TAMALIKA CHAIRA

FUZZY SET AND ITS EXTENSION

THE INTUITIONISTIC FUZZY SET

WILEY

Fuzzy Set and Its Extension

Fuzzy Set and Its Extension

The Intuitionistic Fuzzy Set

Tamalika Chaira
Midnapore (West)
West Bengal, India

WILEY

This edition first published 2019

Registered Office(s)
John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, USA

Editorial Office
111 River Street, Hoboken, NJ 07030, USA

For details of our global editorial offices, customer services, and more information about Wiley products visit us at www.wiley.com.

Wiley also publishes its books in a variety of electronic formats and by print-on-demand. Some content that appears in standard print versions of this book may not be available in other formats.

Library of Congress Cataloging-in-Publication data applied for

ISBN: 9781119544197

Cover Design: Wiley
Cover Image: © kentoh / Shutterstock

Set in 10/12pt Warnock by SPi Global, Pondicherry, India

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

To my parents

Barid Baran Chaira

and

Puspa Chaira

Contents

Preface

Since Lofti A. Zadeh introduced fuzzy set theory about 50 years ago, i.e. in 1965, theory of fuzzy sets has evolved in many directions and has received more attention from many researchers. Applications of the theory can be found ranging from pattern recognition, control system, image processing, decision making, operations research, robotics, and management.

This book discusses on connections between fuzzy set and crisp set, fuzzy relations, operations on fuzzy sets, various aggregation operators using fuzzy sets, fuzzy numbers, arithmetic operations on fuzzy numbers, fuzzy integrals, fuzzy matrices and determinants, and fuzzy groups. Applications on decision making and image processing is also given.

Apart from fuzzy set, intuitionistic fuzzy set is also discussed in this book. Since its inception by K. Atanassov in 1985, intuitionistic fuzzy set theory has also received attention but to limited number of researchers as compared to fuzzy set. Though its use in application is not as comparable as that of fuzzy set, but still research studies are carried out in the areas that use fuzzy set. In intuitionistic fuzzy set, computational complexity is more as two types of uncertainties are used. But, for obtaining better result, where uncertainty present is more, especially in diagnosis of medical images, accurate result is very much important compromising the computational complexity. So, researchers try to use it on real-time application.

The book discusses the basics of intuitionistic fuzzy set, intuitionistic fuzzy relations, operations on intuitionistic fuzzy sets, various intuitionistic fuzzy aggregation operators, intuitionistic fuzzy numbers, arithmetic operations on intuitionistic fuzzy numbers, intuitionistic fuzzy integrals, and intuitionistic fuzzy matrices. Also, application in decision making and image processing using intuitionistic fuzzy set is also given.

This book is an attempt to unify both fuzzy/intuitionistic fuzzy set and their existing work in application. The primary goal of this book is to help the readers to know the mathematics of both fuzzy set and intuitionistic fuzzy set so that with both these concepts, they can use either fuzzy/intuitionistic fuzzy set in their applications.

Finally, I would like to acknowledge the authors of the papers that have been referred in the book. I acknowledge my beloved daughter, Shruti De, for giving the title of the book. I acknowledge my parents for their continuous support while writing this book. I am also indebted to John Wiley & Sons, Inc. for making the publication of this book possible.

Tamalika Chaira

Organization of the Book

The book contains 10 chapters. Each chapter begins with an introduction, theory, and also several examples that will help the readers to understand the chapters in a better way. *Chapter 1* starts with preliminaries of fuzzy sets and relations. Different types of membership function, composition of fuzzy relation, and fuzzy binary relation that includes symmetric, reflexive, transitive, and equivalent relations are explained with examples. Similar to fuzzy set, intuitionistic fuzzy sets, operations, relations, and compositions are also explained with examples.

Chapter 2 deals with fuzzy numbers. Zadeh's extension principle is explained that states how an image of a fuzzy subset is formed using a function. Using the extension principle, arithmetic operations on fuzzy numbers are explained. Fuzzy numbers with α-cut, operations on fuzzy numbers, and L–R representation of fuzzy numbers are explained with examples. Intuitionistic fuzzy numbers such as triangular and trapezoidal fuzzy numbers, along with operations, are also explained with examples.

Chapter 3 details fuzzy similarity measures and measures of fuzziness. Similarity measures on fuzzy sets and fuzzy numbers are discussed. More emphasis is given on similarity measure on fuzzy numbers. Different types of similarity measures based on the center of gravity, area, perimeter, and graded mean integration of fuzzy numbers are discussed in detail. Measures of fuzziness and different types of entropy are also explained. Intuitionistic fuzzy similarity measures, distance measures, and entropy are also discussed.

