$ equivalent to $\mathbb{P}$ via $\mathrm{d}\overline{\mathbb{Q}} := \mathcal{E}(\overline{N})_T\mathrm{d}\mathbb{P}$, we deduce that $\overline{L} := M - \langle M, \overline{N}\rangle$ is in $\mathcal{M}_{\mathrm{loc}}(\overline{\mathbb{Q}})$ and

$$(RY)_{\tau\wedge t} \ge (RY)_{\tau\wedge T} + \int_{\tau\wedge t}^{\tau\wedge T} R_s f_s(0,0,0)\,\mathrm{d}s - (\overline{L}_T^\tau - \overline{L}_t^\tau), \quad t \in [0, T],$$

for all stopping times τ. This yields the required lower bound. □

Again, we can specify explicit conditions on the generator function that are sufficient to ensure the more abstract assumptions of the previous result.

Proposition 3.11 *Let $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ be a solution to the BSDE (ξ, f) with ξ in $L^\infty(\mathcal{F}_T)$, f being Lipschitz continuous w.r.t. (y, z) with Lipschitz constant $K_f^{y,z}$ such that $f_\cdot(0,0,0)$ is bounded. Assume that one of the following conditions holds:*

1. (finite activity) f *satisfies* ($\boldsymbol{A_{fin}}$).
2. (infinite activity) f *satisfies* ($\boldsymbol{A_{infi}}$) *and* $U * \widetilde{\mu}$ *is a* BMO($\mathbb{P}$)*-martingale.*

Then $|Y_t| \le \exp\big(K_f^{y,z}(T-t)\big)\big(|\xi|_\infty + (T-t)|f_s(0,0,0)|_\infty\big)$ *holds for all* $t \le T$, *in particular* $|Y|_\infty \le \exp\big(K_f^{y,z}T\big)\big(|\xi|_\infty + T|f_s(0,0,0)|_\infty\big)$.

Proof This follows directly from Proposition 3.10 and Example 3.8, since f satisfies condition ($\mathbf{A}_\gamma$) (resp. ($\mathbf{A}'_\gamma$)) using Eq. (3.5). □

In the last part of this section we apply our comparison theorem for more concrete generators. To this end, we consider a generator f being truncated at bounds $a < b$ (depending on time only) as

$$\widetilde{f}_t(y,z,u) := f_t\big(\kappa(t,y),\ z,\ \kappa(t,y+u)-\kappa(t,y)\big), \tag{3.6}$$

with $\kappa(t,y) := \big(a(t)\vee y\big)\wedge b(t)$. Next, we show that if a generator satisfies ($\mathbf{A}_\gamma$) within the truncation bounds, then the truncated generator satisfies ($\mathbf{A}_\gamma$) everywhere.

Lemma 3.12 *Let* f *satisfy (3.4) for* Y, U *such that*

$$a(t) \le Y_{t-},\, Y_{t-}+U_t(e),\, Y_{t-}+U'_t(e) \le b(t), \quad t \in [0,T]$$

and let γ *satisfy one of the conditions of Example 3.3 for the martingale property of* $\mathcal{E}(\overline{\gamma} * \widetilde{\mu})$. *Then* $\widetilde{f}$ *satisfies (3.4). Especially, if* f *satisfies* ($\boldsymbol{A_{fin}}$) *on the set where* $a(t) \le y,\, y+u \le b(t)$ *then* $\widetilde{f}$ *is Lipschitz in* (y,z), *locally Lipschitz in u and satisfies* ($\boldsymbol{A}_\gamma$).

Proof Using monotonicity of $x \mapsto \kappa(t,x)$, we get that

$$\widetilde{f}_t(Y_{t-},Z_t,U_t) - \widetilde{f}_t(Y_{t-},Z_t,U'_t)$$

equals

$$\begin{aligned}
&f_t\big(\kappa(t,Y_{t-}),Z_t,\kappa(t,Y_{t-}+U_t)-\kappa(t,Y_{t-})\big) - f_t\big(\kappa(t,Y_{t-}),Z_t,\kappa(t,Y_{t-}+U'_t)-\kappa(t,Y_{t-})\big)\\
&\quad\le \int_E \overline{\gamma}_t(e)\big(\kappa(t,Y_{t-}+U_t(e))-\kappa(t,Y_{t-}+U'_t(e))\big)\,\zeta(t,e)\,\lambda(\mathrm{d}e)\\
&\quad\le \int_E \overline{\gamma}_t(e)\big(\mathbb{1}_{\{\overline{\gamma}\ge 0,U\ge U'\}}+\mathbb{1}_{\{\overline{\gamma}<0,U<U'\}}\big)\big(U_t(e)-U'_t(e)\big)\,\zeta(t,e)\,\lambda(\mathrm{d}e).
\end{aligned}$$

