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Jacques Sesiano

Magic Squares

Their History and Construction from Ancient Times to AD 1600

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Jacques Sesiano
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Preface

The original purpose of this book was to provide an English translation of my study on the science of magic squares in Islamic countries, published first in French (*Les carrés magiques dans les pays islamiques*, Lausanne 2004) and then in Russian (*Magicheskie kvadraty na srednevekovom vostoke*, Saint Petersburg 2014). It was to follow my *Magic squares in the tenth century* published quite recently (2017), which edited the two earliest texts then known, by Anṭākī and Būzjānī.

These two texts were of a rather different nature. Whereas the second author had produced an easily readable text, providing a didactic introduction to the science of magic squares, the first gave, between a set of propositions mostly taken from Euclid and a collection of problems, a very concise study on the construction of certain types of magic square. First of all, it was surprising to consider the difference in level between the three parts of Anṭākī's text: while the first and the third gave the impression of having been compiled without much insight by a second-rate mathematician, the middle one, on magic squares, was of an amazingly high level and obviously the result of serious mathematical studies. At that time I already expressed my conviction that Anṭākī could not be the author of it, and that he was just copying a text ultimately going back to Greek antiquity. Next, the difference in level between Anṭākī's and Būzjānī's texts was also surprising: although their authors were contemporaries, it seemed as if the second was providing the basic elements of a nascent science, whereas the first treated specialized topics certainly not accessible to general readers. Not having any explanation for this discrepancy, I could only write that "the tenth century, for the history of magic squares, gives the impression of being both a beginning and an end".

The discovery in a London manuscript of several fragments of the earliest Arabic writing on magic squares, that by T̲h̲ābit ibn Qurra (836-901), explained these incongruities. First, it appeared that Anṭākī had just copied verbatim T̲h̲ābit's text (without making any mention of him) and that Anṭākī's work could thus certainly not be considered as a tenth-century achievement. Furthermore, from T̲h̲ābit's introduction, it appeared that his source was indeed one Greek text —obviously, then, a particular study devoted to just specific aspects of magic squares, perhaps indeed a culmination in antique times although we lack any elements of comparison. Since this text does not explain the basic elements of the science of magic squares, Būzjānī's treatise is an attempt to do it.

The origin of Arabic studies on magic squares is thus Thābit's translation. Incidentally, we may note that the transmission of this Greek study was itself some kind of miracle: as Thābit tells us at the beginning of his translation, it was preserved by two manuscripts, both in rather poor condition, of which he could make sense only because what was damaged in one copy was legible in the other.

This discovery had of course an impact on the present translation since it changed drastically the earliest history of magic squares. The immediate effect was to change the title, originally to be 'Magic squares, their history and construction from early times to AD 1600'; for I could now safely replace 'from early times' by 'from ancient times'. As to my choice of the upper limit, that is because the seventeenth century saw the beginning of a new epoch in the history of magic squares, as Europe began, with Fermat, to study magic squares on new bases. In the two centuries before, Europe had known only two sets of seven magic squares of different sizes, all received from late mediaeval Latin translations of Arabic texts, where each square, being associated with one of the seven planets then known, could have, used as a talisman, the same influence as its planet. There was no single indication on the construction of these few squares, but lengthy descriptions of their use for good or evil purposes. This first European encounter with these figures and their magical use thus gave rise to their modern designation (which also accounts for the little consideration they often subsequently received). That had not always been the case: the Arabic denomination *wafq al-a'dād*, 'harmony of numbers' —corresponding perhaps to something like ἁρμονία τῶν ἀριθμῶν— did far more justice to the science of these squares.

Table of contents

Chapter I. Introduction

§1. Definitions . . . 1
§2. Categories of order . . . 4
§3. Banal transformations of ordinary magic squares . . . 6
§4. Historical outline . . . 8
§5. Main sources considered . . . 13
§6. Squares transmitted to the Latin West . . . 16

Chapter II. Ordinary magic squares

A. Squares of odd orders . . . 21

§1. First attempts . . . 21
§2. Method of diagonal placing . . . 24
1. Description . . . 24
2. Discovery of this method . . . 25
3. Another way . . . 28
4. A related method . . . 30
5. Modifying the square's aspect . . . 31
§3. A method brought from India . . . 32
§4. Separation by parity . . . 34
§5. Use of the knight's move . . . 36
§6. Principles of these methods . . . 39

B. Squares of evenly-even orders . . . 44

§7. The square of order 4 . . . 44
§8. Method of dotting . . . 45
§9. Exchange of subsquares . . . 47
§10. Generalization . . . 48
§11. Continuous filling . . . 50
§12. Principles of these methods . . . 51

§13. Filling according to parity . . . 57
§14. An older method . . . 58
§15. Crossing the quadrants . . . 60
§16. Descent by the knight's move . . . 63
1. Square of order 4 . . . 63
2. Squares of higher orders . . . 65
§17. Filling pairs of horizontal rows . . . 69
§18. Four knight's routes . . . 71
§19. Knight's shuttle . . . 73
§20. Filling by knight's and bishop's moves . . . 74
§21. Filling according to parity . . . 77
§22. Filling the subsquares of order 4 . . . 79
Particular cases . . . 85
C. Squares of evenly-odd orders . . . 88
§23. Exchanges in the natural square . . . 88
§24. An older method . . . 91
§25. Principles of these methods . . . 93
§26. Method of the cross . . . 95
§27. Construction of a border . . . 102
§28. Method of the central square . . . 104

Chapter III. Composite magic squares

A. Equal subsquares displaying different sums . . . 107
§1. Composition using squares of orders larger than 2 . . . 107
§2. Composition using squares of order 2 . . . 116
B. Equal subsquares displaying equal sums . . . 120
§3. Examples . . . 120
C. Division into unequal parts . . . 127
§4. Cross in the middle . . . 127
§5. Central square . . . 131