Chapter 4 outlines fuzzy measures and fuzzy integrals. Definition and properties of fuzzy measures are discussed. Sugeno measure is a special type of fuzzy measure is discussed with examples. Other types of fuzzy measures such as belief measure, possibility measure, plausibility measure, and necessity measure are discussed. Fuzzy integrals such as Choquet and Sugeno integrals are explained with figures and example on decision making problem is also provided. Intuitionistic fuzzy Choquet integral is also discussed.

Chapter 5 discusses on fuzzy operators where different types of fuzzy operators are used. Fuzzy algebraic operations such as complement, sum, difference, bounded sum, bounded difference, union, and intersection are explained with examples. Fuzzy set theoretic operations that include fuzzy triangular norms (t-norms) and triangular conorms (t-conorms) are explained. Triangular norms suggested by different authors are discussed. Fuzzy/intuitionistic fuzzy aggregation operators that combine different pieces of information into a single object in a same set are explained. Different types of aggregation operators such as fuzzy/intuitionistic fuzzy generalized ordered weighted averaging, hybrid averaging operator, quasi-arithmetic weighted averaging operator, fuzzy/intuitionistic fuzzy-induced generalized averaging operator, fuzzy/intuitionistic fuzzy Choquet and induced Choquet ordered aggregation operator are explained with examples on decision making.

Chapter 6 examines matrices and determinants of a fuzzy matrix. Fuzzy matrix/determinant operations are explained with properties and examples. Adjoint and determinants of a fuzzy matrix, and inverse of a fuzzy matrix are discussed with examples. Intuitionistic fuzzy determinants and matrices are also discussed.

Chapter 7 outlines fuzzy linear equation. It is a continuation of Chapter 6 where an unknown vector is computed using general equation method and also using Cramer's rule. Finding inverse of a fuzzy matrix is discussed with examples. Fuzzy linear equation using L–R-type fuzzy numbers is also discussed with examples where left and right spread of a fuzzy number are considered.

Chapter 8 is dedicated to fuzzy subgroups. Definition of fuzzy subgroup is provided along with properties. Many examples of fuzzy subgroups are mentioned. Other types of fuzzy subgroups such as fuzzy-level subgroup and fuzzy normal subgroup are also discussed with examples. Definition of fuzzy subgroup with respect to fuzzy t-norm is also included. Product of fuzzy subgroups with respect to t-norm with propositions are explained.

Chapter 9 is based on the application on image processing. Introduction on image processing along with image enhancement, segmentation, clustering, edge detection, and morphology are explained with examples using both fuzzy and intuitionistic fuzzy set. Results on medical images for detection of abnormal lesions/clot/hemorrhage in CT scan brain image are shown.

Lastly, the book ends up with *Chapter 10* where Type-2 fuzzy set is explained. Introduction and representation of Type-2 fuzzy set is discussed. Operations on Type-2 fuzzy set along with examples are provided.

1

Fuzzy/Intuitionistic Fuzzy Set Theory

1.1 Introduction to Fuzzy Set

A classical set is normally defined as a collection of objects or elements x in $X = \{x_1, x_2, x_3, \dots, x_n\}$ that are finite. Each element or object either belongs or does not belong to a set. Most of our traditional tools, modeling, and methods are based on crisp set theory where elements are deterministic and precise. By crisp we mean, answer is "yes" or "no" rather "a little bit less or a little bit more" type. This means that the statement is either "true or false" and in mathematics it may be defined as either "0 or 1." Elements have a Boolean state of nature that means either belongs to the set or does not belong to the set. This belongingness to a set may be termed as "membership value" or the degree of belongingness. So, if an element in a set is present, then its membership value is "1" else its membership value is "0." The membership function or the degree of belongingness of an element "x" in the set is denoted by $\mu(x)$.

But in reality, this crisp theory does not follow. This deterministic and precise theory does not work. It may happen many times in day-to-day situation, when the terms like "less," "more or less," or "very high" are required. To deal with such type of situations, fuzzy set theory is used. In fuzzy set theory, instead of having precise or sharp values, gradually varying values are used. Prof. L.A. Zadeh in 1965 [1] introduced the concept of fuzzy set theory on the basis of principles of uncertainty, ambiguity, and vagueness. He suggested that in real world, classes of objects do not always have precisely defined membership values. The objects do not have a rigid demarcating boundary, i.e. either present or do not present and thus there is a gradual transition from zero to unity membership. These sets are named as fuzzy sets and the elements in these sets have membership values lying between 0 and 1, i.e. $0 \le \mu(x) \le 1$. These fuzzy sets have been extensively used in many application areas such as image processing,

Fuzzy Set and Its Extension: The Intuitionistic Fuzzy Set, First Edition. Tamalika Chaira.

pattern recognition, decision-support systems, artificial intelligence, control system, and so on to model uncertainties, imprecision, and vagueness inherent in them.

