History of Mathematics Education

Dirk De Bock
Geert Vanpaemel

Rods, Sets and Arrows

The Rise and Fall of Modern Mathematics in Belgium

Springer

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ISSN 2509-9736 ISSN 2509-9744 (electronic)
History of Mathematics Education
ISBN 978-3-030-20598-0 ISBN 978-3-030-20599-7 (eBook)
https://doi.org/10.1007/978-3-030-20599-7

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

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Overall Book Abstract and Abstracts for the Ten Chapters of the Book

The introduction of modern mathematics in Belgian secondary schools in September 1968 was one of the most radical education reforms that the country has ever seen. From the very start, the reform was controversial among mathematicians and mathematics educators, and until now, it continues to be considered as either a wonderful experiment or a dramatic failure. This monograph is the first attempt to present a comprehensive overview of the reform in its wider context, and to make a critical assessment of its impact both on the national and the international level.

Rods, Sets, and Arrows describes in detail the rise and fall of modern mathematics in Belgium from its early phases driven by the technological optimism of the post-War era until its demise around the end of the twentieth century. It puts the modern mathematics reform in a broad perspective, comparing it to other variants of mathematical instruction methodologies such as the movement for intuitive geometry, the didactical use of teaching aids, or the Dutch *Realistic Mathematics Education* alternative. Apart from its central focus on curriculum reform, the book also attempts to uncover some of the political and ideological motives behind the modern mathematics movement and its origins in the post-War euphoria for science and mathematics.

The Belgian reform was strongly embedded in international movements. Not only were international events, such as the famous Royaumont Seminar in 1959, of the utmost importance for the advancement of the reform in Belgium, but Belgian mathematicians and mathematics teachers also played crucial roles at the international level. These Belgian contributions are still much under-exposed in the scholarly literature. The book focuses on the contributions made by distinct personalities, such as Paul Libois, Willy Servais, Frédérique Lenger, and Georges Papy. In particular, an analysis is offered of the groundbreaking textbook series *Mathématique Moderne* by Papy, which reshaped the content of secondary school mathematics and heavily influenced national and international debates during the implementation phase of the reform.

The book is subdivided into three parts. The first part follows the early reform movement and its many sources of inspiration: The *Reform Pedagogy* of Ovide Decroly, the Marxist views on man and modern civilization, and the debates on the use of teaching aids within the International Commission for the Study and Improvement of Mathematics Teaching. It ends with the consolidation of reform views at the OEEC and ICMI conferences in Royaumont, Aarhus, Zagreb-Dubrovnik, and Athens. In the second part of the book, the focus is on the work of Georges Papy, his textbook series and the creation of the Belgian Centre for Mathematics Pedagogy in 1961. It also includes an analysis of the many classroom experiments undertaken by Papy and his collaborators. The fall of modern mathematics, starting in the 1980s, and the search for alternatives are discussed in the third and final part. This period coincides with national reform in Belgium, which placed government responsibility for education at the regional level. The book follows the different approaches taken in the aftermath of the modern mathematics reform by the Flemish (Dutch-speaking) Community and the French Community. It is argued that the reaction against the modern mathematics reform may have been instrumental in the genesis of mathematics education as a scholarly field in Belgium.

The book is based on the analysis of a wide range of original sources, including some from private archives. It also presents some rare photographs of its main protagonists and provides a full bibliography of primary and secondary literature.

Chapter 1: Reform Pedagogy and the Introduction of Intuitive Geometry in Secondary School Mathematics

In the aftermath of World War II, Belgian intellectuals participated in the *Comité d'Initiative pour la Rénovation de l'Enseignement en Belgique*. Their aim was to renew education for 6- to 16-year-olds in all disciplines. Main inspiration was found in the work of Ovide Decroly, a protagonist of Reform Pedagogy. This reform movement led to new curricula in the late 1940s, including for mathematics a course of intuitive geometry in the first years of secondary school aimed at providing students with a practical geometrical knowledge base and preparing them for a deductive approach in subsequent years. The most influential advocate of this new approach to geometry was the mathematician, Paul Libois. For Libois, intuitive geometry was closely connected to his epistemological conception of geometry, considering geometry as a part of physics. His views also bear a clear parallel to his political position as a prominent Marxist communist. Libois' ideas were influential in Belgium until the end of the 1950s when the modern mathematics movement emerged.

Chapter 2: Revival of International Collaboration in Mathematics Education During the 1950s

In the early 1950s, Caleb Gattegno, who held doctorates in both mathematics and psychology, took the initiative to organize regular meetings of internationally renowned psychologists, mathematicians, and mathematics teachers, and the International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM) was born. During the 1950s, Belgians including Frédérique Lenger, Louis Jeronnez, and Willy Servais played a prominent role within CIEAEM. The work of CIEAEM also had a major influence on Belgian mathematics education. Much attention was given to the study and stimulation of students' learning processes by concrete models and other new teaching aids, among them geoboards, mathematical films, electrical circuits, and the Cuisenaire rods. A confrontation with Bourbaki's mathematical structures and their assumed relation with the basic structures of early mathematical thinking, as revealed by Jean Piaget, led to a call to experiment with some elements of modern mathematics at the secondary school level.

Chapter 3: Search for National Identity: Willy Servais and the Belgian Society of Mathematics Teachers

In 1953, the Belgian Society of Mathematics Teachers was founded, ensuring a structural relation between the work of the CIEAEM and the community of Belgian mathematics teachers. The Society brought together a few hundred mathematics teachers from both linguistic communities (French and Dutch). It immediately started its own professional journal *Mathematica & Paedagogia* that rapidly became a main forum for national and international exchange in mathematics education. From the mid-1950s on, the trend toward modern mathematics became clearer. The prominence of the Society and its journal in mathematics education debates during the 1950s was largely due to the leadership and versatile contacts of Willy Servais, the most influential Belgian mathematics educator of that time, both in his home country and on the international scene. Also, in the next two decades, Servais' reputation could stand, thanks to his open-mindedness, broad experience, and international outlook.