Setting $\overline{\gamma^*} := \overline{\gamma}\big(\mathbb{1}_{\{\overline{\gamma}\ge 0,U\ge U'\}}+\mathbb{1}_{\{\overline{\gamma}<0,U<U'\}}\big)$ we see that the stochastic exponential $\mathcal{E}\big(\int \beta \mathrm{d}B + \overline{\gamma^*} * \widetilde{\mu}\big)$ is a martingale for all bounded and predictable processes β and $\widetilde{f}$ satisfies (3.4). The latter claim easily follows from the fact that if f satisfies ($\mathbf{A_{fin}}$) on $a(t) \le y,\, y+u \le b(t)$ then f satisfies (3.4) on $a(t) \le Y_{t-}, Y_{t-}+U_t(e), Y_{t-}+U'_t(e) \le b(t)$ using Example 3.8. The Lipschitz properties of $\widetilde{f}$ follow from the fact that κ is a contraction and f is Lipschitz within the truncation bounds. □

Concrete L^∞-bounds for bounded solutions to BSDE (ξ, f) with suitable $\widehat{f}$-part are provided by

Proposition 3.13 *Let f be a generator of the form (2.6) with $\left|\widehat{f_t}(y,z)\right| \le K_1 + K_2|y|$ for some $K_1, K_2 \ge 0$, $g_t(y,z,0,e) \equiv 0$ and $\xi \in L^\infty(\mathcal{F}_T)$ with $c_1 \le \xi \le c_2$ for some $c_1, c_2 \in \mathbb{R}$. Assume that there are solutions a and b to the ODEs $y'(t) = K_1 + K_2|y(t)|$, $y(T) = c_1$ and $y'(t) = -(K_1 + K_2|y(t)|)$, $y(T) = c_2$ respectively, such that $a \le b$ on $[0, T]$. If the truncated generator $\widetilde{f}$ in (3.6) satisfies $(\boldsymbol{A}_\gamma)$ and is Lipschitz in (y, z), then any solution $(\widetilde{Y}, \widetilde{Z}, \widetilde{U}) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ to the JBSDE $(\xi, \widetilde{f})$ also solves the JBSDE (ξ, f) and satisfies $a(t) \le \widetilde{Y}_t \le b(t)$, $t \in [0, T]$.*

Proof We set $Y_t := \kappa(t, \widetilde{Y}_t)$, $Z_t := \widetilde{Z}_t$, $U_t(e) := \kappa(t, \widetilde{Y}_{t-} + \widetilde{U}_t(e)) - \kappa(t, \widetilde{Y}_{t-})$ and

$$f_t^i(y,z,u) := \widehat{f_t}^i\big(\kappa(t,y),z\big) + \int_E g_t\big(\kappa(t,y), z, \kappa(t, y+u) - \kappa(t,y), e\big)\,\zeta(t,e)\,\lambda(\mathrm{d}e)$$

with $\widehat{f_t}^1(y,z) := -(K_1 + K_2|y|)$, $\widehat{f_t}^2(y,z) := \widehat{f_t}(y,z)$ and $\widehat{f_t}^3(y,z) := K_1 + K_2|y|$. By the assumptions on the ODEs, we have that $(a(t), 0, 0)$ solves the BSDE (c_1, f^1) and $(b(t), 0, 0)$ solves the BSDE (c_2, f^3). Taking into account that $\widetilde{f}^1 \le \widetilde{f}^2 \le \widetilde{f}^3$, $c_1 \le \xi \le c_2$ and $\widetilde{f}^2$ satisfies $(\mathbf{A}_\gamma)$, comparison theorem Proposition 3.1 yields $a(t) \le \widetilde{Y}_t \le b(t)$. Hence, Y and $\widetilde{Y}$ are indistinguishable, $U = \widetilde{U}$ in $\mathcal{L}^2(\widetilde{\mu})$ and $(\widetilde{Y}, \widetilde{Z}, \widetilde{U})$ solves the BSDE (ξ, f). □

In the next section, we apply these results to two situations: Using Corollary 4.4, we give an alternative proof of Theorem 3.5 of [7] via a comparison principle instead of an argument with stopping times. Moreover, the estimates in Corollary 4.6 are applied to solve the power utility maximization problem via a JBSDE approach in Sect. 5.2.