Example 1 Consider a following set X and its subset A as:

$$X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} \text{ and } A = \{x_1, x_3, x_5, x_6, x_7\}.$$

Considering the set to be a crisp set, subset A may be represented using membership function (either 0 or 1):

$$\mu_A(x_1) = 1,\ \mu_A(x_3) = 1,\ \mu_A(x_5) = 1,\ \mu_A(x_6) = 1,\ \mu_A(x_7) = 1.$$

Thus, A may be written as

$$A = \{(x_1, 1), (x_2, 0), (x_3, 1), (x_4, 0), (x_5, 1), (x_6, 1), (x_7, 1)\}.$$

Now, let us imagine a situation where the membership degrees of the elements in a set take any value in the interval [0,1]. That means each element in the set has a fractional membership, depending on the degree of its presence in the set that may be partially, moderately, or fully present. Elements having partial membership in the set have membership values that lie between $0 < \mu_A(x_i) < 1$ and elements with full membership have membership value $\mu_A(x_i) = 1$. This membership concept may also be represented as:

$$\mathrm{A} = \{(x_1/0.7), (x_2/0.8), (x_3/0.9), (x_4/1), (x_5/0.7), (x_6/0.6), (x_7/0.3)\},$$

where x_i is an element of the set A, followed by the membership value of the element x_i that lies between 0 and 1. It is a measure of the degree of belongingness of the element in the set.

An example relating to person's height is shown.

It is observed that in crisp set theory, there is a sharp transition of height. If the membership degree is 0, i.e. $\mu = 0$, then the person is not tall and if the membership degree is $\mu = 1$, then the person is tall. So, membership degree does not have any role, if a person's height is 6 ft or 7 ft. They are simply both tall. But there is a significant difference in the heights. So, crisp set works better in binary mathematics but not in real-world situation.

Fuzzy approach to the set leads to a better approximation of a person's height as shown in Figure 1.1b. The figure shows a continuous function. Persons with different heights do not have same membership degree. So, if a person whose membership degree is 0.4 whereas a person whose membership degree of 0.55, then their heights are different. That means the second person is little bit taller than the first person. If another person has membership degree 0.95, then this person is considered to be significantly tall.

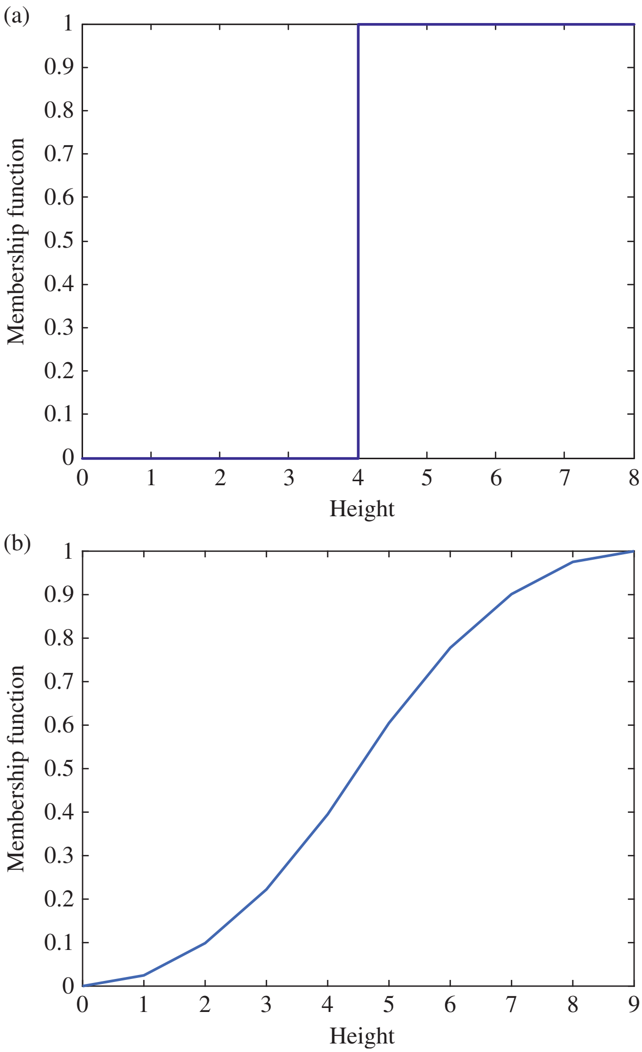


Figure 1.1 Crisp set versus fuzzy set. (a) Crisp set. (b) Fuzzy set.

1.2 Mathematical Representation of Fuzzy Sets

If X be a collection of objects denoted by x, then a fuzzy set A in X is defined as:

$$A = \{(x, \mu_A(x)) \mid x \in X\},$$

where $\mu_A(x)$ is the degree of membership of x in A. The degree of membership lies between [0,1]. Membership degree "zero" means no presence and membership degree "1" means full presence of that element in a set. In between values means partial presence of that element in a set.