Chapter 4: From Royaumont to Athens: Belgian Reformers on the International Scene

In 1959 the Organisation for European Economic Co-operation (OEEC) organized a major international seminar on "New Thinking in School Mathematics" at the Cercle Culturel de Royaumont in Asnières-sur-Oise (France). The Royaumont Seminar soon acquired an almost iconic status among mathematics reformers and came to be seen as a decisive turning point in the history of the modern mathematics reform. During the seminar a consensus was forged between mathematics reformers on the basic tenets of what modern mathematics stood for, and the first steps toward a new curriculum were taken. The conclusions of Royaumont served as a manifesto for curriculum reformers around the world. The Royaumont Seminar was followed by other, more specialized conferences, in Zagreb-Dubrovnik (1960) and Athens (1963), during which concrete proposals for a new mathematics curriculum were worked out. Also, these constituted milestones in the history of modern mathematics, moments when theoretical debates finally turned into actions. We describe the role of the Belgian delegates to the conferences of Royaumont, Zagreb-Dubrovnik, and Athens and evaluate the interactions between the national and the international movements.

Chapter 5: Preparing for the Introduction of Modern Mathematics into the Classroom: Experimentation and Teacher Training

The 1960s were characterized by a wide range of activities aimed at assisting the actual implementation of modern mathematics into the classroom: experimentation with different target groups, related to the development of new curricula, and large-scale programs of teacher re-education. After a first experiment, Georges Papy, a professor of algebra, was consulted, his task being to promote the quality of the experimental actions. Papy engaged himself completely and soon became the architect and undisputed leader of the modern mathematics reform in Belgium. He designed and carried out audacious experiments, developed new programs and teaching materials, and engaged mathematics teachers through large-scale in-service education programs. Papy's actions were coordinated by the newly-founded Belgian Centre for Mathematics Pedagogy, of which he became the chairman. A very different approach to the modernization of the teaching of mathematics was advocated by Paul Libois whose collaborators conducted their own experiments at the *École Decroly*.

Chapter 6: *Mathématique Moderne:* A Pioneering Belgian Textbook Series Shaping the Modern Mathematics Reform of the 1960s

In 1963 the Belgian mathematician and mathematics educator Georges Papy published the first volume of his groundbreaking textbook series entitled *Mathématique Moderne* (in collaboration with Frédérique Papy-Lenger), intended for students from 12 to 18 and based on several years of classroom experimentation. It marked a revolution in the teaching of mathematics and in the art of textbook design. Papy reshaped the content of secondary school mathematics by basing it upon the unifying themes of sets, relations, and algebraic structures. Meanwhile, he proposed an innovative pedagogy using multicolored arrow graphs, playful drawings, and "visual proofs" by means of drawings of film strips. During the 1960s and early 1970s, translations of the volumes of *Mathématique Moderne* appeared in European and non-European languages and were reviewed in mathematics education journals of that time. Papy's "MMs" influenced the national and international debates and became major guides for shaping the modern mathematics reform in several countries.

Chapter 7: Modern Mathematics in Belgian Secondary and Primary Education: Between Radicalism and Pragmatism

After a period of 10 years of experimentation and confusion about the future direction of school mathematics, a political decision clarified the situation: from 1968 on, modern mathematics was compulsorily introduced in all Belgian secondary schools and a few years later also in primary schools. For more than 20 years, it was the dominant paradigm for the teaching and learning of mathematics. The reform was quite radical, although some traditional subjects and methods were maintained. Modern mathematics led both to new mathematical content and to a modernization of teaching methods. Proper notations and symbols, the use of the correct vocabulary, and theory development received increased attention, barriers between mathematical subdomains were largely eliminated, and geometry education was redirected toward transformation and vector geometry.

Chapter 8: From Critique to Math War: A Divided Community of Belgian Mathematics Teachers

The developments during the 1960s seem to suggest that within the Belgian mathematics education community there was a kind consensus about the modernization efforts and the way they were led by Papy and his CBPM. The reality was however different: during the 1960s, a real anti-modern mathematics movement originated, the opposition being headed by Léon Derwidué, professor at the Faculty of Engineering in Mons, and by MATEC, an organization of mathematics teachers in technical schools. During the 1970s, the Belgian mathematics education community was remarkably silent, and the math war seemed to have been fought. However, in the early 1980s, this silence was broken by pedagogues and mathematics educators who firmly criticized the starting points of modern mathematics and the way it was introduced and dictated at the primary and secondary level. These critics demanded that the abstract language and aberrations of modern mathematics be left behind and that there be a return to a realistic, concrete, and basic teaching of mathematics.

Chapter 9: The Fall of Modern Mathematics in Flanders: From Structuralism to Eclecticism

During the mid-1980s and the 1990s, the modern mathematics model was gradually adapted and finally abandoned. These developments no longer took place in a unitary Belgian context. By the end of the 1980s, Belgium had become a Federal State consisting of three communities—the Flemish, the French and the (small) German-speaking community. Each became fully responsible for educational matters within its community. In this chapter, we discuss the post-modern mathematics developments in Flanders. Flemish mathematics educators and teachers at that time were strongly inspired by the Dutch model of *Realistic Mathematics Education*, conceiving mathematics as a human activity and emphasizing, among other things, the role of rich contexts, applications, and modeling. At the same time, some elements of Belgian's own tradition were maintained. It resulted in a more-or-less balanced approach to mathematics education with influences from the mechanistic and realistic traditions, with still some elements of the structural modern mathematics vision.

Chapter 10: A Joint Action to Reshape Mathematics Education in the French Community of Belgium

In the late 1970s, a "reform of the reform" was launched in the French Community of Belgium, more modestly and receiving less media attention than the modern mathematics revolution of the 1960s. It was the time of a new generation of mathematics educators with Nicolas Rouche as a main figurehead. They pleaded, among other things, for students' guided construction of knowledge by confronting them with substantial problem situations that can give meaning to concepts and theorems prior to their mathematical conceptualization, and for a global and coherent view on mathematics education "from kindergarten to university." Several small working groups of teachers and mathematics educators were established, among them the *Groupe d'Enseignement Mathématique*, preceding the creation in 1992 of the *Centre de Recherche sur l'Enseignement des Mathématiques*, an institute for the study and development of mathematics education that joined actors from all educational levels and networks in the French Community of Belgium.

Preface to the Series

Books in Springer's series on the history of mathematics education comprise scholarly works on a wide variety of themes, prepared by authors from around the world. We expect that authors contributing to the series will go beyond top-down approaches to history, so that emphasis will be placed on the learning, teaching, assessment and wider cultural and societal issues associated with schools (at all levels), with adults and, more generally, with the roles of mathematics within various societies.

