

Synthese Library 412

Studies in Epistemology, Logic, Methodology,  
and Philosophy of Science

Stefania Centrone

Sara Negri

Deniz Sarikaya

Peter M. Schuster *Editors*

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# Mathesis Universalis, Computability and Proof



Springer

# Synthese Library

Studies in Epistemology, Logic, Methodology,  
and Philosophy of Science

Volume 412

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Stefania Centrone • Sara Negri • Deniz Sarikaya  
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Editors

# Mathesis Universalis, Computability and Proof

 Springer

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# Preface

The present volume emerged from the Humboldt-Kolleg “Proof Theory as *Mathesis Universalis*”, which is held from 24 to 28 July 2017 at the German-Italian Centre for European Excellence, Villa Vigoni (Loveno di Menaggio, Como). This meeting brought together 39 scholars from 12 countries (Germany, Finland, France, Italy, Japan, the Netherlands, Norway, Austria, Switzerland, Sweden, the UK and the USA) on three continents (Europe, Asia and America). Both the conference and the volume exemplify the ideals of the Alexander von Humboldt Foundation, “open societies”, “excellent research”, “European integration” and “scientific network” beyond any linguistic, religious and political differences, as Enno Aufderheide puts it in the first contribution of this volume.

The conference was originally conceived as the capstone of a project supported by the German Research Association (DFG) on the development of Leibniz’s ideas on the *mathesis universalis* by the mathematician and philosopher Bernard Bolzano (1781–1848) and by the founder of phenomenology and mathematician by training Edmund Husserl (1859–1938). However, the development of the *mathesis universalis* in Bolzano and in Husserl is connected to certain programmatic requirements for proofs, in order to satisfy the criteria to be considered “rigorous” (“streng” in German). Hence, the focus of the conference was proof theory, in general the branch of mathematics that is concerned with the problem of articulating the formal conditions to which proofs must conform to be rigorous.

Generous financial support came first and foremost from the Alexander von Humboldt Foundation and furthermore from the following associations: Altonaer Stiftung für philosophische Grundlagenforschung (ASPGF), Deutsche Mathematiker-Vereinigung (DMV), Deutsche Vereinigung für Mathematische Logik und für Grundlagenforschung der Exakten Wissenschaften (DVMLG) and Gesellschaft für Analytische Philosophie (GAP).

The editors are indebted to all the contributors and referees for the time and energy they devoted to writing and reviewing the papers. The encouragement at a decisive moment and the friendly advice from Otávio Bueno, editor in chief of *Synthese Library*, and from Palani Murugesan, Springer's project coordinator, were truly invaluable.

Berlin, Germany  
Helsinki, Finland  
Hamburg, Germany  
Verona, Italy  
March 2019

Stefania Centrone  
Sara Negri  
Deniz Sarikaya  
Peter M. Schuster

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## About the Editors

**Stefania Centrone** is currently *privatdozent* at the University of Hamburg and holds a *Heisenberg-Stelle* at the Technical University of Berlin. In 2012, she was awarded a DFG-Eigene Stelle for the project “Bolzanos und Husserls Weiterentwicklung von Leibnizens Ideen zur Mathesis Universalis” at the Carl von Ossietzky University of Oldenburg, where she remained as a research assistant until 30 September 2018. In 2016, she was deputy professor of Theoretical Philosophy at the Georg-August-Universität Göttingen. She is also author, among others, of the volumes *Logic and Philosophy of Mathematics in the Early Husserl* (Springer 2010) and *Studien zu Bolzano* (Academia Verlag 2015) and is editor, among others, of *Versuche über Husserl* (Mainer 2013), *Essays on Husserl’s Logic and Philosophy of Mathematics* (Springer 2017) and *Reflections on the Foundations of Mathematics: Univalent Foundation, Set Theory and General Things* (Springer 2019) (with Deborah Kant and Deniz Sarikaya).

**Sara Negri** is professor of Theoretical Philosophy at the University of Helsinki, where she has been a docent of Logic since 1998. After a PhD in Mathematics in 1996 at the University of Padova and research visits at the University of Amsterdam and Chalmers University of Technology, she has been a research associate at the Imperial College in London, a Humboldt fellow in Munich and a visiting scientist at the Mittag-Leffler Institute in Stockholm and Hausdorff Research Institute for Mathematics in Bonn. Her research interests range from mathematical logic and philosophy of mathematics to proof theory and its applications to constructivism, philosophical logic and formal epistemology.

**Deniz Sarikaya** is PhD student of Philosophy and studied mathematics and philosophy at the University of Hamburg with experience abroad at the Universiteit van Amsterdam and Universidad de Barcelona. He stayed a term as a visiting student researcher at the University of California, Berkeley, developing a project on the Philosophy of Mathematical Practice concerning the philosophical impact of the usage of automatic theorem prover, and as a RISE research intern at the University of British Columbia. He is mainly focusing on philosophy of mathematics and logic.

**Peter M. Schuster** is associate professor for Mathematical Logic at the University of Verona. After both doctorate and habilitation in mathematics, he was privatdozent at the University of Munich and lecturer at the University of Leeds. Apart from constructive mathematics at large, his principal research interests are about Hilbert's programme in abstract mathematics, especially the computational content of classical proofs in algebra and related fields, in which transfinite methods, such as Zorn's lemma, are invoked.

# Chapter 1

## Introduction: *Mathesis Universalis*, Proof and Computation



Stefania Centrone

**Abstract** By “*mathesis universalis*” Descartes and Leibniz understood a most general science built on the model of mathematics. Though the term, along with that of “*mathesis universa*”, had already been used during the seventeenth century, it was with Descartes and Leibniz that it became customary to designate it as a universal mathematical science that unifies all formal *a priori* sciences. In his *Dissertatio de arte combinatoria* (1666), early Leibniz writes that the *mathesis* is not a discipline, but unifies parts from different disciplines that have quantity as their subject. A while later, in a fragment entitled *Elementa Nova Matheseos Universalis* (1683?) he writes “the *mathesis* [ . . . ] shall deliver the method through which things that are conceivable can be exactly determined”; in another fragment he takes the *mathesis* to be “the science of all things that are conceivable.” The more mature Leibniz considers all mathematical disciplines as branches of the *mathesis* and designs the *mathesis* as a general science of forms applicable not only to magnitudes but to every object that exists in our imagination, i.e. that is possible at least in principle. As a general science of forms the *mathesis* investigates possible relations between “arbitrary objects” (“*objets quelconques*”). It is an abstract theory of combinations and relations among objects whatever.