Example 2 Suppose there are 10 agricultural fields and they are classified based on their fertility. Let $X = \{1,2,3,4,5,6,7,8,9,10\}$ be the agricultural lands. Now a fuzzy set for "fertile land" may be described as:

$$A = \{(1,0.3),(2,0.4),(3,0.5),(4,0.7),(6,1.0),(7,0.8),(8,0.6),(10,0.4)\}.$$

It is observed that the fertility of lands 5 and 9 is 0, i.e. infertile as their membership is "0" and land 6 is the most fertile as the membership degree is 1. The fertility of other lands depends on the membership degree. The higher the membership value, the more is the fertility.

Example 3 An integer close to "8" is written as:

$$A = \{(5,0.2),(6,0.6),(7,0.8),(8,1),(9,0.8),(10,0.7),(11,0.5)\}.$$

So, the elements around "8" has more membership value than those elements are away from "8" and "8" has membership degree 1.

Here, we present a definition of convex fuzzy set.

Convex fuzzy set – A fuzzy set A is said to be convex if [2]

$$\mu_A[\lambda x_1 + (1-\lambda)x_2] \geq \min\{\mu_A(x_1),\mu_A(x_2)\}, x_1,x_2 \in A, \lambda \in [0,1].$$

Figure 1.2 shows an example of a convex and non-convex fuzzy set.

Suppose for a particular value of $\lambda \in [0,1]$, let $\lambda x_1 + (1 - \lambda)x_2 = x_3$, which lies between x_1 and x_2. Then the condition $\mu_A(x_3) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}$ fulfils for the convex fuzzy set but the condition is not fulfilled for the non-convex fuzzy set.

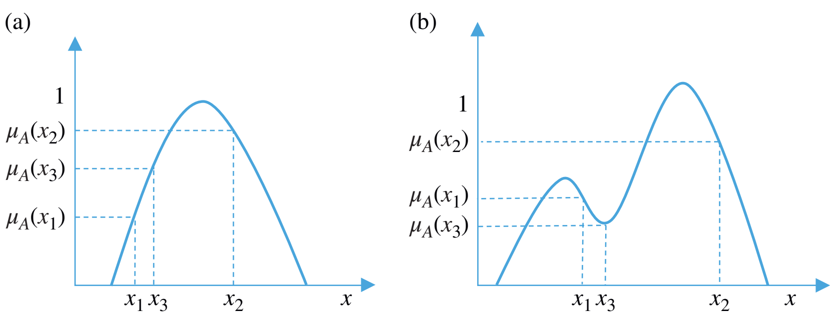


Figure 1.2 (a) Convex fuzzy set. (b) Non-convex fuzzy set.

There are few terms associated with the membership function. These are explained below.

i) Core – The core of a membership function of a fuzzy set A is the region characterized by complete and full membership in a fuzzy set A.
   The core thus consists of only those elements whose membership values $\mu_A(x) = 1$.
ii) Support – The support of a membership function of a fuzzy set A is the region characterized by nonzero membership. Thus, support consists of those elements whose membership values are greater than 0, i.e. $\mu_A(x) > 0$.
iii) The boundary of membership function comprises the region where the elements possess nonzero membership but not full membership, i.e. $0 < \mu_A(x) < 1$.
iv) Alpha-level set of a fuzzy set A with $\mu_A(x)$ as the membership function is a crisp set of all elements to a degree α is given as:

$$A_\alpha = \{x \in A \mid \mu_A(x) \geq \alpha\}.$$

v) Strong alpha-level set, which is same as the alpha-level set, is a crisp set that consists of all elements to a degree greater than α is given as:

$$A'_\alpha = \{x \in A \mid \mu_A(x) > \alpha\}.$$

Example 4

$$A = \{(1,0.3),(2,0.4),(3,0.5),(4,0.7),(6,1.0),(7,0.8),(8,0.6),(10,0.4)\}$$
$$A_{0.5} = \{(3,0.5),(4,0.7),(6,1.0),(7,0.8),(8,0.6)\},$$
$$A_{0.4} = \{(2,0.4),(3,0.5),(4,0.7),(6,1.0),(7,0.8),(8,0.6),(10,0.4)\}.$$

Strong α-level set for α =0.7 is:

$$A'_{0.7} = \{(6,1.0),(7,0.8)\}.$$

Example 5 Consider a fuzzy set A defined on the interval $x = [0,5]$ of integers by a membership function $\mu(x) = \frac{x}{x+3}$. Find the α-level set corresponding to $\alpha = 0.3$.

