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Gaëtan Chenevier
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Automorphic Forms and Even Unimodular Lattices

Kneser Neighbors of Niemeier Lattices

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Preface

Automorphic forms are functions defined on *adele groups*, derived from harmonic analysis, whose theory forms a far-reaching generalization of that of modular forms. Langlands' famous *functoriality conjecture* predicts unexpected connections between automorphic forms associated with quite different groups. Recent advances confirm part of these general conjectures, as well as their refinements by Arthur, for the classical groups. The technicality of the proofs is formidable, but, in contrast, the statements are fascinating due to their extreme beauty, their wide range of applications, and to some extent their simplicity. Our aim in this book is to reconsider several problems of classical origin, from number theory and the theory of quadratic forms, in light of these recent results.

A special case, in which the Langlands conjectures nevertheless conserve all their flavor while being freed from numerous difficulties present in general, is that where one restricts oneself to studying automorphic forms that are *unramified at all primes*. These forms are also called *level 1* automorphic forms. When one deals with classical or Siegel modular forms, historic examples of automorphic forms if there ever were any, this assumption means that one considers only forms that are modular for the groups $SL_2(\mathbb{Z})$ or $Sp_{2g}(\mathbb{Z})$, and not for general congruence subgroups.

The interest of the case of level 1 automorphic forms does not lie uniquely in the simplifications it provides; it is also very appealing for the number-theorist because of the mix of scarcity and elegance of the examples (here too, think of modular forms for $SL_2(\mathbb{Z})$). Moreover, these forms are linked, sometimes very directly, sometimes much less so, and sometimes only conjecturally, to objects of algebraic geometry (varieties, stacks) that are both proper and smooth over the ring \mathbb{Z} of integers, and even to *motives* over the rational numbers with everywhere good reduction, objects that are as fascinating as they are mysterious.

In this work, we aim to study the conjectures of Arthur and Langlands in the context of level 1 automorphic forms, to give precise formulations of the statements arising from the work of Arthur in this framework, and to illustrate the latter through examples that are more specific but particularly spicy. We will also compare Arthur's results with those derived from more classical constructions, namely *theta series*,

which put numerous examples within reach. Some of these constructions turn out to be even richer, as we discovered, when they are combined with the *triality principle*. Let us emphasize that we wish to work, if possible, with groups of high rank, as they best reveal the richness of the general phenomena, and to move away from the classical examples provided by “small” groups such as GL_2 , which have already been the subject of an extensive literature.

Our illustrations will mainly concern the theory of quadratic forms over \mathbb{Z} that are nondegenerate and positive definite, in other words, the theory of *even (integral)* Euclidean lattices whose determinant is 1 or 2. This condition on the determinant means exactly that the associated projective quadric is smooth over \mathbb{Z} , in which case the associated special orthogonal group is smooth (and even reductive) over \mathbb{Z} . In the dimensions (less than or equal to 25) for which these objects are classified, the concrete problem we are going to address is the determination, for each prime number p , of the number of p -neighborhoods in Kneser’s sense, between the classes of such objects. We will call this the *p-neighbor problem*.

The p -neighbor problem allows for a quite elementary approach: this is the point of view that we chose to follow in the introduction (Chap. 1), and also in the organization of this book, where it will serve as a connecting thread. This will also make it possible to begin by exposing the rich and fascinating history of the subject, and to highlight some simple but striking statements that are consequences of our results (the dimension 16 case, the determination of the *p-neighborhood graphs* in dimension 24, the affirmation of the Nebe–Venkov conjecture on the linear combinations of higher-genus theta series of Niemeier lattices. . .). However, we think it is helpful to explain our original motivation beforehand, which was to test Arthur’s results in a context that is both concrete and of high dimension, a motivation that we will not emphasize in the beginning of the introduction.

In the remainder of this preface, we will explain the place of the p -neighbor problem in the general landscape of Langlands’ conjectures, or even motives, as well as the line of thought that led us to this problem. We hope that this enlightenment (or darkening depending on the viewpoint!) will arouse the interest of the readers who are maybe less sensitive to the appeal of the theory of Euclidean lattices. In any case, this passage will be inevitable in order to understand the ideas of the solution of the p -neighbor problem we propose, which uses, in a crucial way, the aforementioned recent developments. This apparent disproportion between the sophistication of methods and the elementary aspect of the p -neighbor problem is one of the charms of the latter.

The remainder of the preface will be organized as follows. First, we return, in a more precise way, to the notion of level 1 automorphic forms (studied in Chap. 4). After having discussed a few examples, we briefly present Langlands’ conjectures, emphasizing a statement that we call the *Arthur–Langlands conjecture* (Chaps. 6 and 8). We explain how Langlands and Arthur motivate this conjecture by means of a certain hypothetical group, the *Langlands group of \mathbb{Z}* , which we denote by $\mathbf{L}_{\mathbb{Z}}$. When one specializes the statements to *algebraic* automorphic forms, the Langlands group $\mathbf{L}_{\mathbb{Z}}$ can, to a large extent, be replaced by the absolute Galois group of \mathbb{Q} . We will then be in a position to provide the enlightenment we promised above and also

a glimpse at some of the problems still to be solved once Arthur’s results have been “put into the machine.”