Chapter IV. Bordered magic squares

§1. Preliminary observations . . . 141

A. Squares of odd orders 142

§2. Empirical construction 142
§3. Grouping the numbers by parity 146
§4. Placing together consecutive numbers 148
§5. Method of Stifel 149
§6. Zigzag placing 150
§7. Mathematical basis of these general methods 151

B. Squares of evenly-even orders 158

§8. Equalization by means of the first numbers 159
§9. Alternate placing 160
§10. Method of Stifel 161
§11. General principles of placing for even orders 163

C. Squares of evenly-odd orders 167

§12. Equalization by means of the first numbers 167
§13. Cyclical placing 168
§14. Method of Stifel 170
§15. Principles of these methods 171

Chapter V. Bordered squares with separation by parity

§1. The main square and its parts 177

§2. Filling the inner square 179

§3. Filling the remainder of the square by trial and error 180

A. Methodical filling of the oblique square 182

§4. Completing the placing of odd numbers 183

B. Methodical placing of the even numbers 184

§5. Situation after filling the oblique square 185
1. Determining the number of cells remaining empty 185
2. Determining the sum required 186

§6. Rules for placing the even numbers 188
1. First main equalization rule 190
2. Second main equalization rule 191
3. Neutral placings 192

§7. Case of the order $n = 4t + 1$ (with $t \geq 2$) 193
1. First border 193
2. Other borders 195
§8. Case of the order $n = 4t + 3$ (with $t \geq 1$) 201
1. First border 202
2. Other borders 202
C. Particular case of the order 5 208

Chapter VI. Magic squares with non-consecutive numbers

Particular case: The given numbers form arithmetical progressions

1. The numbers form a single progression 211
2. The numbers form n progressions 212
3. Magic square with a set sum 213
4. Magic products 214

General case: Squares with arbitrary given numbers

A. Squares of odd orders 215
§1. Square of order 3 215
1. General observations 215
2. The given numbers are in the first row 216
3. The middle number is in the median lower cell 217
4. The given numbers are in the diagonal 218
5. The given numbers are in the middle row 218
§2. Square of order 5 219
1. The given numbers are in the first row 219
2. The given numbers are in the second row 220
3. The given numbers are in the diagonal 220
4. The given numbers are in opposite lateral rows 221
5. The given numbers are in the first two rows 222
B. Squares of evenly-even orders 224
§3. Square of order 4 224
1. General observations 224
2. The given numbers are in the upper row 226
3. The given numbers are in the second row 229
4. The given numbers are in the end cells of the first row and the median of the second 231

5. The given numbers are in the end cells of the first row and the median of the third . . . 232
6. The given numbers are in the diagonal . . . 232
7. The given numbers are in the corner cells . . . 233
8. Writing in the sum as a whole . . . 234
a. Cases of impossibility . . . 234
b. Filling according to 'substance' . . . 235

§4. Square of order 8 . . . 239

1. The given numbers are in the first row . . . 239
a. Particular case: method of division . . . 239
b. General case . . . 239
c. Filling the second half . . . 240
2. The given numbers are in the first two rows, within the quadrants' diagonals . . . 242
3. The given numbers are in the first and third rows, within the diagonals . . . 243
4. The given numbers are in the first and fourth rows, within the diagonals . . . 244
5. Writing in the global sum . . . 245

C. Squares of evenly-odd orders . . . 248

§5. Square of order 6 . . . 248

1. The given numbers are in the first row . . . 248
2. The given numbers are equally distributed in the lateral rows . . . 249

§6. Squares of higher evenly-odd orders . . . 251

Chapter VII. Other magic figures

§1. Literal squares . . . 253

1. Squares of odd orders . . . 253
a. Case of a prime order . . . 253
b. Case of the order $n = 9$ . . . 254
c. Case of the order $n = 3 \cdot z$, with $z \neq 3$ . . . 255
d. Case of the square order $n = z^2$, with $z \neq 3$ . . . 256
e. Case of the composite order $n = t \cdot z$, with t, z odd $\neq 3$ . . . 257
2. Squares of even orders . . . 263

§2. Squares with one empty cell . . . 265

1. Squares of odd orders with central cell empty . . . 265

- 2. Case of the square of order 3 ... 266
- 3. Squares of evenly-even orders ... 268

§3. Squares with divided cells ... 270

§4. Magic triangles ... 271

§5. Magic crosses ... 273

§6. Magic circles ... 274

- 1. First type ... 274
- 2. Second type ... 276

§7. Magic rectangles ... 277

- 1. Both sides are odd ... 277
- 2. Both sides are evenly even ... 278
- 3. One side evenly even and the other evenly odd ... 278
- 4. Both sides are evenly odd ... 279

§8. Magic cubes ... 280

Appendices ... 281

Bibliography ... 303

Index ... 307

Chapter I

Introduction

§ 1. Definitions

A *magic square* is a square divided into a square number of cells in which natural numbers, all different, are arranged in such a way that the same sum is found in each horizontal row, each vertical row, and each of the two main diagonals.

A square with n cells on each side, thus n^2 cells altogether, is said to have the *order* n. The constant sum to be found in each row is called the *magic sum* of this square. Thus the squares of the figures 1 and 2 have the order 6. But their constant sum is different, with 111 for the first and 591 for the other. The first being filled with the first 36 natural numbers, its sum is the smallest possible for a 6×6 magic square. In the second, the numbers are not consecutive. Though both are magic squares, what is usually considered is the first kind, thus a square of order n filled with the first n^2 natural numbers. In this case the constant sum is easily determined: since the sum of all these numbers equals

$$\frac{n^2\,(n^2+1)}{2},$$

the sum in each row (line, column, main diagonal), thus the magic sum for such a square, will be

$$M_n = \frac{n(n^2+1)}{2}.$$

1	**32**	**34**	**3**	**35**	**6**
30	**8**	**27**	**28**	**11**	**7**
19	**23**	**15**	**16**	**14**	**24**
18	**17**	**21**	**22**	**20**	**13**
12	**26**	**10**	**9**	**29**	**25**
31	**5**	**4**	**33**	**2**	**36**

Fig. 1

131	**101**	**92**	**107**	**36**	**124**
191	**94**	**100**	**115**	**85**	**6**
3	**137**	**50**	**135**	**72**	**194**
192	**123**	**134**	**75**	**62**	**5**
1	**40**	**110**	**69**	**175**	**196**
73	**96**	**105**	**90**	**161**	**66**

Fig. 2

A square displaying the same sum in the $2n + 2$ aforesaid rows is an *ordinary* or *common magic square*. It meets the minimal number of required conditions, and such a square can be constructed for any given order $n \geq 3$. (There cannot be a magic square of order 2 with different

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numbers, as we shall see below.) But there are magic squares which display further properties.