Solution

First, we will compute the membership function in the interval [0,5].

$$\mu(0) = \frac{0}{0+3} = 0, \quad \mu(1) = \frac{1}{1+3} = 0.25,$$
$$\mu(2) = 0.4, \quad \mu(3) = 0.5, \quad \mu(4) = 0.57, \quad \mu(5) = 0.625.$$

So, the α-level set corresponding to $\alpha = 0.3$ is $A_{0.3} = [2,3,4,5]$.

1) Height of A is the largest membership grade obtained by any element in A.
2) A fuzzy set is called normal if its height is 1, i.e. if there is at least one point with $\mu_A(x) = 1$. Otherwise, it is called subnormal.

1.3 Membership Function

In a fuzzy set, the degree of membership of an element signifies the extent to which the element belongs to a fuzzy set, i.e. there is a gradation of membership value of each element in a set. A membership function is a curve that defines how each point in the input space is mapped to a membership value between 0 and 1. There are different types of membership functions that may be viewed as mappings of diverse human choices to an interval [0,1]. To name a few, some membership functions are defined as follows:

i) triangular membership function (Figure 1.3)

$$\mu(x) = \begin{cases} 0, & a \le x \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ \dfrac{c-x}{c-b}, & b \le x \le c \end{cases}$$

ii) trapezoidal membership function (Figure 1.4)

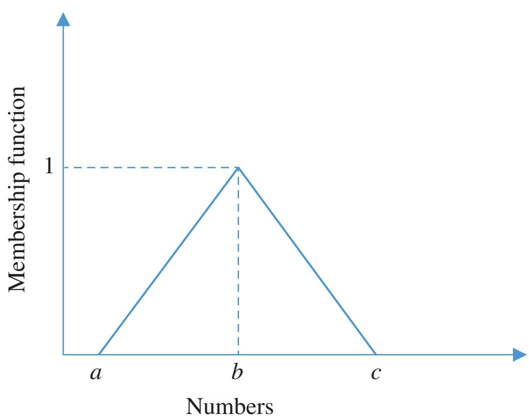
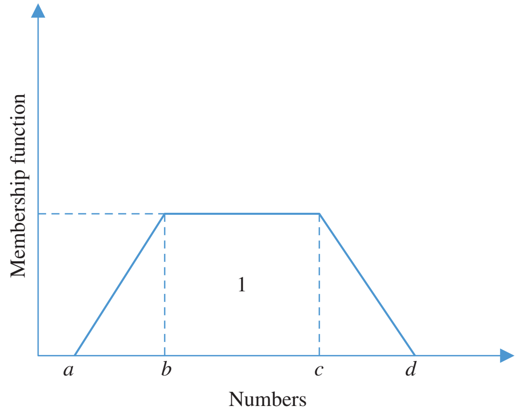


Figure 1.3 Triangular membership function.

Figure 1.4 Trapezoidal membership function.



$$\mu(x) = \begin{cases} 0, & a \le x \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \dfrac{d-x}{d-c}, & c \le x \le d \\ 0 & x > d \end{cases}$$

a, b, c, d are the four parameters, where a, d are at the feet of the trapezium and b, c lie at the shoulders. It is to be noted that if $b = c$, then the trapezoid becomes a triangle.

iii) Gaussian membership function

$$\mu(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-m)^2}{2\sigma^2} \right),$$

where σ, m are the standard deviation and mean, respectively.

iv) S membership function – This membership function is given by Zadeh [3].

$$\mu(x) = \begin{cases} 0, & x < a \\ 2 \cdot \dfrac{(x-a)}{(c-a)^2}, & a < x < k\beta \\ 1 - 2 \cdot \left[\dfrac{(x-a)}{(c-a)} \right]^2, & k\beta < x < c \\ 1, & x > c \end{cases},$$

where $0 < k < 2$, β is a threshold. Generally, $\beta = 0.5x_{max}$ is chosen and x_{min} and x_{max} are the minimum and maximum values of the elements in the set. In most cases, we select $a = x_{min}$, $c = x_{max}$.

Many authors write $k\beta = \dfrac{a+c}{2}$.

v) Membership function from restricted equivalence function [4] – A function $REF: [0,1]^2 \rightarrow [0,1]$ is called restricted equivalence function if it satisfies the following conditions:

i) $REF(x,y) = REF(y,x)$ for all $x, y \in [0,1]$,
ii) $REF(x,y) = 1$ iff $x = y$,
iii) $REF(x,y) = 0$, if and only if $x = 1, y = 0$ or $x = 0, y = 1$,
iv) $REF(x,y) = REF(c(x), c(y))$ for all $x, y \in [0,1]$, c is a strong negation, where $c: [0,1] \rightarrow [0,1]$ is a negation that satisfies the following properties [4]:
   a) $c(0) = 1, c(1) = 0$,
   b) $c(x) < c(y)$, iff $x \geq y$,
   c) $c(c(x)) = x, \forall x \in [0,1]$.

v) For all $x, y, z \in [0,1]$, if $x \leq y \leq z$, then $REF(x,y) \geq REF(x,z)$ and $REF(y,z) \geq REF(x,z)$.