In addition to generating texts on the history of mathematics education written by authors in various nations, an important aim of the series will be to develop and report syntheses of historical research that have already been carried out in different parts of the world with respect to important themes in mathematics education—like, for example, "Historical Perspectives on how Language Factors Influence Mathematics Teaching and Learning," and "Historically Important Theories Which Have Influenced the Learning and Teaching of Mathematics."

The mission for the series can be summarized as:

- To make available to scholars and interested persons around the world the fruits of outstanding research into the history of mathematics education;
- To provide historical syntheses of comparative research on important themes in mathematics education; and
- To establish greater interest in the history of mathematics education.

The present book provides an important addition to the series. The authors tell the story of the history of mathematics curricula in Belgium at a critical time, starting some 60 years ago at the inception of modern mathematics. This book makes available, in English, analyses of events (and an extensive supporting literature) which have previously not been easily available to English speakers. As the text proceeds, readers are shown how thinking about modern mathematics in Belgium waxed and waned, depending not only on the key figures involved, but also on the perceptions and involvement of various stakeholders. Of special interest are the profound effects these events and interactions had on fundamental questions like: "What should be the intended mathematics curricula in schools?", "Should the intended curricula be the same for all learners?" and "Who should be responsible for bringing about changes to school mathematics curricula?" Although the context for the book is mathematics education in Belgium, the book provides an excellent model for future books in this series—studies which address critical periods in the historical evolution of mathematics education in countries around the world.

We hope that the series will continue to provide a multi-layered canvas portraying rich details of mathematics education from the past, while at the same time presenting historical insights that can support the future. This is a canvas which can never be complete, for today's mathematics education becomes history for tomorrow. A single snapshot of mathematics education today is, by contrast with this canvas, flat and unidimensional—a mere pixel in a detailed image. We encourage readers both to explore and to contribute to the detailed image which is beginning to take shape on the canvas for this series.

Any scholar contemplating the preparation of a book for the series is invited to contact Nerida Ellerton (ellerton@ilstu.edu), in the Department of Mathematics at Illinois State University or Melissa James, at the Springer New York office.

Normal, IL, USA

Nerida F. Ellerton
M.A. (Ken) Clements

Preface to the Book

In September 1968, modern mathematics (or "new math") was introduced in Belgian secondary schools, mandatorily and generally. It is considered as one of the most radical education reforms that Belgium had ever seen. From the very start, the modern mathematics reform was controversial among mathematicians and mathematics educators, and until today, it continues to be considered as either a wonderful experiment or a dramatic failure. In this monograph, we describe and analyze the rise of modern mathematics in Belgium during the 1960s and its fall later. Our scope is obviously limited to Belgium, but we situate this history in a broader international context in which, from the beginning, Belgian mathematicians and mathematics teachers played a prominent role. In writing this book, we had four objectives in mind. First, we attempted to put the modern mathematics reform in a broader perspective, which also includes other variants of mathematical instruction methodologies, such as intuitive geometry, the use of teaching aids, or the Dutch *Realistic Mathematics Education* alternative. Second, we wanted to uncover the ideological background behind the modern mathematics movement and its origins in the post-War optimism for science and mathematics. Third, we aimed at highlighting and assessing the important role played by Belgian mathematicians and mathematics teachers on the international level, which remains still under-exposed in the scholarly literature. And finally, we hoped to further our understanding on the genesis of mathematics education as a scholarly field in Belgium.

Obviously, the modern mathematics reform did not come from nowhere. In the first section of this monograph (Chaps. 1, 2, 3 and 4), we discuss the long maturation period of the reform which can be roughly situated between 1945 and 1960. Shortly after the end of World War II, in the euphoria of liberation and the momentum of reconstruction, Belgian intellectuals allied forces to renovate education at all levels and in all disciplines, including mathematics. Inspiration was found in Ovide Decroly's *Reform Pedagogy*, an international pedagogical movement advocating a child-centred approach to teaching with particular attention to bridging the gap between school and society. A protagonist during that period was the mathematician and communist politician Paul Libois, professor of geometry at the *Université Libre de Bruxelles*. This first reform movement led to a new curriculum, including a course on *intuitive geometry* in the first years of secondary school. From the early 1950s, other mathematicians and mathematics educators came to the forefront, among them Willy Servais as the main figure. Servais became very influential at the international scene, particularly in circles of the newly created International Commission for the Study and Improvement of Mathematics Teaching. Students' learning processes and its stimulation by concrete models and new teaching aids became major points of interest. Initially, this development had much in common with intuitive geometry, but gradually, the prefiguration of abstract mathematical ideas, sometimes seen to be in opposition to the spontaneous intuition of the student, became a major goal. By the end of the 1950s, the purely didactical debates became aligned with the cry—in particular by mathematicians—for adding new contents to the curriculum. School mathematics was regarded as being out of tune with modern developments in academic mathematics and needed to be adapted not only in methodology but also in actual content. This opened the way for a debate on which the elements of modern mathematics should be included. On the international level, a consensus seemed to grow on the structural view of mathematics as expounded by the Bourbaki group. Other alternatives, such as a greater emphasis on probability theory, statistics, and other types of applied mathematics, received less attention. The international debates were swiftly taken up by Belgian mathematicians. In 1953, Servais founded the Belgian Society of Mathematics Teachers, which served to disseminate the Commission's ideas and proposals. The Society immediately started its own professional journal *Mathematica & Paedagogia* which became a forum for national (and international!) exchange in mathematics education. A turning point in the ongoing reflections on mathematics education was the Organisation for European Economic Co-operation (OEEC) Seminar, held at Royaumont (Paris) in 1959. The Seminar consolidated much of the work being done during the 1950s and forged a consensus among the leading mathematicians about the direction to be taken. For Belgium, the Royaumont Seminar provided a point of reference in the elaboration of a new mathematics curriculum. In the following years, a modern mathematics curriculum was actively developed.