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By “*mathesis universalis*” Descartes and Leibniz understood a most general science built on the model of mathematics. Though the term, along with that of “*mathesis universa*”, had already been used during the seventeenth century, it was with Descartes and Leibniz that it became customary to designate it as a universal mathematical science that unifies all formal *a priori* sciences.<sup>1</sup> In his *Dissertatio de arte combinatoria* (1666), early Leibniz writes that the *mathesis* is not a discipline, but unifies parts from different disciplines that have quantity as their subject.<sup>2</sup> A while later, in a fragment entitled *Elementa Nova Matheseos Universalis* (1683?) he writes “the *mathesis* [...] shall deliver the method through which things that are conceivable can be exactly determined”; in another fragment he takes the *mathesis* to be “the science of all things that are conceivable.” The more mature Leibniz considers all mathematical disciplines as branches of the *mathesis* and designs the *mathesis* as a general science of forms applicable not only to magnitudes but to every object that exists in our imagination, i.e. that is possible at least in principle. As a general science of forms the *mathesis* investigates possible relations between “arbitrary objects” (“*objets quelconques*”). It is an abstract theory of combinations and relations among objects whatever.

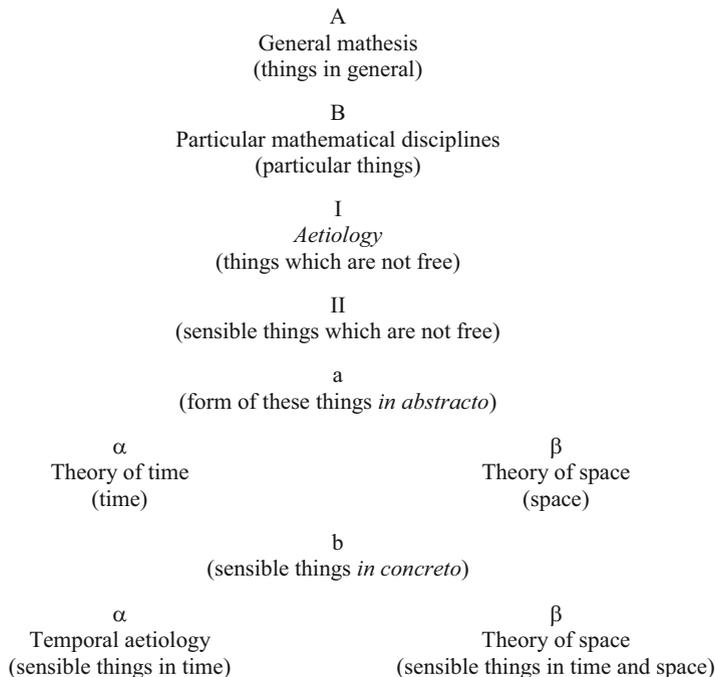
In 1810 the mathematician and philosopher Bernard Bolzano, also known as the “Bohemian Leibniz” or “Leibniz in Bohemia”,<sup>3</sup> published a short book entitled *Contributions to a Better-Founded Presentation of Mathematics*. The first part, *On the Concept of Mathematics and its Classification*, focuses on the search for a definition of mathematics as well as a principle of classification for its different branches; the second part is a discourse *on the mathematical method*. Bolzano defines mathematics as “a science which deals with the general laws (forms) to which things must conform [*sich richten nach*] in their existence [*Dasein*]” (§8), and explains that he understands by “things” not only the ones that actually exist “but also those which simply exist in our *imagination* [...], in other words, *everything which can in general be an object of our representational capacity* [*unseres Vorstellungsvermögens*]” (§8).<sup>4</sup> From this definition of mathematics, Bolzano derives a classification of the *mathesis* in several particular disciplines: *The laws to which things must conform in their existence* are either so general that they are applicable to *all things without exception*, or not. The former laws will constitute the first main part of the mathematics. It can be called “*general mathesis*”; everything else is *particular mathesis* (§11). Bolzano conceives the *mathesis* according the following schema:

<sup>1</sup>Cp. e.g. Descartes Rule IV, AT.X, 372–379, Rule I, AT.X, 361; AT.X, 255; G. W. Leibniz, [1678] *Leibniz an die Pfalzgräfin Elizabeth* (?), in AA II, 1.A, 659–666.

<sup>2</sup>AA VI, 1, 171.

<sup>3</sup>Bolzano had been called thus by the Herbartian philosopher Josef Durđk (1837–1902) in a speech held before a group of Czech-Bohemian intellectuals on the occasion of the centenary of Bolzano’s birth on October 1881.

<sup>4</sup>Hereto cp. a. o. Casari 2004.



The *general mathesis* includes, so Bolzano, arithmetic, the theory of combinations and several other parts. “These parts of mathematics must therefore not be considered as *coordinate with the rest* (chronometry, geometry, etc.); it is rather that the latter are *subordinate* to the general *mathesis* as a whole, as species [Art] of the genus. And because the concept of *number* is one of those of the *general mathesis* it will also appear *frequently* in all these particular parts, but it will not exhaust their content (§11).” “Now, in order to obtain the *particular* or *special* parts of mathematics, we must put the *things themselves*, with whose general forms mathematics is concerned, into certain *classes* (§12).” Thus, each single branch of the *mathesis* appears to be an independent reality with its own mathematical structure.

Everything which we may ever think of as *existing* we must think of as being one or the other: either *necessary or free* (i.e. not necessary) in its existence. That which we think of as something *free* is subject to no conditions and laws in its *becoming* (or *existence*) and is therefore not an object of mathematics. That which we think of as *necessary* in its existence is so, either simply (i.e. in *itself*) or only *conditionally* (i.e. on the presupposition of something else).