4 Existence and Uniqueness of Bounded Solutions

This section studies BSDE with jumps by the monotone stability approach. Building on (straightforward) results for finite activity, the infinite activity case is treated by monotone approximations.

4.1 The Case of Finite Activity

Definition 4.1 A generator function f satisfies condition $(\mathbf{B}_\gamma)$, if it is Lipschitz continuous in (y, z), locally Lipschitz continuous in u (in the sense that $u \mapsto f_t(y, z, -c \vee u \wedge c)$ is Lipschitz continuous for any $c \in (0, \infty)$), $f_\cdot(0, 0, 0)$ is bounded, and f satisfies condition $(\mathbf{A}_\gamma)$.

The next result readily leads to Proposition 4.3, for A in (2.6) with $\lambda(A) < \infty$.

Proposition 4.2 *Let $\xi \in L^\infty(\mathcal{F}_T)$ and f satisfies $(\boldsymbol{B}_\gamma)$. Then there exists a unique solution (Y, Z, U) in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ to the BSDE (ξ, f). Moreover for all $t \in [0, T]$, $|Y_t|$ is bounded by* $\exp\big(K_f^{y,z}(T-t)\big)\big(|\xi|_\infty + (T-t)|f_\cdot(0,0,0)|_\infty\big)$.

Proof Consider the Lipschitz generator $f_t^c(y, z, u) := f_t\big(y, z, (u \vee (-c)) \wedge c\big)$ with $c > 0$ and Lipschitz constant K_{f^c}. By classical fixed point arguments and a-priori estimates (cf. e.g. [7, Propositions 3.2, 3.3]) there is a unique solution $(Y^c, Z^c, U^c) \in \mathcal{S}^2 \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ to the BSDE (ξ, f^c); it satisfies

$$|Y_t^c| \leq C\mathbb{E}\Big(|\xi|^2 + \int_t^T |f_s^c(0,0,0)|^2 \,\mathrm{d}s \,\Big|\, \mathcal{F}_t\Big) \leq C\big(|\xi|_\infty^2 + T|f_\cdot(0,0,0)|_\infty^2\big) < \infty,$$

for some constant $C = C(T, K_{f^c})$. Now Proposition 3.10 implies that $|Y_t^c|$ is dominated by $\exp\big(K_f^{y,z}(T-t)\big)\big(|\xi|_\infty + (T-t)|f_\cdot(0,0,0)|_\infty\big)$ for all $c > 0$. Choosing $c \geq 2\exp\big(K_f^{y,z}T\big)\big(|\xi|_\infty + T|f_\cdot(0,0,0)|_\infty\big)$ we get that (Y^c, Z^c, U^c) with $Y^c \in \mathcal{S}^\infty$ solves the BSDE (ξ, f) since U^c is bounded by c. Uniqueness follows by comparison. □

This leads to a preliminary result on bounded solutions if jumps are of finite activity.

Proposition 4.3 *Let $\xi \in L^\infty(\mathcal{F}_T)$ and let f satisfy $(\boldsymbol{A_{fin}})$ (recall Definition 3.6) with $f_\cdot(0,0,0)$ bounded. Then there exists a unique solution (Y, Z, U) in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ to the BSDE (ξ, f). Moreover for all $t \in [0, T]$, $|Y_t|$ is bounded by* $\exp\big(K_f^{y,z}(T-t)\big)\big(|\xi|_\infty + (T-t)|f_\cdot(0,0,0)|_\infty\big)$.