The Belgian modern mathematics movement soon found its leader in the strong personality of Georges Papy, professor of algebra and Libois' younger colleague at the Brussels University. In the second section (Chaps. 5, 6 and 7), we discuss how the implementation of modern mathematics in the classroom took place during the 1960s. Papy designed, carried out, and evaluated experiments with different target groups, developed new curricula, and trained (*recycled*) teachers through large-scale in-service education programs. These actions were coordinated by the Belgian Centre for Mathematics Pedagogy, which had been founded in 1961, and received ample attention in the international mathematics education community. In 1963, Papy published the first volume of a groundbreaking textbook series entitled *Mathématique Moderne*, based on his experimental trajectory and intended for the teaching of modern mathematics to 12–18-year-olds. Inspired by the work of Bourbaki, Papy reshaped the content of secondary school mathematics by basing it on the unifying themes of sets, relations, and algebraic structures. Meanwhile, he proposed an innovative pedagogy using multi-coloured arrow graphs, playful drawings, and visual proofs by means of film strips. Papy's textbook series influenced the national and international debates and became a major guide for shaping the modern mathematics reform in several countries. From 1968 on, modern mathematics became mandatory in secondary schools and a few years later also in primary schools. For more than twenty years, it was the dominant paradigm for the teaching and learning of mathematics. Proper notations and symbols, the use of the right jargon, and theory development received increased attention, barriers between mathematical subdomains were largely eliminated, and geometry education was redirected toward transformation and vector geometry.

The fall of modern mathematics and the community's search for alternatives are discussed in the third and final section (Chaps. 8, 9 and 10). Already in the 1960s, a real anti-Papy movement originated in Belgium, but it could not stop the introduction of modern mathematics. When during the early 1970s modern mathematics was criticized at international forums, the criticisms were not heard in the Belgian mathematics education community. This silence was broken in the early 1980s: The starting points of modern mathematics and the way it was introduced and dictated at the primary and secondary level were firmly criticized by pedagogues and mathematics educators. They advised urgently to leave the abstract language and aberrations of modern mathematics and to return to a realistic, concrete, and basic teaching of mathematics. At the official level, a real change came only by the end of the 1980s when the Belgian educational landscape was completely redesigned as a result of the political restructuring of the nation. Belgium became a Federal State consisting of three Communities—the Flemish Community, and the French- and (small) German-speaking Communities—with each becoming responsible for its own educational matters.

Although new curricula in the Flemish Community were inspired by the Dutch model of *Realistic Mathematics Education*, emphasizing the role of applications and modeling, the valuable elements of its own traditions were maintained, resulting in a more or less balanced approach to mathematics education. In the French-speaking Community, a new generation of mathematics educators came to the forefront with Nicolas Rouche as a main figurehead. They pleaded, among other things, for a global view on mathematics education and for paying renewed attention to the historical and epistemological roots of mathematical concepts and theories. During the late 1970s and 1980s, several small working groups of teachers and mathematics educators were established, preceding the creation in 1992 of the Research Centre for Mathematics Education, an institute for the study and development of mathematics education that joined actors of all educational levels and networks in the French Community of Belgium.

The main focus of our book is on a definite episode in the history of Belgian mathematics education, but of course, this history is embedded in a broader educational and political history. We mention aspects of these latter histories insofar as they contribute to a better understanding of our main discourse, but obviously, it was not feasible to make large digressions into the educational history of the Belgian school system. We can mention here the competitive animosity between state schools and free (mainly Catholic) schools, resulting in a School War during the 1950s or the cultural differences between the Flemish (Dutch-speaking) and French-speaking communities, which show the different reception of ideas from the Netherlands or France. Also, there was no space to go into a detailed discussion of all the different educational trajectories and streams available to students. We are aware that our book cannot do justice to all the intellectual and social aspects of the modern mathematics reform or to all the personalities involved. We have not attempted to be complete, although we did try to present a balanced view of the reform. Though we follow a time line, this discourse is more thematic than strictly chronological. Time periods of the different chapters partly overlap, and that has caused some repetition. We predominantly relied on written documents of the period under scrutiny, such as legal acts, mathematical curricula and methodological recommendations, textbooks and teachers' courses, protagonists' discourses, and other testimonials. We acknowledge that these sources can only partially grasp the actual classroom realities of the time which are often more complex and varied. We leave it to the reader to judge if we succeeded in sketching a true historical picture.

Writing about history also presupposes a critical distance and a neutral stance of the authors toward the period described. These conditions may not have been sufficiently met for the last two chapters in which the developments in the 1980s and 1990s are discussed. These chapters, basically reporting about what happened after modern mathematics in Flanders and in the French-speaking part of Belgium, were, however, indispensable for the completeness of our review. We take into account that future researchers may question the objectivity of our description of these more recent parts of history.

This monograph integrates the results of research on the history of the Belgian modern mathematics movement and its international connectedness. Parts of this research have already been presented at and published in the Proceedings of international meetings. In particular, presentations have been made at the Second, Third, Fourth, and Fifth International Conferences on the History of Mathematics Education, the Seventh European Summer University on the History and Epistemology in Mathematics Education, and the Thirteenth International Congress on Mathematical Education.

The authors are greatly indebted to many people who helped them in various ways during this enterprise, in particular Francis Buekenhout, Guy and Yolande Noël, Michel Roelens, and Lieven Verschaffel. Others have co-authored preparatory publications, provided documents, or advised us. We mention them, in alphabetical order, and at the risk of omissions for which we apologize: Évelyne Barbin, Jeanne Bartholomé, Assunta Bianchi, Kristín Bjarnadóttir, Anne-Marie Bosteels, Cristina Carruana, Sylvain Courtois, Yves Cuisenaire, Uriel De Grande, Chris De Munter, Marc Depaepe, Johan Deprez, Stéphane Derwidué, Mark D'hoker, Ahmed Djebbar, Christine Docq, Jean Doyen, Raf Feys, Fulvia Furinghetti, Marie-France Guissard, Christiane Hauchart, Sabine Janssen, Dirk Janssens, Robert Kennes, Francis Lowenthal, José Matos, Jean Mawhin, Marta Menghini, Chantal Randour, Nicole Rombouts, Sylvie Rouche, Marie-France Servais, Harm Jan Smid, Etienne Steyaert, Christiane Vandeputte, Wim Van Dooren, Paul Van Praag, Jan Vermeylen, and Bert Zwaneveld.