The *necessary in itself* is called God and is considered in *metaphysics* not as a merely *possible* object but as an *actual* object. Therefore, there remains only the *hypothetically necessary*, which we consider as produced through some ground (*Grund*). Now there are certain *general conditions* according to which everything, which is produced through a ground (*in* or *out* of time), must be *regulated* in its *becoming* or *existence*. These conditions taken together and ordered scientifically will therefore constitute the *first*

main part of the particular *mathesis*, which I call, for want of a better name, the *theory of grounds* (*Grundlehre*) or *aetiology* (§ 13).

“Aetiology” alludes to the Greek word ‘αιτία’ which means whatever is specified in an answer to a why-question. The reference to Aristotle’s theory of the four αιτία is explicit. The *theory of grounds* or *aetiology* is the counterpart of causality at the level of a relation between true propositions. There obtains, according to Bolzano, a *certain objective connection* among the truths that are germane to a certain homogeneous field of objects: some truths are the “grounds” (“*Gründe*”) of others, and the latter are “consequences” (“*Folgen*”) of the former. As in so many other respects, Bolzano stands here on Leibniz’s shoulders. In his *Theory of Science* (1837)<sup>5</sup> Bolzano quotes the following passage by Leibniz:<sup>6</sup>

A reason is a known truth whose connection with some less well-known truth leads us to give our assent to the latter. But it is called a ‘reason’, especially and par excellence, if it is the cause not only of our judgement but of the truth itself. . . . A cause in the realm of things corresponds to a reason in the realm of truths, which is why causes themselves . . . are often called ‘reasons’.

The idea of a dependency relation between the truths that are relative to a certain mathematical reality has a pendant both at the level of proofs and at the level of theories. From the *etiological* point of view a proof is a procedure by which we bring to light the *reasons* of the truth of a proposition, while, on an *epistemic* perspective, a proof is a procedure through which we *ascertain* its truth. At the level of a theory T this contrast reappears as the distinction between a *privileged* presentation of T in which every non-axiomatic truth of T is “etiotologically proven” and the *various* possible logical presentations of T that are epistemically adequate (insofar as every non-axiomatic truth of T is provable in them).<sup>7</sup>

Which programmatic requirements do proofs have to obey, if they are to be rigorous?

As we just said, rigorous proofs should provide the ground(s) of their conclusion. Bolzano recasts this request in a number of formal constraints that proofs should satisfy in order to ground their conclusion. Rigorous proofs should:

- (a) proceed from the general to the particular;
- (b) proceed from the simpler to the more complex;
- (c) be such that all premises are relevant to obtain the conclusion;
- (d) exclude alien intermediate concepts.

In particular, the requirement (b) of increasing-complexity has been taken up in modern logic with the various normalization results given e.g. for calculi of natural deduction or sequent calculi. More exactly, requirement (b) corresponds in modern logic to the top-down deductive style from simple premisses to complex conclusion in cut-free sequent calculi; requirement (d) corresponds to the process

<sup>5</sup>Bolzano 1837, II, 342 (Op.)

<sup>6</sup>Leibniz 1704, Book IV, Chapter xvii, §3.

<sup>7</sup>Hereto cp. Casari 1987, 330.

of normalization/cut-elimination in natural deduction/sequent calculus. *De facto*, all attempts to capture formally requirements (b) and (d) end up with transitivity (or cut) elimination.

It is worth noticing that in some of the following contributions rigorous proofs are sometimes called “analytic proofs”, alluding to the fact that they display an *analysis* of the proposition that is to be proven. This meaning of “analytic” is neither to be confused with the application of ideas and techniques from the analytic number theory nor with the meaning expressed by “analytic” in philosophy from Kant onwards: a proposition is analytic when the predicate-concept is totally included in the subject-concept.

This being said, we end this introduction with some sample questions the following contributions address:

- (i) Where are the “*Anfangsgründe der reinen Mathesis*” to be found?
- (ii) Which are the sources of objectivity and truth in mathematics?
- (iii) How does Leibniz’ idea of *mathesis universalis* relate with his claim for *theoria cum praxis*?
- (iv) Can one develop Bolzano’s ideas about grounds and grounding trees and Gentzen’s ideas about the justification of deductive inferences by way of a well-given definition of the concepts of *legitimate inference* and *ground*?
- (v) Is there a privileged form that definitions should have? What is a good definition?
- (vi) What is the price to be paid in terms of proof complexity when we decide to give up transitivity?
- (vii) How much complexity can be demanded from a proof that shows that the requirement of increasing-complexity (transitivity elimination) is possible?
- (viii) To what extent does the form of a rigorous proof get close to proofs given in common mathematical praxis?
- (ix) What happens if we look at the issue of proof theory as a branch of *mathesis universalis* from the perspective of computer science?
- (x) Does the axiomatic form of proofs mirror the underlying proof?
- (xi) How should a theory T, through which we describe a certain field of objects, account for the complexity of the underlying objects?
- (xii) How is it that finitary proof theory became infinitary?
- (xiii) What do “ground-consequence” proof systems for equational theories of untyped operations (including combinatory logic and  $\lambda$ -calculus) look like?
- (xiv) What are the best solutions to the problem of giving a computational meaning to classical reasoning? Can they be re-examined in connection with algorithms and programming?
- (xv) How do point-free methods account for ideal elements?
- (xvi) How should impredicative methodology be applied to higher inductive types that form the basis of the recent applications in homotopy theory?
- (xvii) In a constructive proof of a negated statement, to which extent may we apply the Law of Excluded Middle?

Such and many other questions are the topics of the next contributions.

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## Chapter 2

# Diplomacy of Trust in the European Crisis: Contributions by the Alexander von Humboldt Foundation



**Enno Aufderheide**

**Abstract** Centrifugal forces are at work in Europe. A quarter of a century after the unification of Europe, it appears that rifts are opening up again, precipitated by financial crises, debt crises and refugee crises. But the rifts do not only run between prosperous or emergent countries and those that, ridden by crisis, are becoming impoverished. The picture is much more complex because the rifts run right the way through the societies themselves.

