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Siyao Xu

Study on Magnetohydrodynamic Turbulence and Its Astrophysical Applications

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Siyao Xu

Study on Magnetohydrodynamic Turbulence and Its Astrophysical Applications

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Supervisor's Foreword

It is my great pleasure to write a foreword to this impressive dissertation book on magnetohydrodynamic (MHD) turbulence and its astrophysical applications, written by my former student and collaborator, Dr. Siyao Xu. Magnetic fields and turbulence are at the core of all astrophysical fluids. During six years of study at Peking University, China, Dr. Xu published ten first-author papers in leading astrophysical journals as a graduate student. These papers cover a wide range of subjects, from developing fundamental MHD turbulence theories to many applications of the theories to various astrophysical phenomena, including cosmic rays, molecular clouds, interstellar medium (ISM), radio pulsars, fast radio bursts (FRBs), and gamma-ray bursts. These publications contain original contributions to several different fields. The impact of these studies has been already felt by researchers in several different disciplines.

This book is a collection of the selected topics in the above list, with the focus on several projects finished during the last three years of her Ph.D. period. The first two chapters describe her major theoretical work on MHD turbulence in a partially ionized medium (Chap. 1) and small-scale turbulence dynamo (Chap. 2). Both are complicated subjects and previously have been tackled mostly numerically by researchers in the field. She developed innovative analytical tools to solve these problems. In particular, the analytical model of turbulent dynamo presents a major advance in the field. The theory solves a problem previously believed only solvable numerically and makes clear predictions that have been confirmed later by numerical simulations. The next three chapters are astrophysical applications of MHD turbulence theories in ISM (Chap. 3), radio pulsars (Chap. 4), and fast radio bursts (Chap. 5). In Chap. 3, she studied the structure function (SF) of the Faraday rotation measure (RM) of the ISM and developed a method of disentangling magnetic fluctuations from density fluctuations. She proposed a natural interpretation of a feature observed in the SF of ISM RM distribution as the transition from the global Kolmogorov turbulence to supersonic turbulence. Chapters 4 and 5 investigate scatter broadening of radio pulses in radio pulsars and FRBs. In the radio pulsar work, she interpreted a break in the observed dispersion measure—scatter broadening timescale also as due to the transition from the Kolmogorov

turbulence to supersonic turbulence. In the FRB work, she interpreted the observed scattering tail of some FRBs as originating from the supersonic turbulence in the host galaxies of the FRBs, consistent with FRBs being born in star-forming galaxies. All three chapters address a self-consistent physical picture invoking supersonic turbulence.

In summary, this is a comprehensive dissertation with breadth, depth, rigor, consistency, and innovative ideas. It will be a good read for students and researchers working in the fields of MHD turbulence, ISM, pulsars, and FRBs. Enjoy!

Las Vegas, USA
February 2019

Prof. Bing Zhang

Abstract

Turbulence and magnetic fields are ubiquitous in the universe. Their importance to astronomy cannot be overestimated. The theoretical advancements in magnetohydrodynamic (MHD) turbulence achieved during the past two decades have significantly influenced many fields of astronomy. Constructing predictive theories of the magnetic field amplification by turbulence and the dissipation of MHD turbulence in a partially ionized medium is the core of the thesis. These fundamental nonlinear problems were believed to be tractable only numerically. This thesis provided comprehensive analytical descriptions in quantitative agreement with existing numerics, as well as theoretical predictions in physical regimes still unreachable by simulations, and explanations of various related observations. The thesis further promoted the astrophysical applications of MHD turbulence theories, including interstellar density fluctuations and the effect on observations, e.g., Faraday rotation, scattering measurements of Galactic and extragalactic radio sources; evolution and importance of magnetic fields during the formation of the first stars and in molecular clouds; scattering and diffusion of cosmic rays. It demonstrates the key role of MHD turbulence in connecting diverse astrophysical processes and unraveling long-standing astrophysical problems, as foreseen by Chandrasekhar, a founder of modern astrophysics.

Keywords Magnetohydrodynamics · Turbulence · Turbulent dynamo · Interstellar medium

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I am very grateful to my advisor Prof. Bing Zhang for his guidance and encouragement. This work would not have been finished without his support. I would also like to thank Prof. A. Lazarian, who has taught me physics of MHD turbulence during my visit in Madison and during my entire Ph.D. study. It has been a great pleasure to work with both Prof. Zhang and Prof. Lazarian on many different projects. I thank the members of my thesis committee: Prof. Di Li, Prof. Lixin Li, Prof. Zhuo Li, Prof. Renxin Xu, and Prof. He Gao for their insights and comments on my thesis. I acknowledge the support from China Scholarship Council during my stay in University of Wisconsin—Madison and the support from the Pilot-B program for gravitational wave astrophysics of the Chinese Academy of Sciences and the Research Corporation for Scientific Advancement during my visit at the Aspen Center for Physics. I am grateful for the financial support provided by Prof. Lixin Li for my visit to Ruhr-University Bochum.