Bustince et al. [4] defined a restricted equivalence function using automorphism which is as follows:
Automorphism in a unit interval $[a,b]$ is a continuously increasing function $\varphi: [a,b] \longrightarrow [a,b]$ with boundary condition $\varphi(a) = a$, $\varphi(b) = b$.
If φ_1 and φ_2 are two automorphisms in a unit interval, then

$$REF(x,y) = \varphi_1^{-1}(1 - |\varphi_2(x) - \varphi_2(y)|) \tag{1.1}$$

is a restricted equivalence function
with $c(x) = \varphi_2^{-1}(1 - \varphi_2(x))$.
If $\varphi_1(x) = \varphi_2(x) = x$,

$$\text{then } REF(x,y) = \varphi_1^{-1}(1 - |x - y|) = 1 - |x - y|. \tag{1.2}$$

Let c be a strong negation and an equilibrium point of fuzzy negation is a value such that
$c(e) = e$ which is obtained as follows:

For $\varphi_2(x) = x$, $c(x) = 1 - x$, then $c(0) = 1$, $c(1) = 0$ and $c(e) = 1 - e$.
If we choose $e = 0.5$, then $c(0.5) = 1 - 0.5 = 0.5$. This implies $c(e) = e$.
For finding the membership function, consider a function $F: [0,1] \rightarrow [e,1]$ such that
$F(x) = 1$ iff $x = 0$,
$F(x) = e$ iff $x = 1$ and $F(x)$ is nonincreasing function.

Then the membership function is defined as [5]:

$$\mu(x) = F(c(REF(x,y))). \tag{1.3}$$

$$\text{Let the function } F(x) = 1 - (1 - e)x \tag{1.4}$$

with $F(0) = 1$, $F(1) = e$.

Substituting Eq. (1.4) in Eq. (1.3), the membership function is written as:

$$\mu(x) = 1-(1-e)c(REF(x,y)). \tag{1.5}$$

If $c(x) = 1 - x$ for all $x \in [0,1]$ and $REF(A,B) = 1 - |x - y|$ and $e = 0.5$, then

$$\mu(x) = 1-0.5\cdot c(1-|x-y|) = 1-0.5|x-y|.$$

d) Chaira [6] defined another type of membership function using restricted equivalence function.

Let $\varphi_2(x) = x$. From the definition of restricted equivalence function, we know:

$$REF(x,y) = {\varphi_1}^{-1}(1-|\varphi_2(x)-\varphi_2(y)|) = {\varphi_1}^{-1}(1-|x-y|),$$

so $REF(x,y) = {\varphi_1}^{-1}(1 - |x - y|)$.
Considering $\varphi_1(x) = \ln[x(e - 1) + 1]$, where $e = \exp(1)$ and using inverse function, we get:

$${\varphi_1}^{-1}(y) = \frac{e^y-1}{e-1}.$$

Then, by mathematical induction we get

$$REF(x,y) = {\varphi_1}^{-1}(1-|x-y|) = \frac{e^{1-|x-y|}-1}{e-1}. \tag{1.6}$$

If we define membership function $\mu : [0,1]$ as:

$$\mu(x) = REF(x,y),$$

then the membership function becomes:

$$\mu(x) = \frac{e^{1-|x-y|}-1}{e-1}. \tag{1.7}$$

e) Gamma membership function [7] – This membership function is derived from the probability density function of Gamma distribution. It is defined as:

$$f(x) = \frac{\left(\frac{x-\nu}{\beta}\right)^{\gamma-1}\exp\left(-\left(\frac{x-\nu}{\beta}\right)\right)}{\Gamma(\gamma)}, x \ge \nu; \gamma, \beta > 0,$$

where ν is a location parameter, β is a scale parameter, Γ is the gamma function, and γ is a shape parameter.

When $\nu = 0$ and $\beta = 1$, then the distribution is a standard gamma distribution

$$f(x) = \frac{(x)^{\gamma-1}\exp(-x)}{\Gamma(\gamma)};$$

if $\gamma = 1$ and $\nu \neq 0$, then $f(x) = \exp(-(x-\nu))$.
This is the Gamma membership function.

1.4 Fuzzy Relations

Fuzzy relation was initially introduced by Zadeh [3] and then by Kaufmann [8]. It represents the strength of association of the elements of fuzzy sets. Fuzzy relations are mapping elements of one universe, say X, to another universe, say Y, through a Cartesian product of two universes. Relation $\mathcal{R}$ can be "x larger than y," or "x taller that y."