Science, on the other side, supports European integration and it is precisely where differences emerge, where rifts threaten to occur, that the connecting, bridge building force of science acquires its importance.

The Alexander von Humboldt-Foundation's activities have been closely linked with the history of Europe, especially during the period when Europe was divided. Its mission is to build bridges with and for science and research. First and foremost, it promotes the development and maintenance of academic networks between Germany and other countries around the world, but also amongst those other countries themselves. And by focussing on the long-term, sustainable impact of our work, the Humboldt-Foundation fosters science as diplomacy of trust – trust in Germany, trust in each other within the Humboldt Network and trust in nations as partners in the peaceful development of humankind. The chapter explains some of the mechanisms by which these goals are achieved and calls for a concerted effort of scientists and scholars to build trust not only between nations, but especially within societies.

*Before speaking about the relationship of research and the European Crisis, let me thank the organizers and especially Stefania Centrone for putting together such a remarkable conference and bringing together people from a dozen countries and three continents. It has been a huge workload for them – and the result can give us hope not only for research on the *Mathesis Universalis*, but for the future of our societies as such.*

---

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Ladies and gentlemen, Humboldtians,

It was with great pleasure that I accepted the invitation by Stefania Centrone to speak about the relationship between research (or researchers) and the European crisis.

Not in spite of, but precisely because of, the fact that the topic of this conference is so specific, I should like to talk this evening about the reasons why this Humboldt Kolleg, this specialist scientific conference, is so closely linked with the European idea and why scientists and scholars should not distance themselves from the European crisis.

And to summarize my answer to this “why”-question, let me already now point out that I am convinced that the *role* and the *gift* of researchers give them opportunities to build a diplomacy of trust,

- to campaign for open societies, as they are both a prerequisite for excellent research and for the European Idea
- to campaign for more trust in research
- to better understand those, who currently do not trust science and research.

Centrifugal forces are at work in Europe. A quarter of a century after the unification of Europe, it appears that rifts are opening up again, precipitated by financial crises, debt crises and refugee crises. But the rifts do not only run between prosperous or emergent countries and those that, ridden by crisis, are becoming impoverished. The picture is much more complex because the rifts run right the way through the societies themselves. The upsurge of many anti-European parties in nearly all European countries and the rise in xenophobic, isolationist populism would hardly have been conceivable on such a broad scale just a few years ago. And whether these isolationist tendencies are successful, as in the case of Brexit, or fail, as they did in the French presidential election, determines the rifts between the countries that support European integration and those that leave or threaten to leave the Union.

Ladies and gentlemen, science supports European integration and it is precisely where differences emerge, where rifts threaten to occur that the connecting, bridge-building force of science acquires its importance. I am certain that everyone of us here, today, would describe academic cooperation in Europe as an unparalleled success in this respect.

At this juncture, I should like to tell you more about the Humboldt Foundation whose activities have been closely linked with the history of Europe, especially during the period when Europe was divided.

As you know, the mission of the Alexander von Humboldt Foundation is to build bridges with and for science and research. First and foremost, we promote the development and maintenance of academic networks between Germany and other countries around the world, but also amongst those other countries themselves. In doing so, we focus on science itself: we tailor our sponsorship programmes to the needs of researchers and steadfastly maintain excellence and the inherent principles of research as our guiding lights. Academic excellence and the highest quality are the most important, not to say the only, criteria we use to select our fellows

and award winners from a large international pool of outstanding candidates. We promote people, individuals, and we have very good evidence to support our claim that this is indeed the best form of support.

And by focussing on the long-term, sustainable impact of our work we serve political and societal goals: by supporting excellent scientists and scholars from other countries and promoting their research cooperation with Germany, the Humboldt Foundation fosters science as diplomacy of trust – trust in Germany, trust in each other within the Humboldt Network and trust in nations as partners in the peaceful development of humankind.

The Alexander von Humboldt Foundation has played its part in cross-border scientific diplomacy for more than 60 years. In 1953, the Foundation was established by the German government. In those days, freedom and the civil liberties of many people in Europe were limited. Berlin was divided by a wall; Europe was divided by the Iron Curtain. The Cold War dominated international politics and often people's minds as well. Distrust was prevalent throughout Europe. Back then, no-one would have imagined that Europe would ever achieve peaceful reunification, the differences appeared unbridgeable. Under these conditions the Humboldt Foundation started to build bridges between people, and it did so with patience and perseverance. Academic dialogue and exchange were the methods with which we strove to overcome boundaries and walls – even real concrete walls. During the long years of the Cold War, many researchers from the countries behind the Iron Curtain crossed into “enemy territory”, Germany, with a fellowship from the Humboldt Foundation. In the 1970s and 1980s, for example, Humboldt Foundation fellowships were extremely popular amongst Polish, Bulgarian, Hungarian and Russian researchers, to name just a few. After German reunification and the fall of the Iron Curtain, some of our alumni were instrumental in restructuring their countries, such as the long-serving Polish Research Minister, Michal Kleiber, the former President of Hungary, László Sólyom, or the current Ambassador of Georgia to Germany, Lado Chanturia.

At present, however, it looks as though integrating Europe into a cosmopolitan, tolerant, peaceful continent is being called into question, as though the basic values of European coexistence, as though freedom, tolerance and openness were not guaranteed. On the contrary, these are achievements that we need to defend, for which we must fight even, and especially in Europe.