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Chapter 1

MHD Turbulence in a Partially Ionized Medium



Abstract Astrophysical fluids are turbulent, magnetized and frequently partially ionized. As an example of astrophysical turbulence, the interstellar turbulence extends over a remarkably large range of spatial scales and participates in key astrophysical processes happening in different ranges of scales. A significant progress has been achieved in the understanding of the magnetohydrodynamic (MHD) turbulence since the turn of the century, and this enables us to better describe turbulence in magnetized and partially ionized plasmas. The modern revolutionized picture of the MHD turbulence physics facilitates the development of various theoretical domains, including the damping process for dissipating MHD turbulence. This chapter is based on the work by Xu et al. (ApJ 810:44, 2015, [1]), Xu et al. (ApJ 826:166, 2016, [2]), Xu et al. (New J Phys 19:065005, 2017, [3]).

1.1 Turbulent, Magnetized, and Partially Ionized Interstellar Medium

Astrophysical plasmas, e.g., in the low solar atmosphere and molecular clouds, are commonly partially ionized and magnetized (see the book by [4] for a list of the partially ionized interstellar medium phases). The presence of neutrals affects the magnetized plasma dynamics and induces damping of MHD turbulence (see studies by e.g., [5, 6]).

On the other hand, astrophysical plasmas are characterized by large Reynolds numbers, and therefore they are expected to be turbulent (see e.g., [7–9]). This expectation is consistent with the turbulent spectrum of electron density fluctuations measured in the interstellar medium (ISM) [10, 11], and other ample observations from such as the Doppler shifted lines of HI and CO (e.g., [11–14]), synchrotron emission and Faraday rotation [15–17], as well as in-situ turbulence measurements in the solar wind [18].

The theory of MHD turbulence has been a subject under intensive research for decades (e.g., [19–21]). The actual breakthrough in understanding its properties came with the pioneering work by Goldreich and Sridhar [22] (henceforth GS95), where the properties of incompressible MHD turbulence have been formulated. Later research

extended and tested the theory [23–26], and generalized it for the compressible media [27–30].

In this thesis we focus on the MHD theory based on the foundations established in GS95, but do not consider the modifications of the theory that were suggested after GS95, which were motivated by the departure from the GS95 prediction of the turbulent spectral slope observed in some simulations (see [31–33]). We believe that the difference between these numerical simulations and the GS95 predictions can stem from MHD turbulence being somewhat less local compared to its hydrodynamic counterpart [34], which induces an extended bottleneck effect that can flatten the spectrum. Therefore the measurements of the actual Alfvén turbulence spectrum require a large inertial range to avoid the numerical artifact due to an insufficient inertial range. This idea seems to be in agreement with higher resolution numerical simulations [35, 36], which show consistency with the GS95 expectations.

The properties of MHD turbulence in a partially ionized gas derived on the basis of the GS95 theory have been addressed in the theoretical works by Lithwick and Goldreich [27], Lazarian et al. [37], but these studies did not cover the entire variety of the regimes of turbulence and damping processes. MHD simulations in the case of a partially ionized gas are more challenging than the case of a fully ionized gas, and therefore to establish the connection between the theoretical expectations and numerical results on the MHD turbulence in a partially ionized gas is difficult. The two-fluid MHD simulations in e.g., [38–41] exhibit more complex properties of turbulence compared to the MHD turbulence in a fully ionized gas.

A significant improvement in the understanding of MHD turbulence in a partially ionized gas has been achieved in the recent theoretical studies, in particular, in Xu et al. [1] (henceforth XLY15) where the damping of MHD turbulence was considered in order to describe the different linewidth of ions and neutrals observed in molecular clouds and to relate this difference with the magnetic field strength. The analysis of turbulent damping has been significantly extended in the later paper, namely, in Xu et al. [2] (henceforth XYL16), which dealt with the propagation of cosmic rays in partially ionized ISM phases. XYL16 presented a more in-depth study of ion-neutral decoupling and damping arising in the fast and slow mode cascades.¹