Let us consider two universes, X and Y. In a crisp set, a set of ordered pairs is the product set $X \times Y$:

If $X = \{x_1, x_2, x_3, x_4\}$, $\quad Y = \{y_1, y_2\}$, then

$$X \times Y = \{(x_1,y_1),(x_1,y_2),(x_2,y_1),(x_2,y_2),(x_3,y_1),(x_3,y_2),(x_4,y_1),(x_4,y_2)\}.$$

The notation of relation in crisp set can also extended to fuzzy set. In fuzzy set, the Cartesian product is given as follows:

Consider two fuzzy sets: $X = \{(3,0.5), (5,0.4), (6,0.1)\}$, $Y = \{(4,0.3), (8,0.4)\}$.

The product is the set of pairs from X and Y with minimum memberships. So,

$$\begin{aligned} X \times Y = \{&[(3,4),\min(0.5,0.3)],[(3,8),\min(0.5,0.4)],[(5,4),\min(0.4,0.3)],[(5,8),\\ &\min(0.4,0.4)],[(6,4),\min(0.1,0.3)],[(6,8),\min(0.1,0.4)]\} \\ = \{&[(3,4),0.3],[(3,8),0.4],[(5,4),0.3],[(5,8),0.4],[(6,4),0.1],[(6,8),0.1]\}. \end{aligned}$$

Fuzzy relation is studied by a number of authors, e.g. Zadeh [3], Kaufmann [8], Klir and Yaun [9], and Zimmerman [2]. Suppose P is a product set and μ is the membership grade between x and y, then fuzzy relation $\mathcal{R}$ is a subset of the product set P taking its values of μ. Similar to ordinary sets, fuzzy relations are fuzzy subsets of $X \times Y$.

The relation $\mathcal{R}$ can be defined as:

$$\mathcal{R} = \{(x,y),\mu_R(x,y)\},\ \forall (x,y) \in X \times Y,\ \forall x \in X, \forall y \in Y$$

where the membership matrix of a $m \times n$ binary fuzzy relation has the form:

$$\begin{bmatrix} \mu_{\mathcal{R}}(x_1,y_1) & \mu_{\mathcal{R}}(x_1,y_2) & \dots & \mu_{\mathcal{R}}(x_1,y_n) \\ \mu_{\mathcal{R}}(x_2,y_1) & \mu_{\mathcal{R}}(x_2,y_2) & \dots & \mu_{\mathcal{R}}(x_2,y_n) \\ & \cdot & & \\ & \cdot & & \\ & \cdot & & \\ \mu_{\mathcal{R}}(x_m,y_1) & \mu_{\mathcal{R}}(x_m,y_2) & \dots & \mu_{\mathcal{R}}(x_m,y_n) \end{bmatrix}$$

$\mu_{\mathcal{R}}(x_1,y_1)$ is the ordered pair in the product space $X \times Y$ and it denotes the membership grade between x_1 and y_1.

Let X and Y be two discrete fuzzy universes and the relation $\mathcal{R}$ is given as:

$$\mathcal{R}(X,Y)=\left\{\frac{(x_1,y_1)}{0.2},\frac{(x_1,y_2)}{0.3},\frac{(x_1,y_3)}{0.5},\frac{(x_2,y_1)}{0.3},\frac{(x_2,y_2)}{0.5},\frac{(x_2,y_3)}{0.6},\frac{(x_3,y_1)}{0.1},\frac{(x_3,y_2)}{0.3},\frac{(x_3,y_3)}{0.4}\right\}.$$

The membership matrix may be represented as:

$$\begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.5 & 0.6 \\ 0.1 & 0.3 & 0.4 \end{bmatrix}.$$

Just like fuzzy set, there are also few basic operations on fuzzy relation.

Let us consider two fuzzy relations $\mathcal{R}_1$ and $\mathcal{R}_2$.

If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two fuzzy relations in the same product space, the following operations are defined as follows:

i) The union of two fuzzy relations $\mathcal{R}_1$ and $\mathcal{R}_2$ is a new relation $\mathcal{R}_1 \cup \mathcal{R}_2$,

$$\mathcal{R}_1 \cup \mathcal{R}_2 = \int_{x\times y} \frac{\left[\mu_{\mathcal{R}_1}(x,y) \vee \mu_{\mathcal{R}_2}(x,y)\right]}{(x,y)},$$

where the membership function of $\mathcal{R}_1 \cup \mathcal{R}_2$ is given as:

$$\mu_{\mathcal{R}_1\cup\mathcal{R}_2}(x,y)=\mu_{\mathcal{R}_1}(x,y)\vee\mu_{\mathcal{R}_2}(x,y)=\max\left[\mu_{\mathcal{R}_1}(x,y),\mu_{\mathcal{R}_2}(x,y)\right].$$