The Alexander von Humboldt Foundation's Philipp Schwartz Initiative, for example, does indeed defend the values of freedom and tolerance. Since 2016, the Foundation has sponsored universities and research institutions in Germany under this programme, allowing them to host threatened researchers from abroad who are at risk from war and political persecution. Under the Philipp Schwartz Initiative, more than 100 individuals are currently being sponsored, most of them from Syria and Turkey. The programme is financed by funding from the Federal Foreign Office and substantial donations from important foundations in Germany. And not only from Germany: Just four weeks ago, we received a particularly large donation from the United States: the American Andrew W. Mellon Foundation is supporting the Philipp Schwartz Initiative to the tune of more than a million US dollars. This is, in my opinion, a really remarkable gesture with which the Mellon Foundation is

putting its own mission, that is, to defend the essential American values of openness and freedom, into practice here in Europe, as well.

However, it is not just in the context of programmes for endangered researchers that the sciences can contribute to a cosmopolitan Europe. Last year, in a widely acclaimed speech,<sup>1</sup> the President of the German Research Foundation, Professor Strohschneider, argued that open, pluralistic societies constantly address issues that are new to them and scrutinise the things with which they are familiar. This is what characterises them as modern knowledge societies because curiosity about what we do not know, what is foreign to us, and the will to grapple with it are also the drivers of processes which generate scientific insight. In Strohschneider's words: "It is intrinsic to the very essence of research to question all the things we think we know about the natural world and the cultural world."

Hence, societies characterised by pluralism and diversity, by critical analysis and independent thinking, are both a prerequisite for excellent research and preconditioned for realising the European idea, for creating a Europe that sees cultural diversity, openness and civil liberties as the guarantors of democracy and the quality of life.

This clearly defines the goal we have to work for: a culture of discourse and reasoning.

And at this point, I believe, the catchphrase "science as diplomacy of trust" acquires a new meaning or a new responsibility, because outside the academic community we time and again – and unfortunately ever more often – encounter mistrust towards science. There is a tendency to relativise facts and scientifically-sound findings or to simply deny the evidence or even present lies as equally valid. In many places, populist movements and some political decision makers threaten the work and the values not just of the sciences. It is as though yet another rift were opening up in this respect: between loud voices hostile to science and the science-driven majority.

Given this situation, we are challenged to conduct a "diplomacy of trust" on behalf of science. We must strengthen trust in science and campaign for it because policies based on unproven, unprovable claims, policies which suppress consideration and criticism of political decisions can never be in a society's interest. It should, of course, be emphasised that science has no ambitions to pre-empt political decisions, nor does it claim to hold the definitive solutions. But we certainly must come to a broadly-based consensus in society that research and verifiable knowledge, deriving both from the natural sciences as well as the humanities and social sciences, should flow into political and social decision-making.

Together with the German Alliance of Science Organisations, the Humboldt Foundation is involved in various strategies on the topic of communicating science. But we are certain that the greatest potential for building trust in research is to be found in our Humboldt Network that now embraces more than 27,000

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<sup>1</sup>[http://www.dfg.de/download/pdf/dfg\\_im\\_profil/reden\\_stellungnahmen/2016/160705\\_rede\\_strohschneider\\_festveranstaltung.pdf](http://www.dfg.de/download/pdf/dfg_im_profil/reden_stellungnahmen/2016/160705_rede_strohschneider_festveranstaltung.pdf)

members in over 140 countries. Every individual in this world-spanning network is an ambassador for science or for the humanities. It follows that the Humboldt Network is unique in its diversity, including countless nations and cultures, all with their own experiences, ideas and beliefs, shaped by the Humboldtians who share their knowledge and commitment within it. In the Humboldt Network, or as Humboldtians themselves like to call it, the Humboldt Family, we see the ideals of tolerance, openness and freedom as well as respect for human dignity being put into practice every day. Men and women, Muslims, Christians, agnostics, Hindus and Jews, black and white, young and old cultivate their relationships with one another. The effects of this togetherness are manifold and – without doubt – we must continue to work together, to share our knowledge about the world around us with others and to try and understand our world and ourselves better.

The many examples of courageous Humboldtians who engage with their countries and their societies constantly encourage us at the Foundation to keep working for open, knowledge-driven, free societies.

At the same time, I should like to encourage all of you to take every possible opportunity to speak in public about how important research is and about what you as researchers do. Without compromising facts, there are ways of reaching out to the man and woman in the street and showing them by example what an essential role education and science play. Campaign for trust in science!

In this campaign, we must engage in dialogue, even if it is sometimes rejected, even if we are not always taken seriously, even if the wind is against us. And for this dialogue, it is not enough to talk about research and knowledge. We also need this dialogue to better understand those who do not trust research. We must understand them so that their needs are not left to the promises of populists, but that we can help serve these needs in an evidence based way.

If we succeed in this type of dialogue, trust in the European idea will be restored. I look forward to discussing these thoughts with you.

# Chapter 3

## Mathesis Universalis and Homotopy Type Theory



Steve Awodey

**Abstract** The present paper investigates the use of impredicative methods for the construction of inductive types in homotopy type theory. Inductive types have been constructed impredicatively in other systems of type theory in the past, but these fail to have the correct rules. Using new methods, the paper shows how to repair these prior constructions, and extend the impredicative methodology to include also the newly discovered higher inductive types that form the basis of the recent applications in homotopy theory. This present work refines and extends the traditional logistic approach to foundations of mathematics to encompass both arithmetic and geometry in a comprehensive logistic system that also admits a computational implementation on modern computing machines. The connection to the idea of mathesis universalis is thus quite direct.

In the Preface of his groundbreaking work *Begriffsschrift*, Frege writes:

Leibniz's ... idea of a universal characteristic, of a *calculus philosophicus* or *ratiocinator*, was so gigantic that the attempt to realize it could not go beyond the bare preliminaries. ... But, even if this worthy goal cannot be reached in one leap, we need not despair of a slow, step-by-step approximation. ... It is possible to view the notation of arithmetic, geometry, and chemistry as realizations, for specific fields, of Leibniz's idea. The *Begriffsschrift* proposed here adds a new field to these, indeed the central one, which borders on all the others. If we take our departure from there, we can ... proceed to fill in the gaps in the existing formula languages, connecting their hitherto separated fields into a single domain,

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and extend this domain to include fields that up to now have lacked such a language. I am confident that my *Begriffsschrift* can be successfully used wherever special value must be placed on the validity of proofs, as for example when the foundations of the differential and integral calculus are established. **It seems to me to be easier still to extend the domain of this formula language to include geometry. We would have only to add a few signs for the intuitive relations that occur there. In this way we would obtain a kind of *analysis situs*.** — G. Frege, *Begriffsschrift*, 1879 (emphasis added) Frege (1879)

“Analysis situs” is of course the nineteenth-century term for what we now call topology.