Proof Noting that local Lipschitz continuity in u follows from the absolute continuity of g in u with locally bounded density function, the claim follows from Propositions 3.11 and 4.2. □

Corollary 4.4 *Let $\xi \in L^\infty(\mathcal{F}_T)$ and let f be a generator satisfying $(\boldsymbol{A_{fin}})$, with $g_t(y, z, 0, e) \equiv 0$ and $|\widehat{f}_t(y, z)| \leq K_1 + K_2|y|$ for some $K_1, K_2 \geq 0$. Set*

$$b(t) = \begin{cases} (|\xi|_\infty + \frac{K_1}{K_2})\exp(K_2(T-t)) - \frac{K_1}{K_2}, & K_2 \neq 0 \\ |\xi|_\infty + K_1(T-t), & K_2 = 0. \end{cases}$$

Then there exists a unique solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ to the BSDE (ξ, f) and moreover $|Y_t| \leq b_t$ for $t \in [0, T]$. Finally $\int Z\,\mathrm{d}B$ and $U * \widetilde{\mu}$ are BMO($\mathbb{P}$)-*martingales.*

Proof By Lemma 3.12 and Proposition 4.3, there is a unique solution (Y, Z, U) in the space $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ to the BSDE $(\xi, \widetilde{f})$. By Proposition 3.13, it also solves the BSDE (ξ, f) and $-b(t) \leq Y_t \leq b(t)$, $\forall t \in [0, T]$. Uniqueness follows from the fact that one can apply the comparison Theorem 3.9 for generators satisfying $(\mathbf{A_{fin}})$. The BMO property follows from Lemma 2.3. □

Remark 4.5 Corollary 4.4 is similar to Theorem 3.5 in [7], but its proof is different: It relies on previous comparison results for JBSDEs instead of stopping arguments. The stochastic integrals of the BSDE solution are BMO-martingales under the assumptions for Lemma 2.3, which hold e.g. under the conditions for [7, Theorem 3.6]

Corollary 4.6 *Let $\xi \in L^\infty(\mathcal{F}_T)$ with $\xi \geq C$ for some constant $C > 0$, $K \geq 0$ and set $a(t) := C\exp(-K(T-t))$ and $b(t) = |\xi|_\infty \exp(K(T-t))$, $\forall t \in [0,T]$. Assume f satisfies ($\boldsymbol{A_{fin}}$) for $c \leq y, y+u \leq d$ for all $c, d \in \mathbb{R}$ with $0 < c < d$, and that $\left|\widehat{f}_t(y,z)\right| \leq K|y|$ and $g_t(y,z,0,e) = 0$. Then there exists a unique solution $(Y,Z,U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ to the BSDE (ξ, f) with $Y \geq \epsilon$ for some $\epsilon > 0$. Moreover, it holds $a(t) \leq Y_t \leq b(t)$ and $\int Z\,\mathrm{d}B$ and $U * \widetilde{\mu}$ are* BMO($\mathbb{P}$)*-martingales.*

Proof This can be shown with a similar argument for the uniqueness as above: Let (Y', Z', U') be another solution to the BSDE (ξ, f) with $Y' \geq \epsilon$ for some $\epsilon > 0$. Then f satisfies ($\mathbf{A_{fin}}$) for $a(t) \wedge \epsilon \leq y,\ y+u \leq b(t) \vee |Y'|_\infty$; hence the solutions coincide by comparison. □

Example 4.7 As a special case of Corollary 4.6 to be applied in Sect. 5.2, setting $K := \big(\gamma|\varphi|^2_\infty\big)/\big(2(1-\gamma)^2\big)$ for some $\gamma \in (0,1)$ and some predictable and bounded process φ we define

$$
\begin{aligned}
f_t(y,z,u) &:= \widehat{f}_t(y,z) + \int_E g_t(y,u,e)\,\zeta(t,e)\,\lambda(\mathrm{d}e) \\
&:= \frac{\gamma}{2(1-\gamma)^2}|\varphi_t|^2 y + \int_E \left(\frac{1}{1-\gamma}((u(e)+y)^{1-\gamma}y^\gamma - y) - u(e)\right)\zeta(t,e)\,\lambda(\mathrm{d}e).
\end{aligned}
$$

From $\frac{\partial g}{\partial y}(t,y,u,e) = \left(\frac{u+y}{y}\right)^{1-\gamma} + \frac{\gamma}{1-\gamma}\left(\frac{u+y}{y}\right)^{-\gamma} - \frac{1}{1-\gamma}$, we see that f is Lipschitz in y within the truncation bounds. Moreover, g is continuously differentiable with bounded derivatives and we have $\frac{\partial g}{\partial u}(t,y,u,e) = \left(\frac{u+y}{y}\right)^{-\gamma} - 1 > -1$, for $c \leq y, y+u \leq d$.