Turbulence also provides magnetic field generation via the turbulent dynamo. The corresponding theory can be traced to the classical study of [42]. The predictive kinematic turbulent dynamo theory, which describes an efficient exponential growth of magnetic field via stretching field lines by random velocity shear, was suggested by Kazantsev [43], Kulsrud and Anderson [44]. When the growing magnetic energy becomes comparable to the turbulent energy of the smallest turbulent eddies, the velocity shear driven by these eddies is suppressed due to the magnetic back reaction, and the turbulent dynamo proceeds to the nonlinear stage. Numerical studies demonstrated the nonlinear turbulent dynamo is characterized by a linear-in-time

¹The hint justifying the treatment of cascades separately can be found in the original GS95 study and in Lithwick and Goldreich [27] with more quantitative predictions. More theoretical justifications together with the numerical proofs are provided in Cho and Lazarian [28, 29] as well as in further studies by Kowal and Lazarian [30].

growth of magnetic energy, with the growth efficiency much smaller than unity (see [45–48] for a review). The study in Xu and Lazarian [49] (hereafter XL16) provided an important advancement of both kinematic and nonlinear dynamo theories in both a conducting fluid and a partially ionized gas. They identified new regimes in the kinematic dynamo stage and provided the physical justification for earlier numerical findings on the nonlinear dynamo stage.

The magnetic turbulence and turbulent dynamo theories are interconnected. On one hand, turbulent dynamo inevitably takes place in turbulence with dominant kinetic energy over magnetic energy. On the other, magnetic turbulence is an expected outcome of the nonlinear turbulent dynamo. Simulations in Lalescu et al. [50] found the coexistence of both processes, namely, the conversion of magnetic energy into kinetic energy and the generation of magnetic energy via the turbulent dynamo. Besides, the viscosity-dominated regime with the magnetic energy spectrum k^{-1} is present in both MHD and dynamo simulations at a high magnetic Prandtl number [51–53]. Therefore, it seems synergetic to consider both processes in a unified picture.

1.2 Properties of MHD Turbulence Cascade

1.2.1 General Considerations

Dealing with MHD turbulence in a partially ionized gas, we consider both the rate of nonlinear interactions that arise from turbulent dynamics and the rate of ion-neutral collisional damping. Therefore our first step is to consider the turbulent cascading rate, which can be obtained by studying the properties of MHD turbulence in a fully ionized gas. It is evident that this description is applicable to both cases when neutrals and ions are well coupled and therefore they move together as a single fluid and when ions move independently from neutrals in the decoupled regime. The better defined boundaries between different coupling regimes will be further established in the text.

MHD turbulence in a conducting fluid is a highly nonlinear phenomenon, as the turbulent energy cascades toward smaller and smaller scales down to the dissipation scale [54]. It is well known that weak MHD perturbations can be decomposed into Alfvén, slow, and fast modes [55]. It had been believed that such a decomposition is not meaningful within the strong compressible MHD turbulence due to the strong coupling of the modes [56]. However, physical considerations in GS95 as well as numerical simulations show that the cascade of Alfvén modes can be treated independently of compressible modes owing to the weak back-reaction from slow and fast magnetoacoustic modes [29]. In fact, [28, 29] dealt with perturbations of a substantial amplitude and showed that the statistical decomposition works with trans-Alfvénic turbulence, i.e. for magnetic field perturbed at the injection scale with the amplitude comparable to the mean magnetic field. As we discuss later, these results can be generalized for selected ranges of scales of sub-Alfvénic and super-Alfvén turbulence.

A potentially more accurate decomposition was suggested by Kowal and Lazarian [30]. This approach uses wavelets which are aligned with the local magnetic field direction. Their study confirmed the results in Cho and Lazarian [29].

1.2.2 Weak and Strong Cascades of Alfvénic Turbulence

The pioneering studies of Alfvénic turbulence were carried out by Iroshnikov [57] and Kraichnan [58] for a hypothetical model of isotropic MHD turbulence. Later studies (see [19–21, 59]) pointed out the anisotropic nature of the energy cascade and paved the way for the breakthrough work by GS95. As we mentioned earlier, the original GS95 theory was also augmented by the concept of local systems of reference ([23], hereafter LV99; [24–26]), which specifies that the turbulent motions should be viewed in the local system of reference related to the corresponding turbulent eddies. Indeed, for the small-scale turbulent motions the only magnetic field that matters is the magnetic field in their vicinity. Thus this local field, rather than the mean field, should be considered. Therefore when we use wavenumbers k_{\parallel} and k_{\perp} , they should be seen as the inverse values of the parallel and perpendicular eddy sizes l_{\parallel} and l_{\perp} with respect to the local magnetic field, respectively. With this convention in mind we will use wavenumbers and eddy sizes interchangeably.