*Homotopy type theory* is a new field devoted to a recently discovered connection between Logic and Topology—more specifically, between constructive type theory, which was originally invented as a constructive foundation for mathematics and now has many applications in the theory of programming languages and formal proof verification, and homotopy theory, a branch of algebraic topology devoted to the study of continuous deformations of geometric spaces and mappings.

The basis of homotopy type theory is an interpretation of the system of intensional type theory into abstract homotopy theory. As a result of this interpretation, one can construct new kinds of models of constructive logic and study that system semantically, e.g. proving consistency and independence results. Conversely, constructive type theory can also be used as a formal calculus to reason about abstract homotopy. This is particularly interesting in light of the fact that the type theory used underlies several computer proof assistants, such as Coq and Agda; this allows one to use those systems to reason formally about homotopy theory and fully verify the correctness of definitions and proofs using these computer proof systems. Potentially, this could provide a useful tool for mathematicians working in fields like homotopy theory and higher category theory. Finally, new logical principles and constructions based on homotopical and higher categorical intuitions can be added to the system, providing a way to formalize many classical spaces and sophisticated mathematical constructions. Examples include the so-called *higher inductive types* and the *univalence axiom*. Our focus here will be mainly on the former, rather than the latter.

An application and expansion of the original idea of homotopy type theory is the ambitious program of *univalent foundations*, which was later proposed by Voevodsky. This is a program for new foundations of mathematics, based on homotopy type theory and intended to capture a very broad range of mathematics (I am reluctant to use the phrase “All of Mathematics”, but there is nothing in particular that could not, in principle, be done). The new univalence axiom, which roughly speaking implies that isomorphic structures can be identified, and the general point of view that it promotes sharpen the expressiveness of the system and make it more powerful, so that new concepts can be isolated and new constructions can be carried out, and others that were previously ill-behaved (such as quotients) can be better controlled. The system is not only more expressive and powerful than previous type- and set-theoretic systems of foundations; it also has two further, distinct novelties: it is still amenable to computer formalizations, and it captures

a conception of mathematics that is distinctly “structural”. These two seemingly unrelated aspects, one practical, the other philosophical, are in fact connected in a rather subtle way. The structural character of the system, which the univalence axiom requires and indeed strengthens, permits the use of a new “synthetic” style of foundational axiomatics which is quite different from conventional axiomatic foundations. One might call the conventional, set-theoretic style of foundations an “analytic” (or perhaps “bottom-up”) approach, which “analyses” mathematical objects into constituent *material* (e.g. sets or numbers), or at least constructs appropriate “surrogate objects” from such material—think of real numbers as Dedekind cuts of rationals. By contrast, the “synthetic” (or “top-down”) approach permitted by univalent foundations is based on describing the fundamental *structure* of mathematical objects in terms of their universal properties, which in type theory are given by rules of inference determining directly how the new objects map to and from all other ones.

This fundamental shift in foundational methodology has the practical effect of simplifying and shortening many proofs by taking advantage of a more axiomatic approach, as opposed to the more laborious analytic constructions. Indeed, in a relatively short time, a large amount of classical mathematics has already been developed in this new system: basic homotopy theory, category theory, real analysis, the cumulative hierarchy of set theory, and many other topics. The proofs of some very sophisticated, high-level theorems have now been fully formalized and verified by computer proof assistants—a foundational achievement that would be very difficult to match using conventional, “analytic” style foundational methods.

Indeed, this combination of a synthetic foundational methodology and a powerful computational implementation has the potential to give new life, and a new twist, to Frege’s idea of reducing mathematics to a purely formal calculus. Explicit formalizations that were once too tedious or complicated to be done by hand can now be accomplished in practice with a combination of synthetic methods and computer assistance. This new formal reduction of mathematics (now including “analysis situs”) raises again the epistemological question of whether, and in what sense, the underlying formal system is purely “logical”, and what this means about mathematics and the nature of a priori knowledge. That is a question of significant philosophical interest, but it is perhaps better pursued independently, once the mathematical issues related to the formalization itself are more settled (but see Awodey 2018b).

### 3.1 Type Theory

In its current form, constructive type theory is the result of contributions made by several different people, working both independently and in collaboration. Without wanting to give an exhaustive history (for one such, see Kamareddine et al. 2004), it

may be said that essential early contributions were made by H. Curry, W. Howard, F.W. Lawvere, P. Martin-Löf, D.S. Scott, and W.W. Tait.

Informally, the basic system consists of the following ingredients:

- **Types:**  $X, Y, \dots, A \times B, A \rightarrow B, \dots$ , including both primitive types and type-forming operations, which construct new types from given ones, such as the product type  $A \times B$  and the function type  $A \rightarrow B$ .
- **Terms:**  $a : A, b : B, \dots$ , including variables  $x : A$  for all types, primitive terms  $b : B$ , and term-forming operations like  $\langle a, b \rangle : A \times B$  and  $\lambda x.b(x) : A \rightarrow B$  associated to the type-forming operations.

One essential novelty is the use of so-called *dependent types*, which are regarded as “parametrized” types or *type families indexed over a type*.