AUTOMORPHIC FORMS OF LEVEL 1

Let us fix an algebraic group (scheme) G defined and *reductive* over the ring \mathbb{Z} of integers. This means that G is connected, smooth over \mathbb{Z} , and that its reduction modulo p is reductive over $\mathbb{Z}/p\mathbb{Z}$ for each prime p . The most important examples are GL_n and the famous Chevalley groups, or the groups that are isogenous to them such as PGL_n , but other examples will also play an important role further on.

The adèle group $G(\mathbb{A})$ is a locally compact topological group in a natural way, it is the *restricted product* of the real Lie group $G(\mathbb{R})$ and of the p -adic Lie groups $G(\mathbb{Q}_p)$ over all primes p ; the subgroup $G(\mathbb{Q})$ is discrete in $G(\mathbb{A})$. We denote by Z the neutral component of the center of $G(\mathbb{R})$ (so Z equals 1 if G is semisimple). The homogeneous space $G(\mathbb{Q})\backslash G(\mathbb{A})/Z$ is endowed with a finite $G(\mathbb{A})$ -invariant Borel measure. A central question is to describe the Hilbert space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z)$ of *square integrable automorphic forms of G* , viewed as a unitary representation of $G(\mathbb{A})$ for the right translations. In accordance with our objectives, we limit ourselves to considering the subspace

$$\mathcal{A}^2(G) = L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z \cdot G(\widehat{\mathbb{Z}}))$$

of automorphic forms of *level 1*, which is nothing but the subspace of $G(\widehat{\mathbb{Z}})$ -invariants of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z)$. This is a Hilbert space equipped with a natural unitary action of the group $G(\mathbb{R})$ and, for each prime p , with an action of the convolution ring

$$H_p(G) = \mathbb{Z}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)] ,$$

whose elements are the *Hecke operators* at p ; all these actions commute pairwise. The aim is to describe $\mathcal{A}^2(G)$ endowed with the commuting actions of the group $G(\mathbb{R})$ and of the commutative ring with unit $H(G) := \otimes_p H_p(G)$.

Denote by $\Pi(G)$ the set of isomorphism classes of objects of the form $\pi_\infty \otimes \pi_f$, with π_∞ an irreducible unitary representation of $G(\mathbb{R})$ and π_f a 1-dimensional complex representation of the ring $H(G)$. Such a π_f may equally be viewed as a collection of ring homomorphisms¹ $\pi_p: H_p(G) \rightarrow \mathbb{C}$; we also talk about *systems of eigenvalues of Hecke operators*. Moreover, denote by $m(\pi)$ the *multiplicity* of π as a subrepresentation of $\mathcal{A}^2(G)$; it is finite according to Harish-Chandra. A *level 1 discrete automorphic representation* of G (from here on, “level 1” will always be dropped from the terminology) is an element π of $\Pi(G)$ with $m(\pi) \neq 0$. Finally, denote by $\Pi_{\mathrm{disc}}(G) \subset \Pi(G)$ the subset of these representations. For general reasons,

¹ Here, we do not follow the tradition according to which π_p rather denotes the isomorphism class of the (irreducible) $\mathbb{C}[G(\mathbb{Q}_p)]$ -submodule of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z)$ generated by an arbitrary nonzero element of π . The difference is, however, artificial, as it is a well-known consequence of the commutativity of $H_p(G)$ that the two definitions contain exactly the same information.

we may write

$$\mathcal{A}^2(G) = \mathcal{A}_{\text{disc}}^2(G) \oplus \mathcal{A}_{\text{cont}}^2(G) \quad \text{with} \quad \mathcal{A}_{\text{disc}}^2(G) \simeq \bigoplus_{\pi \in \Pi_{\text{disc}}(G)}^{\perp} m(\pi) \pi. \quad (1)$$

The space $\mathcal{A}_{\text{disc}}^2(G)$ contains the subspace $\mathcal{A}_{\text{cusp}}^2(G)$ of *cuspidal forms*, whose definition is a natural generalization of that of a cuspidal modular form. We denote by $\Pi_{\text{cusp}}(G) \subset \Pi_{\text{disc}}(G)$ the subset of elements appearing in $\mathcal{A}_{\text{cusp}}^2(G)$. The description of the subsets $\Pi_{\text{cusp}}(G) \subset \Pi_{\text{disc}}(G)$ of $\Pi(G)$, and of the multiplicities $m(\pi)$ above, is the heart of the problem. Indeed, we know since Langlands how to describe the *continuous* part $\mathcal{A}_{\text{cont}}^2(G)$ in terms of the $\mathcal{A}_{\text{cusp}}^2(L)$, where L runs through the Levi subgroups of all the proper parabolic subgroups of G defined over \mathbb{Z} . We will not be interested in $\mathcal{A}_{\text{cont}}^2(G)$ in this book.