A *bordered magic square* is one where removal of the successive borders leaves each time a magic square (Fig. 3). With an odd-order square, after removing each (odd-order) border in turn, we shall finally reach the smallest possible square, that of order 3. With an even-order square, that will be one of order 4 (the border cannot be removed since there is no magic square of order 2). For order $n \geq 5$, bordered squares are always possible.

92	**17**	**4**	**95**	**8**	**91**	**12**	**87**	**16**	**83**
99	**76**	**31**	**22**	**77**	**26**	**73**	**30**	**69**	**2**
1	**20**	**64**	**41**	**36**	**63**	**40**	**59**	**81**	**100**
3	**19**	**67**	**58**	**47**	**51**	**46**	**34**	**82**	**98**
96	**80**	**33**	**52**	**45**	**57**	**48**	**68**	**21**	**5**
7	**78**	**35**	**49**	**56**	**44**	**53**	**66**	**23**	**94**
90	**27**	**62**	**43**	**54**	**50**	**55**	**39**	**74**	**11**
13	**72**	**42**	**60**	**65**	**38**	**61**	**37**	**29**	**88**
86	**32**	**70**	**79**	**24**	**75**	**28**	**71**	**25**	**15**
18	**84**	**97**	**6**	**93**	**10**	**89**	**14**	**85**	**9**

Fig. 3

As seen above, the magic sum for a square of order n filled with the n^2 first natural numbers is

$$M_n = \frac{n(n^2+1)}{2}.$$

Clearly, the *average* sum in each cell is $\frac{n^2+1}{2}$. Accordingly, for m cells, this average sum should be m times that quantity; this will be called the *sum due* for m cells. Thus, the inner square of mth order ($m \geq 5$) within a bordered square of order n, being a separate entity, will contain in each row its sum due, namely

$$M_n^{(m)} = \frac{m(n^2+1)}{2}$$

if the main square is filled with the n^2 first natural numbers. The magic sum in any inner square will therefore differ from the sum in the next one, surrounding it, by n^2+1. With this in mind, we clearly see the structure of bordered squares: the elements of the same border which are opposite (horizontally, vertically, or diagonally for the corner cells), which we shall call *complements*, add up to $n^2 + 1$ if the main square is of order n. For example, in each border of Fig. 3, the sum of opposite elements is 101, and the magic sums of the successive squares are 505, 404, 303, and 202 for the smallest. Again, since for an odd-order bordered square there is just one central cell, it must necessarily contain $\frac{n^2+1}{2}$.

Particular cases of ordinary magic squares

It is a condition for magic squares that the main diagonals will contain the magic sum. But there are ordinary magic squares in which the magic sum is also found in the broken diagonals —thus pairs of diagonal rows, on either side of and parallel to a main diagonal, comprising n cells altogether. Such squares are called *pandiagonal* (Fig. 4 where, for example, the broken diagonals $47, 31, 20, \ldots, 21, 26, 42$ and $47, 59, 18, 8, \ldots, 14$ make the magic sum). They are possible for any odd order from 5 on, and for any even order *divisible by 4*, from 4 on.

11	**22**	**47**	**50**	**9**	**24**	**45**	**52**
38	**59**	**2**	**31**	**40**	**57**	**4**	**29**
18	**15**	**54**	**43**	**20**	**13**	**56**	**41**
63	**34**	**27**	**6**	**61**	**36**	**25**	**8**
3	**30**	**39**	**58**	**1**	**32**	**37**	**60**
46	**51**	**10**	**23**	**48**	**49**	**12**	**21**
26	**7**	**62**	**35**	**28**	**5**	**64**	**33**
55	**42**	**19**	**14**	**53**	**44**	**17**	**16**

Fig. 4

Such squares remain magic if we move any lateral row to the other side: the main diagonals of the new square will be magic since they were already magic as broken diagonals. By repeating such vertical and/or horizontal moves we are able to place any chosen element in any given cell of the square; for example, the cell containing 1 can be made to occupy any place in the square. This feature seems to have been highly prized in the early days of magic squares.

This property may be simply represented in the case of the smallest possible pandiagonal square, that of order 4, one form of which is seen in Fig. 5. Supposing it repeated in the plane, as in Fig. 6, any square of order 4 taken in it will be magic (and pandiagonal).

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

Fig. 5

13	**2**	**7**	**12**	**13**	**2**	**7**	**12**	**13**	**2**	**7**	**12**
3	**16**	**9**	**6**	**3**	**16**	**9**	**6**	**3**	**16**	**9**	**6**
10	**5**	**4**	**15**	**10**	**5**	**4**	**15**	**10**	**5**	**4**	**15**
8	**11**	**14**	**1**	**8**	**11**	**14**	**1**	**8**	**11**	**14**	**1**
13	**2**	**7**	**12**	**13**	**2**	**7**	**12**	**13**	**2**	**7**	**12**
3	**16**	**9**	**6**	**3**	**16**	**9**	**6**	**3**	**16**	**9**	**6**
10	**5**	**4**	**15**	**10**	**5**	**4**	**15**	**10**	**5**	**4**	**15**
8	**11**	**14**	**1**	**8**	**11**	**14**	**1**	**8**	**11**	**14**	**1**
13	**2**	**7**	**12**	**13**	**2**	**7**	**12**	**13**	**2**	**7**	**12**
3	**16**	**9**	**6**	**3**	**16**	**9**	**6**	**3**	**16**	**9**	**6**
10	**5**	**4**	**15**	**10**	**5**	**4**	**15**	**10**	**5**	**4**	**15**
8	**11**	**14**	**1**	**8**	**11**	**14**	**1**	**8**	**11**	**14**	**1**