4.2 The Case of Infinite Activity

For linear generators of the form

$f_t(y,z,u) := \alpha^0_t + \alpha_t y + \beta_t z + \int_E \gamma_t(e)u(e)\,\zeta(t,e)\,\lambda(\mathrm{d}e)$, with predictable coefficients α^0, α, β and γ, JBSDE solutions can be represented by an adjoint process. In our context of bounded solutions, one needs rather weak conditions on the adjoint process. This will be used later on in Sect. 5. The idea of proof is standard, cf. [36, Lemma 1.23] for details.

Lemma 4.8 *Let f be a linear generator of the form above and let ξ be in $L^\infty(\mathcal{F}_T)$.*

1. *Assume that $(Y,Z,U) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ solves the BSDE (ξ, f). Suppose that the adjoint process $(\Gamma^t_s)_{s\in[t,T]} := (\exp(\int_t^s \alpha_u\,\mathrm{d}u)\mathcal{E}(\int \beta \mathrm{d}B + \gamma * \widetilde{\mu})^s_t)_{s\in[t,T]}$*

is in $\mathcal{S}^1$ *for any* $t \le T$ *and* α^0 *is bounded. Then* Y *is represented as* $Y_t = \mathbb{E}\big[\Gamma^t_T \xi + \int_t^T \Gamma^t_s \alpha^0_s \,\mathrm{d}s | \mathcal{F}_t\big]$.

2. *Let* α^0, α, β *and* $\widetilde{\gamma}_t := \int_E |\gamma_t(e)|^2 \zeta(t, e)\lambda(\mathrm{d}e)$, $t \in [0, T]$, *be bounded and* $\gamma \ge -1$. *Then there is a unique solution in* $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ *to the BSDE* (ξ, f) *and Part 1. applies.*

Our aim is to prove existence and uniqueness beyond Proposition 4.3 for infinite activity of jumps, that means $\lambda(A)$ may be infinite in (2.6). To show Theorems 4.11 and 4.13, we use a monotone stability approach of [37]: By approximating a generator f of the form (2.7) (with A such that $\lambda(A) = \infty$) by a sequence $(f^n)_{n\in\mathbb{N}}$ of the form (2.7) (with A_n such that $\lambda(A_n) < \infty$) for which solutions' existence is guaranteed, one gets that the limit of these solutions exist and it solves the BSDE with the original data. As in [37], the monotone approximation approach is perceived as being not easy in execution, a main problem usually being to prove strong convergence of the stochastic integral parts for the BSDE. By Proposition 4.9 convergence works for small terminal condition ξ. That is why we can not apply this Proposition directly to data $(\xi, f^n)_{n\in\mathbb{N}}$. Instead we sum (converging) solutions for small $1/N$-fractions of the desired terminal condition. This is inspired by the iterative ansatz from [45] for a particular generator. For our generator family, we adapt and elaborate proofs, using e.g. a $\mathcal{S}^1$-closeness argument for the proof of the strong approximation step. Compared to [45], the analysis for our general family of JBSDEs adds clarity and structural insight into what is really needed. It extends the scope of the BSDE stability approach [37, 45], in particular with regards to non-Lipschitz dependencies in the jump-integrand, while the proof shows comparable ease for the (usually laborious) strong approximation step in the setup under consideration. Differently to e.g. [23, 45, 57], no exponential transforms or convolutions are needed here, as our generators are "quadratic" in U but not in Z. Despite similarities at first sight, a closer look reveals that Theorem 4.11 is different from [35, Theorem 5.4], both in the method of proof and in scope: They prove existence for small terminal conditions by following the fixed point approach by [56], whereas we show stability for small terminal conditions (Proposition 4.9) and apply a different pasting procedure, approximating not only terminal data but also generators. Here wellposedness of the approximating JBSDEs is obtained directly from classical theory by using comparison and estimates from Sect. 3, which enable us to argue within uniform a-priori bounds for the approximating sequence. Examples in Sect. 5 demonstrate that also the scope of our results is different.