- **Dependent Types:**  $x : A \vdash B(x)$  means that  $B(x)$  is a type for each  $x : A$ , and thus it can be thought of as a function from  $A$  to types. Moreover, one can have iterated dependencies, such as:

$$\begin{aligned} &x : A \vdash B(x) \\ &x : A, y : B(x) \vdash C(x, y) \\ &x : A, y : B(x), z : C(x, y) \vdash D(x, y, z) \\ &\text{etc.} \end{aligned}$$

- **Dependent Type and Term Constructors:** There are special type constructors for dependent types, such as the sum  $\sum_{x:A} B(x)$  and product  $\prod_{x:A} B(x)$  operations. Associated to these are term constructors that act on dependent terms  $x : A \vdash b(x) : B(x)$ , such as  $\lambda x.b(x) : \prod_{x:A} B(x)$ .
- **Equations:** As in an algebraic theory, there are then equations  $s = t : A$  between terms of the same type. Certain distinguished equations between terms constructed using the basic constructors, such as  $(\lambda x.b(x))(a) = b(a) : B(a)$ , are designated as primitive *computation rules*.

The entire system of constructive type theory is a formal calculus of such typed terms and equations, usually presented as a deductive system by formal rules of inference. For one modern presentation, see the appendix to *The Univalent Foundations Program* (2013). This style of type theory is somewhat different from the Frege-Russell style systems of which it is a descendant. It was originally intended as a foundation for *constructive* mathematics, and it has a distinctly “predicative” character—for instance, it is usually regarded as open-ended with respect to the addition of new type- and term-forming operations, such as universes, so that one does not make use of the notion of “all types” in the way that set-theory admits statements about “all sets” via its first-order logical formulation. Type theory is now used widely in the theory of programming languages and as the basis of computerized proof systems, in virtue of its good computational properties.

### 3.1.1 Propositions as Types

The system of type theory has a curious dual interpretation:

- On the one hand, there is the interpretation as **mathematical** objects: the types are some sort of constructive “sets”, and the terms are the “elements” of these sets, which are being built up according to the stated rules of construction.
- But there is also a second, **logical** interpretation: the types are “propositions” about mathematical objects, and their terms are “proofs” of the corresponding propositions, which are being derived in a deductive system.

This is known as the *Curry-Howard correspondence*, and it can be displayed as follows:

0	1	$A + B$	$A \times B$	$A \rightarrow B$	$\sum_{x:A} B(x)$	$\prod_{x:A} B(x)$
$\perp$	$\top$	$A \vee B$	$A \wedge B$	$A \Rightarrow B$	$\exists_{x:A} B(x)$	$\forall_{x:A} B(x)$

For instance, regarded as propositions,  $A$  and  $B$  have a conjunction  $A \wedge B$ , a proof of which corresponds to a pair of proofs  $a$  of  $A$  and  $b$  of  $B$  (via the  $\wedge$ -introduction and elimination rules), and so the terms of  $A \wedge B$ , regarded as a type, are just pairs  $\langle a, b \rangle : A \times B$  where  $a : A$  and  $b : B$ . Similarly, a proof of the implication  $A \Rightarrow B$  is a function  $f$  that, when applied to a proof  $a : A$  returns a proof  $f(a) : B$  (*modus ponens*), and so  $f : A \rightarrow B$ . The interpretation of the existential quantifier  $\exists_{x:A} B(x)$  mixes the two points of view: a proof of  $\exists_{x:A} B(x)$  consists of a term  $a : A$  and a proof  $b : B(a)$ ; so in particular, when it can be proved, one always has an instance  $a$  of an existential statement. In classical logic, by contrast, one can use “proof by contradiction” to establish an existential statement without knowing an instance of it, but this is not possible here. This gives the system a distinctly constructive character (which can be specified in terms of certain good proof-theoretic properties). This is one reason it is useful for computational applications.

### 3.1.2 Identity Types

Under the logical interpretation above we now have:

- **propositional logic:**  $0, 1, A + B, A \times B, A \rightarrow B$ ,
- **predicate logic:**  $B(x), C(x, y)$ , with the **quantifiers**  $\prod$  and  $\sum$ .

It would therefore be natural to add a primitive relation representing equality of terms  $x = y$  as a *type*. On the logical side, this would represent the proposition “ $x$  is identical to  $y$ ”. But what would it be *mathematically*? How are we to continue the above table:

0	1	$A + B$	$A \times B$	$A \rightarrow B$	$\sum_{x:A} B(x)$	$\prod_{x:A} B(x)$	?
$\perp$	$\top$	$A \vee B$	$A \wedge B$	$A \Rightarrow B$	$\exists_{x:A} B(x)$	$\forall_{x:A} B(x)$	$x = y$

We shall add to the system a new, primitive type of *identity* between any terms  $a, b : A$  of the same type  $A$ :

$$\text{Id}_A(a, b).$$

The *mathematical* interpretation of this identity type is what leads to the homotopical interpretation of type theory. Before we can explain that, however, we must first consider the rules for the identity types (due to Per Martin-Löf, see e.g. Martin-Löf (1984).

The **introduction** rule says that  $a : A$  is always identical to itself:

$$r(a) : \text{Id}_A(a, a)$$

The **elimination** rule is a form of what may be called “Lawvere’s Law”<sup>1</sup>:

$$\frac{c : \text{Id}_A(a, b) \quad x:A \vdash d(x) : R(x, x, r(x))}{J(a, b, c, d) : R(a, b, c)}$$

That may look a bit forbidding when seen for the first time. Informally, it is saying something like, for any relation  $R(x, y)$ :

$$a = b \ \& \ \forall x R(x, x) \ \Rightarrow \ R(a, b).$$

Omitting the proof terms, this characterizes identity by saying that it is the *least* (or better: *initial*) reflexive relation.

The rules for identity types are such that if  $a$  and  $b$  are *syntactically equal* as terms,  $a = b : A$ , then they are also *identical* in the sense that there is a term  $p : \text{Id}_A(a, b)$ . But the converse is not true: distinct terms  $a \neq b$  may still be propositionally identical  $p : \text{Id}_A(a, b)$ . This is a kind of *intensionality* in the system, in that terms that are identified by the propositions of the system may nonetheless remain distinct syntactically, e.g. different polynomial expressions may determine the same function. Allowing such syntactic distinctions to remain (rather than including a “reflection rule” of the form  $p : \text{Id}_A(a, b) \Rightarrow a = b$ , as is done in “extensional type theory”), gives the system its good computational and proof-theoretic properties. It also gives rise to a structure of great combinatorial complexity.