Fig. 6

Another class of ordinary squares is that of *composite* squares (Fig. 7): the subsquares arranged within the main magic square are themselves magic with each having its own magic sum. Clearly, the possibility of such an arrangement depends on the divisibility of the order of the main square. Moreover, it will appear that the smallest possible composite square is that of order 9. Composite magic squares of lesser order are not possible or particular: that of order 8 may well be divided into four subsquares, but, because of the impossibility of a 2×2 magic square, they cannot have different magic sums; as to that of order 6, its four subsquares filled with different numbers could not even make equal sums.

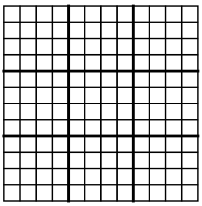

Fig. 7

§ 2. Categories of order

As mentioned above, no magic square of order 2 is possible. Indeed (Fig. 8) the conditions would be

$$a + d = b + c = b + d = c + d = c + a = a + b$$

from which we infer that $a=b=c=d$. A magic square of order 2 is therefore not possible with numbers which are to be different.

a	b
c	d

Fig. 8

$*$	$\dagger$	$\circ$
$\square$	r	$\square$
$\circ$	$\dagger$	$*$

Fig. 9

Consider now a square of order 3 (Fig. 9). Surrounding the central cell is a border comprising four pairs of opposite cells, horizontally ($\square$), vertically ($\dagger$) or diagonally ($*, \circ$). Clearly, with M designating the magic sum and r the central element, the sum in each pair of opposite cells must equal $M - r$. The sum in the whole square must therefore equal $4\,(M - r) + r$; it also equals $3M$. Equating these two quantities gives $M = 3r$.

4	**9**	**2**
3	**5**	**7**
8	**1**	**6**

Fig. 10

Let us now fill the 3×3 square with the nine first natural numbers. Their sum being 45, the magic sum M_3 will be 15; then $r = 5$ and each pair of opposite cells must add up to $M_3 - 5 = 10$. Let us begin by considering the place of 1. If it is in a corner, 9 will be in the opposite corner and we shall need, to complete the two lateral rows meeting in 1, two pairs of numbers adding up to 14. Now there is just one such pair available, 6 and 8. Therefore 1 must necessarily be in the middle of a lateral row, with 6 and 8 as its neighbours. With these four numbers in place the remainder of the square is determined (Fig. 10).

8	**3**	**4**
1	**5**	**9**
6	**7**	**2**

Fig. 11

6	**1**	**8**
7	**5**	**3**
2	**9**	**4**

Fig. 12

2	**7**	**6**
9	**5**	**1**
4	**3**	**8**

Fig. 13

We could exchange the places of 6 and 8, but this would just invert the figure. As a rule, we shall not consider as different the seven squares obtained from a given one by rotation or by inversion, as here, from Fig. 10, Fig. 11-13 and Fig. 14-17. We infer from that and the above stepwise construction that there must be just one form of the magic square of order

3 filled with the nine first natural numbers. This is an exception: for all other squares the possibilities are numerous.

2	9	4
7	5	3
6	1	8

Fig. 14

4	3	8
9	5	1
2	7	6

Fig. 15

8	1	6
3	5	7
4	9	2

Fig. 16

6	7	2
1	5	9
8	3	4

Fig. 17

We have thus far considered individually the squares of order 2 and order 3. Now it is not necessary to do that for higher orders. For there exist *general construction methods*, that is, ways of directly constructing a square of given order, either ordinary or bordered, once the empty square is drawn: a few more or less easily remembered instructions will enable us to place the sequence of consecutive numbers without any computation or recourse to trial and error. These general methods, though, are not applicable to all orders but, as a rule, to just one among three main categories, which are:

— The squares of *odd* orders, also called *odd squares*, thus with orders 3, 5, 7, ..., generally $n = 2k + 1$, of which the smallest is the square of order 3.

— The squares of *evenly-even* orders, or *evenly-even squares*, thus with orders 4, 8, 12, ..., generally $n = 4k$, of which the smallest is the square of order 4.

— The squares of *evenly-odd* orders, or *evenly-odd squares*, thus with orders 6, 10, 14, ..., generally $n = 4k + 2$, of which the smallest is the square of order 6.

Note, though, that these general methods may require some adapting for squares of lower orders, such as 3 or 4, sometimes also 6 and 8.

§ 3. Banal transformations of ordinary magic squares

As said above, rotations and inversions of the same square are not seen as different. On the other hand, there are for *ordinary* squares of orders larger than 3 some transformations which modify their aspect, and the number of possibilities increases with the order.[1]

First, each number i may be replaced by $n^2 + 1 - i$. The result corresponds to placing the numbers in reverse order. For a *symmetrical magic square*, in which the sum of two diagonally opposite elements is n^2+

[1] We omit here the particular feature of pandiagonal squares seen above (p. 3).

1, the square will just be rotated by 180°. See Fig. 18*a* & *b*. Otherwise, the square will be different, with some of the numbers having new neighbours. See Fig. 19*a* & *b*.