In more detail, the task for the next Theorem 4.11 is to construct generators $(f^{k,n})_{1\le k\le N, n\in\mathbb{N}}$ and solutions $(Y^{k,n}, Z^{k,n}, U^{k,n})$ to the BSDEs with data $(\xi/N, f^{k,n})$ for N large enough such that $(Y^{k,n}, Z^{k,n}, U^{k,n})$ converges if $n \to \infty$ and $(Y^n, Z^n, U^n) := \sum_{k=1}^N (Y^{k,n}, Z^{k,n}, U^{k,n})$ solves the BSDE (ξ, f^n). In this case (Y^n, Z^n, U^n) converges and its limit is a solution candidate for the BSDE (ξ, f). For this program, we next show a stability result for JBSDE.

Proposition 4.9 *Let* $(\xi^n) \subset L^\infty(\mathcal{F}_T)$ *with* $\xi^n \to \xi$ *in* $L^2(\mathcal{F}_T)$ *and* $(f^n)_{n\in\mathbb{N}}$ *be a sequence of generators with* $f^n(0, 0, 0) = 0$, $\forall n$, *having property* (B_{γ^n}) *such that* $K^{y,z}_f := \sup_{n\in\mathbb{N}} K^{y,z}_{f^n} < \infty$. *Denote by* $(Y^n, Z^n, U^n) \in \mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$

the solution to the BSDE (ξ, f^n) *with* Y^n *bounded by* $|\xi|_\infty \exp(K^{y,z}_{f^n} T)$ *and set* $\tilde{c} := |\xi|_\infty \exp(K^{y,z}_f T)$. *Assume that* Y^n *converges pointwise,* $(Z^n, U^n) \to (Z, U)$ *converges weakly in* $\mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ *and* $|f^n_t(0,0,u)| \le \widehat{K}|u|_t^2 + \widehat{L}_t$ *for all* n *and* u *with* $|u| \le 2\tilde{c}$, $\widehat{K} \in \mathbb{R}_+$ *and* $\widehat{L} \in L^1(\mathbb{P} \otimes \mathrm{d}t)$. *Then* (Z^n, U^n) *converges to* (Z, U) *strongly in* $\mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$, *if* $|\xi|_\infty \equiv \tilde{c}\exp(-K^{y,z}_f T) \le \exp(-K^{y,z}_f T)/(80 \max\{K^{y,z}_f, \widehat{K}\})$.

Proof We note that (Y^n, Z^n, U^n) is uniquely defined by Proposition 4.2. To prove strong convergence of $(Z^n)_{n\in\mathbb{N}}$ and $(U^n)_{n\in\mathbb{N}}$ we consider $\delta Y := Y^n - Y^m$, $\delta Z := Z^n - Z^m$, $\delta U := U^n - U^m$ and apply Itô's formula for general semimartingales to $(\delta Y)^2$ to obtain

$$
\begin{aligned}
(\delta Y_0)^2 = (\delta Y_T)^2 &+ \int_0^T 2\delta Y_{s-}(f^n_s(Y^n_{s-}, Z^n_s, U^n_s) - f^m_s(Y^m_{s-}, Z^m_s, U^m_s))\mathrm{d}s \\
&- \int_0^T \|\delta Z_s\|^2 \mathrm{d}s - 2\int_0^T \delta Y_{s-}\delta Z_s\,\mathrm{d}B_s - \int_0^T\!\!\int_E (\delta Y_{s-} + \delta U_s(e))^2 - (\delta Y_{s-})^2\,\widetilde{\mu}(\mathrm{d}s, \mathrm{d}e) \\
&- \int_0^T\!\!\int_E (\delta Y_{s-} + \delta U_s(e))^2 - (\delta Y_{s-})^2 - 2\delta Y_{s-}\delta U_s(e)\,\nu(\mathrm{d}s, \mathrm{d}e).
\end{aligned}
$$