Brussels, Belgium
March, 2019

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Geert Vanpaemel

Part I

From Intuitive Geometry Toward Modern Mathematics: Call for Educational Reform in the Aftermath of World War II

Chapter 1

Reform Pedagogy and the Introduction of Intuitive Geometry in Secondary School Mathematics

Abstract: In the aftermath of World War II, Belgian intellectuals participated in the *Comité d'Initiative pour la Rénovation de l'Enseignement en Belgique*. Their aim was to renew education for 6- to 16-year-olds in all disciplines. Main inspiration was found in the work of Ovide Decroly, a protagonist of Reform Pedagogy. This reform movement led to new curricula in the late 1940s, including for mathematics a course of intuitive geometry in the first years of secondary school aimed at providing students with a practical geometrical knowledge base and preparing them for a deductive approach in subsequent years. The most influential advocate of this new approach to geometry was the mathematician, Paul Libois. For Libois, intuitive geometry was closely connected to his epistemological conception of geometry, considering geometry as a part of physics. His views also bear a clear parallel to his political position as a prominent Marxist communist. Libois' ideas were influential in Belgium until the end of the 1950s when the modern mathematics movement emerged.

Key Words: Alexis-Claude Clairaut; Camille Huysmans; Concrete material; Emma Castelnuovo; Federigo Enriques; Guido Castelnuovo; Intuitive geometry; Louis Jeronnez; Marxism; Mechanistic approach; Ovide Decroly; Paul Langevin; Paul Libois

In the Footsteps of Ovide Decroly

As in many other European countries, debates on the improvement of education flared up in Belgium around the end of World War II. One of the earliest initiatives was the foundation in January 1945 of the *Comité d'Initiative pour la Rénovation de l'Enseignement en Belgique* (CIREB) [Committee of Initiative for the Renovation of Education in Belgium], a think tank originating from circles of the resistance, pursuing a global and democratic reform of the Belgian educational system (Noël, 2018). The *Comité* was chaired by the physicist Frans Van den Dungen, vice rector of the *Université Libre de Bruxelles* [Free University of Brussels]. Most other members of the CIREB were also free-thinking intellectuals associated with or who had graduated from the *Université Libre de Bruxelles*, among them also the professor of geometry Paul Libois (1901–1991), an important voice among communist intellectuals (Schandevyl, 1999), and Adolphe Festraets, an influential teacher of mathematics and physics at the *Athénée Royal d'Ixelles* (Louryan, 2011; SBPMef, 1992). In an introductory manifesto *L'École de 6 à 16 ans*

D. De Bock, G. Vanpaemel, *Rods, Sets and Arrows*, History of Mathematics Education,
https://doi.org/10.1007/978-3-030-20599-7_1

[The school for 6- to 16-year-olds] (CIREB, 1945a), the *Comité* proposed an extension of the period of compulsory schooling, at that time in Belgium limited to the age of 14, up to the age of 16. To that end, the manifesto insisted, a completely new type of school had to be created, a synthesis of the existing school systems. "Schools whose task would not only consist in the teaching of some basic techniques (reading, writing, calculating), but the complete formation of the future citizen[1] (p. 3)." These new schools could initially coexist with the existing ones, but should gradually replace them, leading to a unique post-War model of education. In the first issue of *L'École*, a loose-leaf educational magazine in which the ideas of CIREB were further concretized, the initiators explained their project in more detail:

> The CIREB project *The school for 6- to 16-year olds* has been conceived taking into account, as much as possible, current needs and possibilities, as well as the wishes expressed by the diverse democratic organizations of the country.
> As an immediate objective, CIREB proposes to the Government: The creation of schools (for 6- to 16-years olds) in which every child will receive a complete general education (physical and intellectual, moral and civic, technical and scientific, artistic and literary).
> This new education should considerably raise the general level of our society and at the same time promote the training of our elites by creating, in particular for the best gifted, developmental opportunities that are unknown to the school of today.
> The realization of a good general education of 10 years will require some years of serious effort, years during which the country will have the opportunity to gain the experience that will enable it to decide when and how the School for all can take a next step. (CIREB, 1945b, p. 1)

The project of the CIREB, in particular its call for pedagogical innovation, was largely inspired by the work of the Belgian psychologist and pedagogue Dr. Ovide Decroly (1871–1932). Decroly was one of the main protagonists of the so-called *New Education* or *Reform Pedagogy*, an international educational movement that flourished between 1890 and World War II. Figure 1.1 shows a practical example of Decroly's pedagogy aimed at a harmonic and broad child development based on societal involvement, interdisciplinarity, and active learning processes induced by interactions with the surrounding environment (Depaepe, Simon, & Van Gorp, 2003; Van Gorp, 2005). In Belgium, Decroly's ideas were very influential in the pre-War period and had led in 1936 to new curricula for the primary level, putting a strong emphasis on child-centredness and on connecting school matter with children's concrete, daily-life experiences (Centrale Raad voor het Katholiek Lager Onderwijs, 1936a, 1936b; Ministerie van Openbaar Onderwijs, 1936). For mathematics, these reform-based curricula promoted an approach which showed similarities with what later, in the 1970s, would be called "Realistic Mathematics Education" (see Chapter 9). It was, for instance, stated that arithmetic is not a goal in itself but should always be connected to a concrete reality, that long and tedious computations should be avoided, and that word problems should be inspired by pupils' activities and interests. Likewise, in the domain of measurement, it was recommended only to use measures that the children would also use in everyday life.

[1] Unless otherwise stated, all translations were made by the authors.

Figure 1.1. O. Decroly evoking children's spontaneous interests, n.d. (Centre d'Études decrolyennes).

It must however be acknowledged that, due to several factors, the educational practice in Belgian schools did not change fundamentally in those days (Depaepe, De Vroede, & Simon, 1991). With respect to the teaching of arithmetic, for instance, commonly used textbooks continued to pay a lot of attention to long series of bare sums, without "meaning" and unconnected to any applied context (De Bock, D'hoker, & Vandenberghe, 2011).

The project of the CIREB basically aimed at revitalizing Decroly's *New School* ideas in the post-War era for all children aged 6 to 16 (Depaepe et al., 2003). At that time, these ideas were uncompromisingly put into practice at the *École de l'Ermitage* (later named *École Decroly*), a comprehensive school for primary and secondary education founded in 1907, located in an urban centre of Brussels (Uccle) and attracting much support from leftist intellectual circles in the capital, in particular with connections to the *Université Libre de Bruxelles*. The *École* was run by an enthusiastic and committed team of teachers, some of them member of the CIREB Committee. The pedagogical practice at the *École* was considered as a model for the future School for all. A central idea of Decroly—and of the pedagogical approach at the *École*—was to eliminate the artificial subject divisions and thereby to concentrate the teaching of subject matter and related school activities around central themes that corresponded to dominant interests of the children, the so-called method of centres of interest. According to the CIREB (1945a) manifesto, these centres of interest should have different characteristics depending on the age of the students.