The union of the two fuzzy relations is formed by taking the maximum of the two membership grades of the corresponding elements of the two matrices.

ii) The intersection of two fuzzy relations $\mathcal{R}_1$ and $\mathcal{R}_2$ is a new relation,

$$\mathcal{R}_1 \cap \mathcal{R}_2 = \int_{x\times y} \frac{\left[\mu_{\mathcal{R}_1}(x,y) \wedge \mu_{\mathcal{R}_2}(x,y)\right]}{(x,y)}, \tag{1.8}$$

where the membership function of $\mathcal{R}_1 \cap \mathcal{R}_2$ is given as:

$$\mu_{\mathcal{R}_1\cap\mathcal{R}_2}(x,y)=\mu_{\mathcal{R}_1}(x,y)\wedge\mu_{\mathcal{R}_2}(x,y)=\min\left[\mu_{\mathcal{R}_1}(x,y),\mu_{\mathcal{R}_2}(x,y)\right]. \tag{1.9}$$

The intersection of the two fuzzy relations is formed by taking the minimum of the two membership grades of corresponding elements of the two matrices.

Algebraic product of two fuzzy relations is a new fuzzy relation whose membership function is given as:

$$\mu_{\mathcal{R}_1\cdot\mathcal{R}_2}(x,y)=\mu_{\mathcal{R}_1}(x,y)\cdot\mu_{\mathcal{R}_2}(x,y). \tag{1.10}$$

Algebraic sum of two relations is a new relation whose membership function is given as:

$$\mu_{\mathcal{R}_1+\mathcal{R}_2}(x,y)=\mu_{\mathcal{R}_1}(x,y)+\mu_{\mathcal{R}_2}(x,y)-\mu_{\mathcal{R}_1}(x,y)\cdot\mu_{\mathcal{R}_2}(x,y). \tag{1.11}$$

Complement of a relation is a new relation whose membership function is given as:

$$\mu_{\bar{\mathcal{R}}_1}(x,y) = 1 - \mu_{\mathcal{R}_1}(x,y). \tag{1.12}$$

Example 6 Consider two fuzzy relations:

$$\mathcal{R}_1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 \\ 0.7 & 0.2 & 0.8 \\ 0.2 & 0.6 & 0.4 \end{bmatrix}, \mathcal{R}_2 = \begin{bmatrix} 0.3 & 0.1 & 0.2 \\ 0.5 & 0.0 & 0.3 \\ 0.7 & 0.3 & 0.7 \end{bmatrix}.$$

Union: $\mu_{\mathcal{R}_1 \cup \mathcal{R}_2}(x,y) = \begin{bmatrix} 0.3 & 0.3 & 0.5 \\ 0.7 & 0.2 & 0.8 \\ 0.7 & 0.6 & 0.7 \end{bmatrix}$,

Intersection: $\mu_{\mathcal{R}_1 \cap \mathcal{R}_2}(x,y) = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.5 & 0.0 & 0.3 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}$,

Complement: $\mu_{\bar{\mathcal{R}}_1}(x,y) = \begin{bmatrix} 0.9 & 0.7 & 0.5 \\ 0.3 & 0.8 & 0.2 \\ 0.8 & 0.4 & 0.6 \end{bmatrix}$,

$$\mu_{\mathcal{R}_1 \cdot \mathcal{R}_2}(x,y) = \begin{bmatrix} 0.03 & 0.03 & 0.1 \\ 0.35 & 0.0 & 0.24 \\ 0.14 & 0.18 & 0.28 \end{bmatrix}.$$

Fuzzy relation can be represented in different forms. Suppose the fuzzy relation is “x is taller than y.” It can be represented in (i) membership matrix, (ii) tabular form, (iii) linguistically “x is taller than y,” and (iv) taking the union of fuzzy singletons.

Example 7 For the two fuzzy relations, let us define two relations as: $\mathcal{R}_1$ = “x is larger than y” and $\mathcal{R}_2$ = “x is very close to y” where

$$\mathcal{R}_1 = \begin{bmatrix} & y_1 & y_2 & y_3 \\ x_1 & 0.9 & 0.1 & 0.1 \\ x_2 & 0.0 & 0.7 & 0.8 \\ x_3 & 1.0 & 0.1 & 0.7 \end{bmatrix}, \quad \mathcal{R}_2 = \begin{bmatrix} & y_1 & y_2 & y_3 \\ x_1 & 0.3 & 0.1 & 0.8 \\ x_2 & 0.9 & 0.2 & 0.4 \\ x_3 & 0.3 & 0.0 & 0.7 \end{bmatrix}.$$