23	**2**	**19**	**6**	**15**
4	**8**	**25**	**12**	**16**
17	**21**	**13**	**5**	**9**
10	**14**	**1**	**18**	**22**
11	**20**	**7**	**24**	**3**

Fig. 18*a*

3	**24**	**7**	**20**	**11**
22	**18**	**1**	**14**	**10**
9	**5**	**13**	**21**	**17**
16	**12**	**25**	**8**	**4**
15	**6**	**19**	**2**	**23**

Fig. 18*b*

22	**18**	**1**	**14**	**10**
9	**5**	**13**	**21**	**17**
15	**6**	**19**	**2**	**23**
16	**12**	**25**	**8**	**4**
3	**24**	**7**	**20**	**11**

Fig. 19*a*

4	**8**	**25**	**12**	**16**
17	**21**	**13**	**5**	**9**
11	**20**	**7**	**24**	**3**
10	**14**	**1**	**18**	**22**
23	**2**	**19**	**6**	**15**

Fig. 19*b*

Second, we may, for an even order, exchange the quadrants diagonally (Fig. 20*a* & *b*). If the order is odd, part of the median rows must be exchanged as well (Fig. 21*a* & *b*).

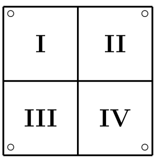

Fig. 20*a*

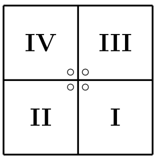

Fig. 20*b*

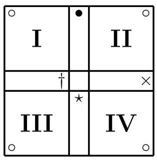

Fig. 21*a*

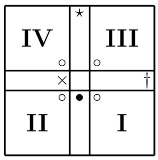

Fig. 21*b*

Third, considering two pairs of rows, with both the horizontal and vertical ones at the same distance from their median axis, we may exchange them in turn (Fig. 22*a* & *b*).

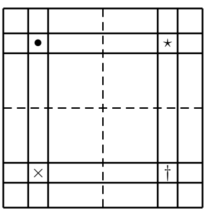

Fig. 22*a*

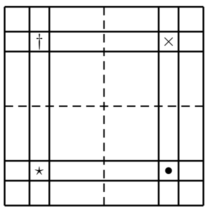

Fig. 22*b*

The magic property will hold since lines and columns keep their elements, and diagonals as well. For a symmetrical square, we may perform just *one* exchange of two symmetrical rows: the two pairs of diagonally opposite elements will move to the other diagonal, but this will not change the sum in the diagonals since each such pair adds up to $n^2 + 1$.

Applying the second transformation to bordered squares will not preserve their character. As to the first, it will modify the places of elements within the border rows (Fig. 23*a* & *b*). Now this is a banal change: we may arbitrarily modify the place of elements *within* the border rows of such a square provided each opposite undergoes the same change.

4	**24**	**23**	**8**	**6**
19	**10**	**17**	**12**	**7**
21	**15**	**13**	**11**	**5**
1	**14**	**9**	**16**	**25**
20	**2**	**3**	**18**	**22**

Fig. 23*a*

22	**2**	**3**	**18**	**20**
7	**16**	**9**	**14**	**19**
5	**11**	**13**	**15**	**21**
25	**12**	**17**	**10**	**1**
6	**24**	**23**	**8**	**4**

Fig. 23*b*

§ 4. Historical outline

When the science of magic squares began is not known. No ancient Greek text dealing with magic squares is still extant, no allusion whatsoever is made to them in antiquity, and the only Greek text preserved is a Byzantine work by Manuel Moschopoulos (ca. 1300), who appears to draw his knowledge from some Arabic or Persian source.[2] Furthermore, allusions to Greek studies by later Arabic authors seem unreliable, such as the attribution of the discovery of magic squares to Archimedes by the 13th-century geographer Qaswīnī[3] despite the fact that the list of Archimedes' works is well known since Greek times.

Still, it now appears certain that there existed Greek books on the subject, and at a remarkable stage of advancement. This we know from a text attributed to the prolific translator of Greek works <u>Th</u>ābit ibn Qurra (836-901). He tells us at the beginning how he became acquainted with the science of magic squares, obviously hardly known at that time, and the difficulty he had to interpret the text of this (apparently anonymous) work.[4] *My first acquaintance with the subject of magic square was the*

[2] Moschopoulos' text has been edited and translated by Tannery; on its origin, see our study of it.

[3] See Wiedemann's *Beitrag V*, pp. 451-452.

[4] This and other parts of <u>Th</u>ābit's translation are found in the anonymous MS.

3×3 figure mentioned by Nicomachos in the Arithmetic.[5] *Then Abū al-Qāsim al-Ḥijāzī came across a 4×4 square, (filled with the numbers) from 1 to 16 with constant difference, which caused his admiration.*[6] *Next I found a 6×6 figure on the back of Euclid's book* (of the *Elements*) *in the handwriting of Isḥāq.*[7] *Afterwards I came across a book containing three or four figures less than (the order) 10. Then I found in the Royal library* (in Baghdad) *two books, for the greater part damaged by insects so that one could understand just a small part of them; the notice about them was in al-Māhānī's handwriting and the first page for the most part in Ḥusayn ibn Mūsā al-Nawbakhtī's handwriting.*[8] *Then I examined them, found elucidating them very laborious, and it seemed to me that it should be possible to make sense of those parts which had been damaged in one by what had been preserved in the other.*

With Thābit's translation general methods for constructing bordered and composite squares may thus be considered as available in the late ninth century. For ordinary squares, however, we do not know of any ancient source, and the documents we have are all, as it seems, originally

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London BL Delhi Arabic 110, a 12th-century compilation of works by various authors from the 9th to 12th centuries. For the Arabic text of this passage, see fol. 60^{v} in this MS or below (p. 281), Appendix 1. (Note our use of slanted writing, not italics, throughout for quotations.)

[5] Nicomachos' *Introduction to arithmetic* (Ἀριθμητικὴ εἰσαγωγή), written towards the end of the first century, is —as indicated by its title— an elementary introduction to number theory (such is indeed the Greek meaning of 'arithmetic'). It does, though, depart from the sound mathematical works of the classical Greek period in that its author tends at times to take his conjectures for theorems. Still, the *Introduction* was one of the main channels of transmission of Greek mathematics, and for early mediaeval Europe the only one, namely through the Latin adaptation by Boëtius, while in the East it was translated twice: into Syriac and, by Thābit ibn Qurra also, into Arabic (ed. Kutsch). There is no mention of a magic square in it (or in the Arabic translation), but a 3×3 square may have appeared as a marginal addition. In a recent study, N. Vinel has quoted (pp. 551-552) a fragment by Iamblichos, a 4th-century commentator on Nicomachos, which seems to allude to the 3×3 magic square.