- For 6- to 8-year-olds, the centres of interest are immediate, occasional, and of short duration. Activities related to observation, realization, manual work, modeling, drawing, music, and native language were included in the general activities of the centre of interest.
- For 8- to 12-year-olds, the centres of interest are organized around the needs of the child: to feed, to fight against the bad weather, to defend itself against enemies, to work in solidarity, to rest, and to recreate. The child comes in closer contact with the environment.
- For 12- to 16-year-olds, the centres of interest are related to the needs of man and society. The great themes, nutrition, protection, defense, and action extend the themes of the previous level.

From the age of 12 on, it was indicated to organize the objects of study in two major courses that were developed in parallel: *Science and Technology* and *History and Languages*. The CIREB also put forward a basic program bringing together a set of essential concepts and skills that all children, not segregated into level classes, should have achieved by a cross-curricular approach at the age of 16.

For mathematics, the CIREB proposed a rather concise program, divided into five sections:

- *Numerical calculations*
  The four basic operations, the arithmetic mean, powers, and square roots performed on whole, fractional, and decimal numbers. Divisibility rules for 2, 3, 4, 5, 6, 9, 10, and 11. The greatest common divisor (GCD) and the least common multiple (LCM). The numerical value of an algebraic expression. The slide rule.
- *Geometrical figures*
  Intuitive knowledge and construction of common geometric shapes, using various instruments and materials. Study of these shapes: lengths, angles, areas, volumes, symmetries, sections, developments, and centres of gravity. Trigonometric ratios. Tables of trigonometric functions.
- *Transformations of figures*
  Reproduction of concretely given geometric shapes, full size and on scale. Similar figures: essential properties and cases of similarity. Reproduction of a specific graph by changing the units.
- *Spatial representations*
  Monge projections, dimensioned plan, and quick perspective.
- *Functions*
  Sketching and using graphs. Concept of function of one and more variables (continuous or not). Graphical representations of the functions $ax + b$, x^2, $1/x$.

Although this "program" was probably nothing more than a first draft, several interesting elements stand out. First, nothing was mentioned about algebra, although some algebraic skills seem to be necessary for the study of functions, the fifth section of the program. Second, for geometry, the program proposed an intuitive approach to the study of geometrical figures. The program does not mention a passage from perception to deduction, but a topic such as "cases of similarity" suggests that it was probably the intention. Third, with the heading "transformations of figures," the authors of the program introduced a dynamic element in geometry education, which could be regarded as an innovation compared to traditional approaches. However, the topic appears rather as a means of drawing than one of reasoning. Finally, the program seemed to advocate a practical and applied approach to mathematics education and not a purely theoretical one, mathematics by doing rather than by contemplating. Despite its brevity, and as it will become apparent later, it can be said that the CIREB program for mathematics reflected some of the key ideas of Paul Libois who certainly had a hand in it.

In May 1945, the CIREB project was presented to Auguste Buisseret, the Belgian Minister of Education, but the *Comité* did not await political approval of its suggestions of reform. The CIREB considered its ideas as being not in contradiction with the existing legislation, and therefore, schools were invited to implement the program directly if they wished to do so. To help schools in this endeavor, the CIREB magazine *L'École* published, between December 1945 and February 1951, several detailed dossiers about cross-curricular themes that could be chosen as "centres of interest" for the different age levels between 6 and 16. Most of these dossiers were prepared by scholars of the *Université Libre de Bruxelles* and covered themes such as the "weather," "labor," and "nutrition." Unfortunately, little noteworthy mathematical material could be found in the different volumes of *L'École*, with the exception of an intriguing article by Paul Libois on the exploitation of numerical data obtained during outdoor activities or in the context of educational games (P. Libois, 1947). Clearly, a reform of mathematics education was not a priority among the members of CIREB.

Camille Huysmans' Reform Program for Secondary Education

In March 1947, the Flemish socialist Camille Huysmans (1871–1968) became the new Belgian Minister of Education. Huysmans, a doctor in Germanic philology who himself had served as a teacher for some years, warmly welcomed the CIREB initiative for a democratic reform of the education system and took action. In 1948 (circular of September 20), Huysmans expounded the general outlines of his policy, in particular with respect to a reform of secondary education as organized by the Ministry (Ministère de l'Instruction Publique, 1948; Ministerie van Openbaar Onderwijs, 1948; Noël, 2002). He blew a new wind through the Belgian education system. Key ideas of Huysmans' reform program related to the objectives of secondary education, the role of the teacher, and several requirements of socio-economic nature. A short anthology:

> Our school system suffers from insufficient knowledge of the psychology of the child and the adolescent.
> It is the students' abilities and not the prior requirements of the discipline that will determine the choice of the material.
> Secondary education is no longer just for children of the wealthy bourgeoisie.
> A more generous understanding of the vocation of the woman and of the more important place that has been assigned to her in the social and economic life has given a new direction to the education of the girl.
> The teacher, regardless of his specific position, is primarily an educator, secondly a mother tongue teacher and finally a specialist in his discipline.
> The requirements that the University imposes on its future students are the same as the requirements that the society may impose on all students who have graduated from secondary education.
> By the end of secondary education, the student must have acquired a broad sense of responsibility toward the society and its members.
> It is also appropriate to install a well-founded patriotism which promotes a sense of respect to other people, a deeper character training and a sense of wide tolerance. (Ministerie van Openbaar Onderwijs, 1948, pp. 15–23)

The reform program was further concretized for the different school subjects in a series of booklets, including one for mathematics (Bosteels, 1950; Ministère de l'Instruction Publique, 1955; Ministerie van Openbaar Onderwijs, 1952), and was officially presented, commented, and illustrated in April 1952 at a multi-day pedagogical internship in Nivelles (Ministère de l'Instruction Publique, 1952). The Catholic network followed with a new program for the secondary schools within their network (Fédération Nationale de l'Enseignement Moyen Catholique, 1953). For mathematics, this program was largely the same as that of the Ministry of Education.