Noting that the stochastic integrals are martingales one concludes that

$$
\begin{aligned}
&\mathbb{E}\Big(\int_0^T 2\delta Y_{s-}(f^n_s(Y^n_{s-}, Z^n_s, U^n_s) - f^m_s(Y^m_{s-}, Z^m_s, U^m_s))\,\mathrm{d}s\Big) \\
&= \mathbb{E}\Big(\int_0^T\!\!\int_E \delta U_s(e)^2\,\nu(\mathrm{d}s, \mathrm{d}e)\Big) + \mathbb{E}\Big(\int_0^T \|\delta Z_s\|^2\,\mathrm{d}s\Big) - \mathbb{E}\big((\delta Y_T)^2\big) + \mathbb{E}((\delta Y_0)^2).
\end{aligned}
\tag{4.1}
$$

Using the inequalities $a \le a^2 + 1/4$, $(a+b)^2 \le 2(a^2+b^2)$, $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, the Lipschitz property of f^n in y and z and the estimate for $f^n_t(0,0,u)$, we have

$$
\begin{aligned}
&|f^n_s(Y^n_{s-}, Z^n_s, U^n_s) - f^m_s(Y^m_{s-}, Z^m_s, U^m_s)| \\
&\le K^{y,z}_{f^n}(|Y^n_{s-}| + \|Z^n_s\|) + K^{y,z}_{f^m}(|Y^m_{s-}| + \|Z^m_s\|) + \widehat{K}|U^n_s|_s^2 + \widehat{L}_s + \widehat{K}|U^m_s|_s^2 + \widehat{L}_s \\
&\le K_1 + 2\widehat{L}_s + K_2(\|\delta Z_s\|^2 + \|Z^n_s - Z_s\|^2 + \|Z_s\|^2 + |\delta U_s|_s^2 + |U^n_s - U_s|_s^2 + |U_s|_s^2),
\end{aligned}
\tag{4.2}
$$

where $K_1 := K^{y,z}_f(2\tilde{c} + 1/2) \in \mathbb{R}$, $K_2 := 5\max\{K^{y,z}_f, \widehat{K}\}$ and $|\cdot|_t$ is defined in (2.5). Combing inequalities (4.1) and (4.2) yields

$$
\begin{aligned}
\mathbb{E}\Big(\int_0^T \|\delta Z_s\|^2 + |\delta U_s|_s^2\,\mathrm{d}s\Big) \le 2\mathbb{E}\Big(\int_0^T &|\delta Y_{s-}|(K_1 + 2\widehat{L}_s + K_2(\|\delta Z_s\|^2 + \|Z^n_s - Z_s\|^2 + \|Z_s\|^2 \\
&+ |\delta U_s|_s^2 + |U^n_s - U_s|_s^2 + |U_s|_s^2))\,\mathrm{d}s\Big) + \mathbb{E}\big((\xi^n - \xi^m)^2\big).
\end{aligned}
$$

Let us recall that the predictable projection of Y, denoted by Y^{p}, is defined as the unique predictable process X such that $X_\tau = \mathbb{E}(Y_\tau | \mathcal{F}_{\tau-})$ on $\{\tau < \infty\}$ for all predictable times τ. For Y^n it holds $(Y^n)^{\mathrm{p}} = Y^n_-$. This follows from [31, Proposition I.2.35.] using that Y^n is càdlàg, adapted and quasi-left-continuous, as $\Delta Y_\tau = \Delta U * \widetilde{\mu}_\tau = 0$ on $\{\tau < \infty\}$ holds for all predictable times τ thanks to the absolute continuity of the compensator ν. Noting that $1 - 2K_2|\delta Y_{s-}| \ge 1 - 4K_2\tilde{c} \ge 3/4$ and setting $Y := \lim_{n\to\infty} Y^n$ we deduce by the weak convergence of $(Z^n)_{n\in\mathbb{N}}$ and