Although only the syntactically equal terms  $a = b : A$  are fully interchangeable everywhere, propositionally identical ones  $p : \text{Id}_A(a, b)$  are still interchangeable

<sup>1</sup>See Lawvere (1970) for a closely related principle.

*salva veritate* in the following sense: assume we are given a type family  $x : A \vdash B(x)$  (regarded, if you like, as a “predicate” on  $A$ ), an identity  $p : \text{Id}_A(a, b)$  in  $A$ , and a term  $u : B(a)$  (a “proof of  $B(a)$ ”). Then consider the following derivation, using the identity rules.

$$\frac{u : B(a) \quad \frac{\frac{x : A \vdash B(x)}{x : A, y : B(x) \vdash y : B(x)} \quad x : A \vdash \lambda y. y : B(x) \rightarrow B(x)}{p : \text{Id}_A(a, b) \quad x : A \vdash \lambda y. y : B(x) \rightarrow B(x)} \quad p_* : B(a) \rightarrow B(b)}{p_* u : B(b)}$$

Here  $p_* = \mathbf{J}(a, b, p, \lambda y. y)$ . The resulting term  $p_* u : B(b)$  (which is a derived “proof of  $B(b)$ ”) is called the *transport* of  $u$  along  $p$ . Informally, this just says

$$a = b \ \& \ B(a) \Rightarrow B(b),$$

i.e. that a type family over  $A$  must respect the identity relation on  $A$ . As we shall see below, the homotopy interpretation provides a different view of transport; namely, it corresponds to the familiar *lifting property* used in the definition of a “fibration of spaces”:

$$\begin{array}{ccc} B & u \dashrightarrow & p_* u \\ \downarrow & & \\ A & a \xrightarrow[p]{} & b \end{array} \quad (3.1)$$

## 3.2 The Homotopy Interpretation

Given any terms  $a, b : A$ , we can form the identity type  $\text{Id}_A(a, b)$  and then consider its terms, if any, say  $p, q : \text{Id}_A(a, b)$ . Logically,  $p$  and  $q$  are “proofs” that  $a$  and  $b$  are identical, or more abstractly, “reasons” or “evidence” that this is so. Can  $p$  and  $q$  be different? It was once thought that such identity proofs might themselves always be identical, in the sense that there should always be some  $\alpha : \text{Id}_{\text{Id}_A(a, b)}(p, q)$ ; however, as it turns out, this need not be so. Indeed, there may be many distinct (i.e. non-identical) terms of an identity type, or none at all. Understanding the structure of such iterated identity types is one result of the homotopical interpretation.

Suppose we have terms of ascending identity types:

$$\begin{aligned} a, b & : A \\ p, q & : \text{Id}_A(a, b) \end{aligned}$$

$$\begin{aligned} \alpha, \beta &: \mathbf{Id}_{\mathbf{Id}_A(a,b)}(p, q) \\ \dots &: \mathbf{Id}_{\mathbf{Id}_{\mathbf{Id}} \dots}(\dots) \end{aligned}$$

Then we can consider the following informal interpretation:

Types	$\rightsquigarrow$	Topological spaces
Terms	$\rightsquigarrow$	Continuous maps
$a : A$	$\rightsquigarrow$	Points $a \in A$
$p : \mathbf{Id}_A(a, b)$	$\rightsquigarrow$	Paths $p$ from $a$ to $b$
$\alpha : \mathbf{Id}_{\mathbf{Id}_A(a,b)}(p, q)$	$\rightsquigarrow$	Homotopies $\alpha$ from $p$ to $q$
	$\vdots$	

So for instance  $A$  may be a space with points  $a$  and  $b$ , and then an identity term  $p : \mathbf{Id}_A(a, b)$  is interpreted as a path in  $A$  from  $a$  to  $b$ , i.e. a continuous function  $p : [0, 1] \rightarrow A$  with  $p0 = a$  and  $p1 = b$ . If  $q : \mathbf{Id}_A(a, b)$  is another such path from  $a$  to  $b$ , a higher identity term  $\alpha : \mathbf{Id}_{\mathbf{Id}_A(a,b)}(p, q)$  is then interpreted as a homotopy from  $p$  to  $q$ , i.e. a “continuous deformation” of  $p$  into  $q$ , described formally as a continuous function  $\alpha : [0, 1] \times [0, 1] \rightarrow A$  with the expected behavior on the boundary of the square  $[0, 1] \times [0, 1]$ . Higher identity terms are likewise interpreted as higher homotopies.

Note that, depending on the choice of space  $A$  and points  $a, b \in A$  and paths  $p, q$ , it may be that there are no homotopies from  $p$  to  $q$  because, for example, those paths may go around a hole in  $A$  in two different ways, so that there is no continuous way to deform one into the other. Or there may be many different homotopies between them, for instance wrapping different numbers of times around the surface of a ball. Depending on the space, this can become quite a complicated structure of paths, deformations, higher-dimensional deformations, etc.—indeed, the investigation of this structure is what homotopy theory is all about.

One could say that the basic idea of the homotopy interpretation is just to extend the well-known topological interpretation of the *simply-typed*  $\lambda$ -calculus (Awodey 2000; Awodey and Butz 2000) (which interprets types as spaces and terms as continuous functions) to the *dependently typed*  $\lambda$ -calculus with  $\mathbf{Id}$ -types. The essential new idea is then simply this:

An identity term  $p : \mathbf{Id}_A(a, b)$  is a path in the space  $A$  from the point  $a$  to the point  $b$ .

Everything else essentially follows from this one idea: the dependent types  $x : A \vdash B(x)$  are then forced by the rules of the type theory to be interpreted as *fibrations*, in the topological sense, since one can show from the rules for identity types that the associated map  $B \rightarrow A$  of spaces must have the *lifting property* indicated in diagram (3.1) above (a slightly more intricate example shows that one can “lift” not only the endpoint, but also the entire path, and even a homotopy). The total  $\mathbf{Id}$ -