(Not so) Mechanistic Mathematics Education

The new wind also blew through the mathematics programs. It was argued that the secondary schools did not prepare their students enough for them to be able to enter university, but at the same time, it was conceded that not all students entering secondary school would later proceed toward higher education. Consequently, mathematical instruction at the secondary level served a double goal: On the one hand, it had to prepare students better for higher studies, and, on the other, it had to provide students with skills that would help them in daily life. Therefore, some notable pedagogical considerations and recommendations were made, in particular for the lower grades of secondary education. First, the main goal of mathematics education was the true and in-depth formation of the human mind, the development of an objective and methodical attitude toward various problems that may arise, and of a critical sense for analysis and synthesis. Second, given the fact that that human ideas originate from concrete experiences, it was recommended to avoid premature abstractions. Abstraction should always be preceded by concrete instantiations.

> Abstract concepts will be better understood if they are founded on a simpler and more solid intuitive base. Moreover, the teacher will often experience the need to reinforce in the concrete domain such knowledge which he believed to be definitely acquired and fixed. (Ministère de l'Instruction Publique, 1955, p. 5)

Third, the Socratic teaching method, in contrast with "dogmatic" instructional approaches, was recommended, both for didactical reasons—revealing connections in the mathematical subject and triggering self-discovery learning—and for creating a positive classroom atmosphere which contributes to students' self-confidence, enthusiasm, and sense of investigation. Fourth, the teacher should apply, rank, and compare the main mathematical research methods. In particular, it was recommended to solve the same problem by several methods which offer an opportunity to confront them from the viewpoint of elegance and efficiency. Fifth, the reform program also included some new elements that may foreshadow the structural tendency of and unity view on mathematics which will become dominant during the late 1950s and 1960s (Noël, in preparation):

- Take advantage of every opportunity to impregnate the minds of the students with the important ideas of analogy and symmetry.
- Make use to frequent repetitions, of syntheses after each theory, of comparisons of equivalent theories.
- Show that the different branches of mathematics do not have to be separated by bulkheads, but that they penetrate and help each other.
- Reduce the role of the memory by grouping knowledge around fundamental ideas. Acquire more unity by providing a solid rational basis (Ministère de l'Instruction Publique, 1955, p. 6).

Sixth and last, the reform document emphasized the necessity of a correct expression and accurate use of language in mathematics lessons, without falling into a rigid automatism or formalism that only burdens the memory and excludes any appeal to proper judgment. In this respect, mathematics and the mother tongue could and should positively influence each other.

The mathematics program for the first year aimed at consolidating and extending students' knowledge and skills acquired at the primary school to open the path gradually to the different subdisciplines of secondary school mathematics. In particular, intuition and practical skills were seen as important tools in generating a foundation for abstract understanding. The program consisted of two parts: arithmetic and intuitive geometry.

The arithmetic part included numeration and the four basic operations on natural numbers, powers and square roots (of perfect squares), divisibility rules, prime factor decomposition and computation of the GCD and the LCM of two or more numbers, operations on fractions, decimal numbers, the arithmetic mean, basics of the metric system, problems about length, area and volume calculation, and word problems "taken from everyday life." In the second year, the number concept was extended with negative numbers, and, with the transition from arithmetic calculation to algebra, a further step toward abstraction was taken (although the idea of representing numbers by letters was already applied, earlier). The guidelines for the first year specified that the study of the four arithmetic operations—a return to schemes which the students already had encountered at the primary level—had above all to explain the underlying mechanisms and properties (commutativity, associativity, product of a sum or difference by a number, product of two sums or differences). To practise these properties, fast and mental arithmetic in the form of “commando rekenen” [drill calculation] was recommended:

> To the latter [the practice of the properties of the operations] eight to ten minutes of each lesson in arithmetic will be devoted, in the form that is common in the Netherlands and is called “commando rekenen.” In principle, this method consists in asking the students to perform mentally a series of operations of which they only write down the result in their notebook, one under the other; the students finally add these results in writing. The resulting sum is then, as a control, communicated to the teacher.
> Such exercises, if properly led by the teacher, are very useful. They create in the classroom an atmosphere of competition and allow a quick control of the results. Above all, they offer an opportunity to again recall the properties that are applied. These recommendations are inspired by one of the essential goals assigned to the lower grades of general secondary education: To provide students with sufficient computational techniques in arithmetic and algebra, so that, freed from the obstacles inherent to any laborious calculation, they can easily be initiated in the more abstract studies of the higher grades. (Ministère de l’Instruction Publique, 1955, p. 7–8)

This quotation—like many other recommendations from this reform document—reflects an approach to arithmetic education, and more generally to mathematics education, which has been labelled by Treffers (1987) as “mechanistic.” This label has a negative connotation and has often been used to characterize mathematics education in many places, all over the world, before and shortly after World War II (see, e.g., Van den Heuvel-Panhuizen & Drijvers, 2014). Freudenthal (1991) typified this mechanistic approach as one in which the learner is seen as a computer-like instrument that can be programmed by drill to perform all kinds of arithmetical and algebraic, maybe even geometrical operations. Problems can be solved by recognizing the underlying model and by repeating the procedure that has been applied previously to this type of model. De Bock, Van Dooren, and Verschaffel (2020) summarized the main characteristics of a mechanistic approach to (elementary) mathematics education as follows:

> In a mechanistic approach, the focus of instruction is on factual and procedural knowledge (e.g., knowing how much 6 × 9 is, to know how to add or multiply multi-digit numbers, to know the formulas for computing the perimeter and the area of regular plane figures, etcetera). Learning is primarily seen as the acquisition of this type of factual and procedural knowledge through basic learning principles such as inculcation, memorizing and repeated practice of technical computational skills, principles that were in the same period promoted and theorized by behavioral psychologists (e.g., Thorndike’s law of exercise and law of effect). The instruction is heavily teacher directed, with the teacher being the dispenser or transmitter of the distinct specific pieces of knowledge and specific skills to be learned, as well as the taskmaster who decides what information and instruction the learners get, and when and how these are provided. In a mechanistic approach, there is little or no attention for conceptual understanding (the reasons behind the facts and procedures that are taught) and theory development, nor for ‘realistic’ applications. (p. 42)