$(U^n)_{n\in\mathbb{N}}$, $Y^n_- = (Y^n)^{\mathrm{p}} \uparrow (Y)^{\mathrm{p}}$ as $n \to \infty$ and by Lebesgue's dominated convergence theorem

$$\begin{aligned}
&\frac{3}{4}\mathbb{E}\Big(\int_0^T \|Z^n_s - Z_s\|^2 + |U^n_s - U_s|^2_s \,\mathrm{d}s\Big)\\
&\leq \frac{3}{4}\liminf_{m\to\infty}\mathbb{E}\Big(\int_0^T \|Z^n_s - Z^m_s\|^2 + |U^n_s - U^m_s|^2_s \,\mathrm{d}s\Big)\\
&\leq \liminf_{m\to\infty} 2\mathbb{E}\Big(\int_0^T |\delta Y_{s-}|(K_1 + 2\widehat{L}_s + K_2(\|Z^n_s - Z_s\|^2 + \|Z_s\|^2 + |U^n_s - U_s|^2_s + |U_s|^2_s))\,\mathrm{d}s\Big)\\
&\qquad + \mathbb{E}\big((\xi^m - \xi^n)^2\big)\\
&= 2\mathbb{E}\Big(\int_0^T |Y^n_{s-} - (Y_s)^{\mathrm{p}}|(K_1 + 2\widehat{L}_s + K_2(\|Z^n_s - Z_s\|^2 + \|Z_s\|^2 + |U^n_s - U_s|^2_s + |U_s|^2_s))\,\mathrm{d}s\Big)\\
&\quad + \mathbb{E}\big((\xi - \xi^n)^2\big).
\end{aligned}$$

Noting $3/4 - 2K_2|Y^n_{s-} - (Y_s)^{\mathrm{p}}| \geq 3/4 - 4K_2\tilde{c} \geq 1/2$, one obtains with dominated convergence

$$\begin{aligned}
&\frac{1}{2}\limsup_{n\to\infty}\mathbb{E}\Big(\int_0^T \|Z^n_s - Z_s\|^2 + |U^n_s - U_s|^2_s \,\mathrm{d}s\Big)\\
&\quad\leq \limsup_{n\to\infty} 2\mathbb{E}\Big(\int_0^T |Y^n_{s-} - (Y_s)^{\mathrm{p}}|(K_1 + 2\widehat{L}_s + \|Z_s\|^2 + |U_s|^2_s)\,\mathrm{d}s\Big) + \mathbb{E}\big((\xi^n - \xi)^2\big) = 0.
\end{aligned}$$

□

We will need the following result which is a slight variation of [37, Lemma 2.5].

Lemma 4.10 *Let $(Z^n)_{n\in\mathbb{N}}$ be convergent in $\mathcal{L}^2(B)$ and $(U^n)_{n\in\mathbb{N}}$ convergent in $\mathcal{L}^2(\widetilde{\mu})$. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that*

$$\sup_{n_k}\|Z^{n_k}\| \in L^2(\mathbb{P}\otimes \mathrm{d}t) \text{ and } \sup_{n_k}|U^{n_k}_t|_t \in L^2(\mathbb{P}\otimes \mathrm{d}t).$$

Proof The result for $(Z^n)_{n\in\mathbb{N}}$ is from [37] and the argument for $(U^n)_{n\in\mathbb{N}}$ is analogous. □

Theorem 4.11 [Monotone stability, infinite activity] *Let $\xi \in L^\infty(\mathcal{F}_T)$ and let $(f^n)_n$ be a sequence of generators satisfying condition (B_{γ^n}) with $K^{y,z}_f := \sup_{n\in\mathbb{N}} K^{y,z}_{f^n} < \infty$. Assume that*

1. *there is $(\widehat{Y}, \widehat{Z}, \widehat{U})$ in $\mathcal{S}^\infty \times \mathcal{L}^2(B) \times \mathcal{L}^2(\widetilde{\mu})$ with $\widehat{U}$ bounded and $f^n_t(\widehat{Y}_{t-}, \widehat{Z}_t, \widehat{U}_t) \equiv 0$ for all n,*
2. *for all $u \in L^0(\mathcal{B}(E), \lambda)$ with $|u| \leq |\widehat{U}|_\infty + 2|\xi|_\infty \exp(K^{y,z}_f T)$ there exists $\widehat{K} \in \mathbb{R}_+$ and a process $\widehat{L} \in L^1(\mathbb{P}\otimes \mathrm{d}t)$ such that $|f^n_t(0, 0, u)| \leq \widehat{K}|u|^2_t + \widehat{L}_t$ for each $n \in \mathbb{N}$,*
3. *the sequence $(f^n)_{n\in\mathbb{N}}$ converges pointwise and monotonically to a generator f,*