To understand the nature of the weak and strong Alfvénic turbulence cascade, it is valuable to consider the interaction between wave packets [60, 61]. For the collision of two oppositely moving Alfvénic wave packets with parallel scales l_{\parallel} and perpendicular scales l_{\perp} , the change of energy per collision is

$$\Delta E \sim (du_l^2/dt)\Delta t, \quad (1.1)$$

where the first term represents the energy change of a packet during the collision, and $\Delta t \sim l_{\parallel}/V_A$ is the time for the wave packet to move through the oppositely directed wave packet of the size l_{\parallel} . To estimate the characteristic cascading rate, we assume that the cascading of a wave packet results from the change of its structure during the collision, which takes place at a rate u_l/l_{\perp} . Thus Eq. (1.1) becomes

$$\Delta E \sim \mathbf{u}_l \cdot \dot{\mathbf{u}}_l \Delta t \sim (u_l^3/l_{\perp})(l_{\parallel}/V_A), \quad (1.2)$$

The fractional energy change per collision is approximately the ratio of ΔE to E ,

$$f \equiv \frac{\Delta E}{u_l^2} \sim \frac{u_l l_{\parallel}}{V_A l_{\perp}}, \quad (1.3)$$

which provides a measure for the strength of the nonlinear interaction. Note that f is the ratio between the shearing rate of the wave packet u_l/l_{\perp} and the propagation rate of the wave packet V_A/l_{\parallel} . If the shearing rate is much smaller than the propagation rate, the perturbation of the wave packet during a single interaction is marginal and

$f \ll 1$. In this case, the cascading is a random walk process, which means that

$$\aleph = f^{-2}, \quad (1.4)$$

steps are required for the wave packet to be significantly distorted. That is, the cascading time is

$$t_{\text{cas}} \sim \aleph \Delta t. \quad (1.5)$$

Here we come to the important distinction between different regimes of Alfvénic turbulence. For $\aleph \gg 1$, the turbulence cascades weakly and the wave packet propagates along a distance much larger than its wavelength. This is the regime of *weak* Alfvénic turbulence, where the wave nature of turbulence is evident. In the opposite regime when $\aleph \approx 1$, the cascading happens within a single-wave-packet collision. In this regime the turbulence is strong and it exhibits the eddy-type behavior.

It is well known that the Alfvénic three-wave resonant interactions are governed by the relation of wavevectors, which reflects the momentum conservation, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$, and the relation of frequencies reflecting the energy conservation $\omega_1 + \omega_2 = \omega_3$ [62]. For the oppositely moving Alfvén wave packets with the dispersion relation $\omega = V_A / k_{\parallel}$, where $k_{\parallel} \sim l_{\parallel}^{-1}$ is the parallel component of the wavevector with respect to the local magnetic field, the perpendicular component of the wavevector $k_{\perp} \sim l_{\perp}^{-1}$ increases along with the interaction. The decrease of l_{\perp} with l_{\parallel} being fixed induces the increase of the energy change per collision. This decreases \aleph to its limiting value ~ 1 , breaking down the approximation of the weak Alfvénic turbulence.

For the critical value of $\aleph \approx 1$, the GS95 critical balanced condition

$$u_l l_{\perp}^{-1} \approx V_A l_{\parallel}^{-1} \quad (1.6)$$

is satisfied, with the cascading time equal to the wave period $\sim \Delta t$. Naturally, the value of \aleph cannot further decrease. Thus any further decrease of l_{\perp} , which happens as a result of wave packet interactions, must be accompanied by the corresponding decrease of l_{\parallel} , in order to keep the critical balance satisfied. As l_{\parallel} decreases, the frequencies of interacting waves increase, which at the first glance seems to contradict to the above consideration on the Alfvén wave cascading. However, there is no contradiction, since the cascading introduces the uncertainty in wave frequency ω of the order of $1/t_{\text{cas}}$.

As the turbulent energy cascades, the energy from one scale is transferred to another smaller scale over the time t_{cas} with only marginal energy dissipation. This energy conservation for turbulent energy flux in incompressible fluid can be presented as [63]:

$$\epsilon \approx u_l^2 / t_{\text{cas}} = \text{const}, \quad (1.7)$$

which in the hydrodynamic case provides the famous Kolmogorov law [64]:

$$\epsilon_{\text{hydro}} \approx u_l^3 / l \approx u_L^3 / L = \text{const}, \quad (1.8)$$