Although Belgian arithmetic education in the early 1950s was strongly influenced by mechanistic principles, as evidenced by the official reform program (Ministère de l'Instruction Publique, 1955; Ministerie van Openbaar Onderwijs, 1952) and other contemporary documents, it would be a mistake to label this education as purely mechanistic. First, as we already explained, improving students' computational skills was primarily seen—at least by the developers of the program—as a means to unravel underlying mechanisms and properties to prepare these students for more abstract mathematics. The drill exercises were not an end in themselves. Second, in the context of the solution of (proportional) word problems, the official program warned explicitly for routine behavior by uniquely relying on one standard algorithm. On the contrary, the authors argued for more flexible solution strategies:

> Too often the rule of three is excessively used and its application then takes a routine character which is completely contrary to the formative goal of mathematics. If possible, we must apply methods that are less cumbersome and more adequate, methods with a more direct appeal to intelligence. (Ministère de l'Instruction Publique, 1955, p. 9)

Third and last, the authors of the reform program argued for bringing the school closer to real life by integrating real-life applications in the mathematics lessons (calculation of discount, interest, mixing of products, etc.). Therefore, it was recommended that the teacher should not only take his examples from books but that he also designed mathematical problems, based on available resources. Moreover, it was suggested that students collaborate with their teacher for collecting data and gathering information to create new problems. The teacher was not simply a dispenser of knowledge, but an active agent in the learning process of the students. Needless to say, these recommendations were inspired by Decroly's *New School* ideas.

Intuitive Geometry in Belgian Secondary Schools

In the course on intuitive geometry, the second part of the new program for the first year of secondary schools, the spirit of the reform, particularly the gradual transition from concrete experiences to abstraction, was marked most clearly (François, 1952). Along with activities of observation, construction of geometrical objects, and reproduction in full scale or on a given scale, intuitive geometry included the study of simple geometrical shapes, both in plane (polygons and circles) and in three-dimensional space (cubes, parallelepipeds, cylinders, cones, and spheres). In accordance with the skills learned in arithmetic, plane figures and solids were measured and calculated. Intuitive geometry continued in the second year with the study of more complex shapes (prisms, pyramids, truncated cones, helixes, and ellipses and parabolas as geometric loci), the concepts of congruence and similarity, and the notion of symmetry with informal references to reflections and rotations. Gradually, the intuitive phase was replaced by a geometry based on deductive reasoning. In the third year, geometry was taught exclusively in a deductive manner. According to Levarlet (1959), this element of the reform was due to inspector Jules Richard, and to Professor Paul Libois, a lifelong advocate of an intuition-based teaching of geometry.

But what does intuitive geometry actually stand for, what was the underlying rationale, and where did it originate from? In fact, there is no single definition of intuitive geometry with regard to school mathematics. Historically, the intuitive approach to geometry goes back at least as far as the ideas of the sixteenth-century mathematician Petrus Ramus, and also to Alexis Clairaut's textbook *Élémens de Géométrie* [Elements of Geometry] (1741), in which the traditional order of Euclid's *Elements* was changed to accommodate for a more user-centred learning trajectory. Later, it re-emerged at the beginning of the twentieth century in Germany and Italy (Giacardi, 2006; Steiner, 1988). In general terms, intuitive geometry can be considered as a replacement of the axiomatic presentation followed by Euclid by a more genetic architecture. As Gattegno (1955) observed:

> The use of the term "intuitive geometry" implies nowadays that the aim remains, as before, knowledge of theorems and geometrical facts, but that the presentation of the material will start with wholes which will be analyzed and not with definitions and axioms, with actions that will gradually be formalized and whose validity will be extended, rather than with general statements universally valid from the start (at least as far as the teacher's mind is concerned). (p. 351)

Menghini (2010) also considers intuitive geometry as an alternative to the rational-deductive approach to geometry based on Euclid. According to her, it is characterized by the use of visualizations, perceptions, concrete materials, and mental images in the generation of knowledge. Intuitive geometry attracted a considerable amount of attention at the beginning of the twentieth century, not only in the context of school mathematics but also as part of a methodological "introspective" reflection on the nature of mathematical knowledge. After World War II, a new wave of interest for intuitive geometry emerged, but now more exclusively in the field of didactics. In Italy, Emma Castelnuovo (1913–2014) published a textbook for secondary schools *Geometria Intuitiva* (1948), in which she worked out a complete intuitive approach for the teaching of geometry. She put the essence of intuitive geometry in the active involvement of both teacher and student in the gradual construction of mathematical knowledge. With reference to Clairaut's *Élémens*, she also suggested that the intuitive approach to geometry actually retraced the historical genesis of geometrical knowledge. Whereas Euclid's *Elements* was the polished pronunciation of an already acquired knowledge, the discovery of geometrical truths which necessarily preceded the writing of the *Elements* was based on discovery and intuition. Following this same trajectory, intuitive geometry was therefore best suited to teach geometry to beginning students.

In post-War Belgian school mathematics, the course of intuitive geometry was primarily meant to be a bridge between the elements of geometrical practice the students had learned in primary school (drawing, measurements) and the course of "rational" geometry, which started in the second year of secondary school. Intuitive geometry aimed at making the students familiar with the properties of simple geometrical objects. Students made their own models of geometrical objects on which they could perform various observations and measurements. This way, they accumulated practical geometrical knowledge and became familiar with the corresponding vocabulary. They also learned to use drawing tools such as a ruler, a protractor, and a compass. Much emphasis was put on the active manipulation of real objects, to develop the students' sense of observation and to induce an "appetite" for learning geometry.

> Intuitive geometry is a delicate subject. It's not just a matter of bringing into young students' minds, by avoiding any deductive reasoning as much as possible, the most important geometrical properties through the path of the eyes and the fingers, according to the time-honored principle "*nihil in intellectu quod non prius fuerit in sensu*" [there is nothing in the intellect without first passing through the senses]. Even more, it is essential that this first contact with geometry gives these youngsters "a taste" for geometry in a lasting way. (Debiève & Verhelst, 1957, p. 3)

For example, students would cut a triangle out of a piece of paper and then cut the triangle in three parts. When the pieces were assembled again, with the internal angles of the original triangle next to each other, the students would be astonished to find that the pieces would align themselves in a straight line, no matter the shape of the original triangle (Figure 1.2). This would lead to an intuitive understanding of the fact that the sum of the angles of a triangle is always equal to two right angles. Students were challenged to verify this fact by