[6] Abū'l-Qāsim al-Ḥijāzī was a theologian of the first half of the 9th century. The description being rather naive, Thābit must be quoting him. His name and those of the other scholars appear in the *Fihrist* of Ibn al-Nadīm, written towards the end of the 10th century, which contains in particular a list of Arabic books or translations.

[7] Isḥāq ibn Ḥunayn, a contemporary of Thābit ibn Qurra, translated the *Elements*, a translation then revised by Thābit himself.

[8] Muḥammad ibn ʿĪsā ibn Aḥmad al-Māhānī was a Persian mathematician and astronomer working in Baghdad around the middle of the 9th century; Ḥasan (as commonly) ibn Mūsā al-Nawbakhtī was a theologian and philosopher of the second half of the 9th century, personally acquainted with Thābit (*Fihrist*, ed. Flügel *et al.*, I, 177, 11). Incidentally, Ibn al-Nadīm also mentions the damage caused by 'insects' (or termites: *araḍa*) to manuscripts (I, 243, 27).

Arabic. General methods are seen to appear in the late tenth or early eleventh century for odd and evenly-even orders, and then, at the end of the eleventh, for evenly-odd ones. (With the evenly-odd order presenting most difficulty, such a stepwise development was only to be expected.)

This then leaves open the question of the existence of ordinary magic squares in Greece. Given the superior level of the Greek methods reported in <u>Th</u>ābit's translation, it would be surprising, to say the least, to find the Greeks not acquainted with their construction. My suggestion of twenty years ago may still be considered: that this science remained more or less confined to (neo-Pythagorean?) circles, just as, more than a thousand years before, the teaching of the Pythagorean school had been restricted to its members, and only later began to spread.[9] This, incidentally, might explain why so little of this Greek science has been transmitted.

α̅ ا 1	β̅ ب 2	γ̅ ج 3	δ̅ د 4	ε̅ ه 5	ϛ̅ و 6	ζ̅ ز 7	η̅ ح 8	ϑ̅ ط 9
ι̅ ى 10	κ̅ ك 20	λ̅ ل 30	μ̅ م 40	ν̅ ن 50	ξ̅ س 60	ο̅ ع 70	π̅ ف 80	ϟ̅ ص 90
ρ̅ ق 100	σ̅ ر 200	τ̅ ش 300	υ̅ ت 400	φ̅ ث 500	χ̅ خ 600	ψ̅ ذ 700	ω̅ ض 800	ϡ̅ ظ 900
͵α̅ غ 1000								

Fig. 24

Returning to Arabic times, we do see continuous development in the 11th and early 12th centuries, with the discovery of various other constructions for ordinary magic squares, sometimes pandiagonal as well. Squares filled with non-consecutive numbers also begin to appear. The origin of that has to do with the association of Arabic letters with numerical values —an adaptation of the Greek numerical system (Fig. 24); this adaptation appeared in early Islamic times, before the adoption of the Indian numerals, but remained in use later. Since to each letter of a word, or to each word of a sentence, was attributed a numerical quantity, thus to the word or the sentence a certain sum, this word or sentence could be

[9] *Les carrés magiques*, p. 270; *Magicheskie kvadraty*, pp. 277-278.

placed as such in the cells of a certain row of a square. The task was then to complete the square numerically so that it would display in each row the sum in question —a mathematically interesting problem since this is not always possible.

Meanwhile, however, popular use of magic squares as talismans grew apace. Authors then followed this trend: many late, shorter texts are just not interested in teaching the construction of magic squares —to say nothing of their mathematical fundament. They give the figures of a few magic squares, most commonly squares of the orders 3 to 9 associated with the seven then known planets (including Moon and Sun) of which they embodied the respective, good or evil, qualities —abundantly described and commented in these texts. The reader is taught on what material and when he is to draw each of such squares; for both the nature of the material and the astrologically predetermined time of drawing are presumed to increase the square's efficacy. It merely remains to put this object in the vicinity of the chosen person, beneficiary or victim.

Of such kind were the first Arabic texts translated into Latin in 14th-century Spain. A characteristic example of theory and application is the following, which uses the properties of Jupiter, mostly favourable, and Mars, mainly not.[10]

The figure of Jupiter is square, four by four, with 34 on each side. If you wish to operate with it: make a silver plate in the day and the hour of Jupiter, provided Jupiter is propitious,[11] *and engrave upon it the figure; you will fumigate it with aloes wood and amber. When you carry it with you, people who see you will love you and you will obtain from them whatever you request. If you place it in the store-house of a merchant, his trade will increase. If you place it in a dovecote or in a hive, a flock of birds or a swarm of bees will gather there. If someone unlucky carries it, he will prosper and be always more successful. If you place it in the seat of a prelate, he will enjoy a long prelature and will not fear his enemies, but be successful among them.*

The figure of Mars when unfavourable means war and exactions. It is a square figure, five by five, with 65 on each side. If you wish to operate with it, take a copper plate in the day and hour of Mars when Mars is decreasing in number and brightness, or malefic and retrograding, or in

[10] This and other examples in our *Magic squares for daily life*. For the original Latin text, see below, Appendix 2. The two magic squares represented are those of our figures 27 and 29 below.

[11] Thus on Thursday, 1st and 8th hours of the day and 3rd and 10th of the night, and with Jupiter on a direct course and increasing in brightness.

any way unfavourable, and engrave the plate with this figure; and you will fumigate it with the excrement of mice or cats. If you place it in a new building, it will never be completed. If you place it in the seat of a prelate, he will suffer daily harm and misfortune. If you place it in the shop of a merchant, it will be wholly destroyed. If you make this plate with the names of two merchants and bury it in the house of one of them, hatred and hostility will come between them. If you happen to fear the king or some powerful person, or enemies, or have to appear before a judge or a court of justice, engrave this figure as said above when Mars is favourable, in direct motion, increasing in number and brightness; fumigate it with one drachma (= $\frac{1}{8}$ ounce) *of carnelian stone. If you put this plate in a piece of red silk and carry it with you, you will win in court and against your enemies in war, for they will flee at the sight of you, fear you and treat you with deference. If you place it upon the leg of a woman,*[12] *she will suffer from a continuous blood flow. If you write it on parchment on the day and the hour of Mars and fumigate it with birthwort and place it in a hive, the bees will all fly away.*

It was thus the arrival of such texts in late mediaeval Europe which first aroused interest in, and led to the study of, such squares there (the figures of these squares are given below, § 6). This incidentally explains the use of the term 'magic' —formerly also 'planetary', which we find still employed by Fermat.[13] The original Arabic (perhaps originally Greek) denomination, 'Harmonious arrangement of numbers' (*wafq al-a'dād*), which had a more mathematical connotation, remained unknown, as well as the various constructions abundantly described in Arabic and Persian manuscripts.

We may infer from this that 'magic' applications do not seem to have played a significant rôle at the outset. As the original name indicates, such squares were then being studied as a branch of number theory, just as were the perfect and amicable numbers so highly esteemed by Pythagoreans and their followers in late antiquity. In short, such research was at the time considered as being perfectly serious and had not yet been tainted with the sad reputation later acquired through popular use of these squares.

Finally, we may note that magic squares are also found elsewhere. The magic square of order 3 appears in China at the beginning of our era; to our knowledge, higher-order squares do not occur there before the

[12] Reverting to evil uses.
[13] *Varia opera mathematica*, p. 176; or *Œuvres complètes*, II, p. 194.

13th century, and are clearly of Arabic or Persian origin. The same holds for Indian magic squares.

§ 5. Main sources considered

In the tenth century, ʿAlī ibn Aḥmad al-Anṭākī (d. 987) wrote a commentary on Nicomachos' *Introduction to arithmetic*, of which only the third (and last) part is extant.[14] It contains first a list of definitions and propositions chiefly taken from Euclid's *Elements*, then a section on magic squares, finally a set of recreational problems, namely how to determine through indirect questions a number being thought of by someone. None of all this has any relation to Nicomachos' work, and we have here a compilation of *various* sources from late antiquity. The part on magic squares is taken verbatim from Thābit's translation, but without any mention of it.[15] The construction of bordered magic squares of any order was thus known in the Islamic world in the middle of the tenth century.

The second 10th-century text, entirely devoted to magic squares, is by the Persian Abū'l-Wafā' Būzjānī (940-997/8), who was one of the most famous mathematicians of Islamic civilization. His treatise is clear and mostly didactic, for every step is explained and justified. We are told how to construct ordinary magic squares, but only for small orders and with particular methods, then bordered squares, but with only one general method, namely for odd squares: for the two even orders, the reader is just told how to *find* the elements in each border. At the end of the treatise, the reader is taught how to form composite squares, and then odd-order bordered squares displaying separation by parity, this mainly relying on trial and error. All this is clearly explained, and gives the impression of reporting the early steps in the science of magic squares for the benefit of the reader (above, p. vi). It is thus a nice introduction, but, as inferred from the information provided by the previous treatise, it does not give an overview of the general methods known at that time.[16]

Another famous mathematician was Abū ʿAlī al-Ḥasan ibn al-Hay-

[14] Edited in our *Magic squares in the tenth century*.

[15] Thus in our edition we could only surmise that the whole of it was taken from ancient Greek sources (above, p. v). Our $\mathcal{A}$.II.7 - $\mathcal{A}$.II.35 are found on fol. 80^{v}-84^{v} of the London manuscript Delhi Arabic 110, while $\mathcal{A}$.II.44 - $\mathcal{A}$.II.54 occur on fol. 99^{r}-100^{v}, followed by larger squares not reproduced by Anṭākī (fol. 102^{v}-107^{r}). Other sources are also used in this manuscript, including three of the authors mentioned below (Būzjānī, Ibn al-Haytham, Asfizārī); this may account for sections omitted from Thābit's translation, with the author selecting his excerpts.

[16] The most significant parts are found in our *Magic squares in the tenth century*; for the complete edition, see our *Traité d'Abū'l-Wafā'*.

tham (b. ca. 965 in Basra, d. 1041 in Egypt). We do not know his treatise on magic squares, but significant excerpts from it are found in an anonymous 12th-century source which considered him to be one of the best authors on the subject.[17] Ibn al-Haytham is the first to have discovered, or reported, or justified, some general methods for constructing ordinary squares, obviously unknown to the two previous authors. After him, mathematical justification of methods seems to have been neglected.

The anonymous *Harmonious arrangement of the numbers*, from the early 11th century as it seems, is one of our main sources.[18] Its (unknown) author tells us that he turned to the science of magic squares as a source of relief from certain disappointments. He further tells us that the main part of his work is his own findings, while anything taken over from his 'predecessors' was ameliorated by him. In short, *oratio pro domo*. His treatise teaches various ways of constructing bordered squares for all three orders, ordinary squares for odd and evenly-even orders, composite squares, and finally squares with a set of given numbers occupying one of the rows. To this last part, which is very elaborate and complete, we shall give particular attention in our sixth chapter.

Contemporary with it, also anonymous but much shorter, is the *Brief treatise teaching the harmonious arrangement of numbers*.[19] The extent of its author's knowledge is much the same, but he restricts himself to one method, sometimes different, for each kind.

With the Persian Abū Ḥātim Muẓaffar Asfizārī (d. before 1121/2), we reach the end of the 11th century. We know two of his methods from the anonymous 12th-century source already mentioned and from the next author to be mentioned. The first is a method for constructing ordinary evenly-even squares, said to be *easier than that by Anṭākī and that by Ibn al-Haytham*, and the second teaches how to obtain an odd-order ordinary magic square displaying separation by parity.[20]

His compatriot Jamāl al-Zamān 'Abd al-Jabbār Kharaqī (d. 1138/9) does not claim any originality, but his treatise is the first to teach general methods for all three types of the two main categories of magic squares, thus now including the previously omitted case of ordinary squares of evenly-odd orders. His text thus shows that the problem of constructing

[17] Publication of this source in our *Une compilation arabe*. This source has other excerpts than the London manuscript (above, n. 4).

[18] See our *Un traité médiéval*.

[19] Our *L'Abrégé enseignant la disposition harmonieuse des nombres*.

[20] Obtaining this separation is easy with ordinary squares —unlike in the case of bordered ones. All these methods will be discussed below.

ordinary magic squares was completely solved around 1100.[21]

Another Persian author writing about magic squares in the next century is ʿAbd al-Wahhāb ibn Ibrāhīm Zanjānī. His short treatise is limited to the construction of bordered squares as well as that of order 4 containing in its upper row four given numbers or, if this arrangement is not possible, their sum. Its importance is not due to any originality but to its success, as witnessed by the number of manuscript copies extant. Zanjānī himself tells us in the introduction that he wrote it at the request of some friends, who just wanted to know the rules for constructing squares of any order.[22] This indeed illustrates the later characteristics of treatises on magic squares: written in response to public demand, they are to teach methods aiming at results, without bothering the reader with questions of fundaments or feasibility.

The first trace of magic squares in the West is the treatise of the Byzantine Manuel Moschopoulos, written at the very beginning of the 14th century (above, p. 8). He constructs ordinary squares of odd orders, of evenly-even orders (taken by him, as it seems, to mean $n = 2^k$), but none of evenly-odd orders. It would seem that he attempted to reconstruct the methods from examples of squares he saw in some manuscript, either Persian or Arabic.

The same happened later, from the 14th century on, in Western Europe, when the first examples of squares arrived through Latin translations of magic texts (see above, pp. 11-12, and below, § 6). They contained two sets of seven squares attributed to the seven planets then known (in the geocentric system, thus including Sun and Moon), and the only explanations were, as we have seen, about their magic use. No Arabic text describing methods was then translated, or studied, and the first attempts to explain or generalize them were based solely on these examples. With these first studies are associated the names of Cornelius Agrippa, Girolamo Cardano and Claude-Gaspar Bachet de Méziriac in the 16th and early 17th centuries. A curious exception is the case of Michael Stifel (1487-1567), who gives instructions for constructing bordered squares of all orders;[23] there is no mention of any source, nor does he give himself any credit, and his methods are not exactly like the ones we know from Arabic sources.

Meanwhile, treatises expounding the construction of magic squares

[21] See our *Herstellungsverfahren III*.
[22] See *Herstellungsverfahren II, II′*.
[23] In his *Arithmetica integra*, fol. 24$^{\text{v}}$-30$^{\text{r}}$.

continued to be written in Arabic, which makes it all the more strange that these methods remained unknown in Europe (though some were rediscovered there in the following centuries). A very complete and mostly clear treatise is the *Rising of the illumination for arranging magic squares*, written around 1600 by the Egyptian Muḥammad Shabrāmallisī.[24] This does not make, though, the author highly competent: sometimes he describes at length something very simple, sometimes he wrongly attempts to generalize a particular method (see below, p. 95). He carefully transmits known constructions, but cannot go beyond that. Through his work we may nevertheless learn the complete treatment of other magic figures, such as literal squares and magic circles. He thus helped to keep alive the knowledge of magic squares, as witnessed by the references to him more than one century later by the Sudanese Muḥammad al-Fulānī al-Kishnāwī (d. 1741) in his own work.[25]

§ 6. Squares transmitted to the Latin West

The following magic squares are those which were transmitted to Europe in the 14th century and frequently copied or printed during the 15th to 17th centuries.[26]

4	9	2
3	5	7
8	1	6

Fig. 25

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

Fig. 26

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Fig. 27

The 3×3 square of Fig. 25 appears both as such and in inverted (right-to-left) form. As to the 4×4 square, that of Fig. 26 appears mostly in the inverted form while the second (which is the previous one rotated and with median columns switched) will be used by Dürer in his *Melencolia* in order to date it (1514).

The two 5×5 squares of Fig. 28 and 29 result from well-known constructions, as we shall see later (II, §§ 2 and 4). Here they are, however, quite different in aspect, for the second square separates the odd numbers from the even ones, all placed in the corners.

[24] *Ṭawāliʿ al-ishrāq fī waḍʿ al-awfāq*, MS. Paris BNF Arabe 2698.
[25] MS. 65496 of the London School of Oriental and African studies.
[26] Their occurrences in manuscripts is given in Folkerts' *Zur Frühgeschichte*.

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

Fig. 28

14	10	1	22	18
20	11	7	3	24
21	17	13	9	5
2	23	19	15	6
8	4	25	16	12

Fig. 29

The first 6×6 square (Fig. 30), which results from a well-known method of construction (p. 91), occurs in inverted form. The second arrangement is more particular, and seems to have been obtained stepwise.[27]

1	35	34	3	32	6
30	8	28	27	11	7
24	23	15	16	14	19
13	17	21	22	20	18
12	26	9	10	29	25
31	2	4	33	5	36

Fig. 30

1	32	34	3	35	6
30	8	27	28	11	7
20	24	15	16	13	23
19	17	21	22	18	14
10	26	12	9	29	25
31	4	2	33	5	36

Fig. 31

22	47	16	41	10	35	4
5	23	48	17	42	11	29
30	6	24	49	18	36	12
13	31	7	25	43	19	37
38	14	32	1	26	44	20
21	39	8	33	2	27	45
46	15	40	9	34	3	28

Fig. 32

The configuration of this 7×7 square (Fig. 32), also found in inverted form, is the most commonly seen in mediaeval Arabic texts (II, § 2).

[27] In the way described in our *Les carrés magiques dans les pays islamiques*, p. 258 (Russian edition, pp. 266-267).