TWO EXAMPLES

The representations π in $\Pi_{\text{disc}}(G)$ have very different concrete manifestations depending on the nature of their *Archimedean component* π_{∞} . If U is an arbitrary irreducible unitary representation of $G(\mathbb{R})$, and if we set $\mathcal{A}_U(G) := \text{Hom}_{G(\mathbb{R})}(U, \mathcal{A}^2(G))$, then we have

$$\mathcal{A}_U(G) = \text{Hom}_{G(\mathbb{R})}(U, \mathcal{A}_{\text{disc}}^2(G)) \simeq \bigoplus_{\{\pi \in \Pi_{\text{disc}}(G) \mid \pi_{\infty} \simeq U\}} m(\pi) \pi_f.$$

This is an $H(G)$ -module in an obvious way, and a finite-dimensional complex vector space according to Harish-Chandra. It is equivalent to describe the whole of $\Pi_{\text{disc}}(G)$ or the $H(G)$ -modules $\mathcal{A}_U(G)$ when U runs through the unitary dual of $G(\mathbb{R})$.

In order to illustrate these notions, it is instructive to specify them in the special case of the group $G = \text{PGL}_2$.² If U is a *discrete series* representation, say with lowest weight the (even) integer $k > 0$, then $\mathcal{A}_U(G)$ can be naturally identified with the space of cusp forms of weight k for $\text{SL}_2(\mathbb{Z})$ endowed with the action of the standard Hecke operators on the latter. If $U := U_s$ is a *principal or complementary series*, parametrized in the usual way by an element $s \in i\mathbb{R} \cup [0, 1[$, then $\mathcal{A}_{U_s}(G)$ can be identified with the Hecke-module of cuspidal *Maass forms* with eigenvalue $(1 - s^2)/4$ for the action of the Laplace operator on the Poincaré upper half-plane. Contrary to the previous case, these spaces are very mysterious: Selberg has proved³ $\mathcal{A}_{U_s}(G) = 0$ for $s > 0$, but we do not know any exact value of s such that $\mathcal{A}_{U_s}(G)$ is nonzero, or whether the latter can be of dimension greater than 1. Finally, according to Bargmann, the unique remaining unitary representation of $\text{PGL}_2(\mathbb{R})$ is the trivial representation 1, and we obviously have $\dim \mathcal{A}_1(G) = 1$ (consider the constant functions).

² Following our definitions, we have a canonical isomorphism $\mathcal{A}^2(\text{PGL}_n) \xrightarrow{\sim} \mathcal{A}^2(\text{GL}_n)$.

³ This is an Archimedean analog of Ramanujan’s conjecture, still open for general congruence subgroups.

Let us now discuss the example that will be of great importance in this book. Let $n \geq 1$ be an integer and \mathbb{R}^n the standard Euclidean space of dimension n . It turns out that the (compact) special orthogonal group of \mathbb{R}^n is of the form $G(\mathbb{R})$ with G reductive over \mathbb{Z} if and only if the integer n is congruent to $-1, 0,$ or $+1$ modulo 8. Let us describe such a G under the assumption $n \equiv 0 \pmod 8$. It is well known that in this case, \mathbb{R}^n has *even unimodular lattices*. Such a lattice L is naturally endowed with an integral quadratic form, positive definite and nondegenerate over \mathbb{Z} . The associated orthogonal group (scheme) O_L is smooth over \mathbb{Z} , and its neutral component SO_L is semisimple over \mathbb{Z} , with real points $SO(\mathbb{R}^n)$.

We denote by \mathcal{L}_n the set of even unimodular lattices in \mathbb{R}^n . Any two elements of \mathcal{L}_n are in the same *genus*, that is, are isometric over \mathbb{Z}_p for every prime p (hence over the rationals as well, according to Hasse and Minkowski). This implies, first, that the space $\mathcal{A}^2(SO_L)$ depends in a nonessential way on the choice of the lattice L . In order to fix ideas, in this book, we will focus on the group $SO_n := SO_{E_n}$, where E_n denotes the *standard* even unimodular lattice generated by $\frac{1}{2}(1, \dots, 1)$ and the n -tuples of integers (x_1, \dots, x_n) with $\sum_i x_i$ even. Another consequence is that we have a natural identification

$$\mathcal{L}_n \xrightarrow{\sim} SO_n(\mathbb{Q}) \backslash SO_n(\mathbb{A}) / SO_n(\widehat{\mathbb{Z}})$$

that is compatible with the obvious actions of $SO_n(\mathbb{R})$ on both sides. If 1 denotes the trivial representation of $G(\mathbb{R})$, and if $\widetilde{X}_n = SO(\mathbb{R}^n) \backslash \mathcal{L}_n$ denotes the finite set of proper isometry classes of elements in \mathcal{L}_n , we therefore have natural isomorphisms

$$\mathcal{A}_1(SO_n) \simeq \{f : \widetilde{X}_n \rightarrow \mathbb{C}\} \simeq \bigoplus_{\{\pi \in \Pi_{\text{disc}}(SO_n) ; \pi_\infty \simeq 1\}} m(\pi) \pi_f .$$

The vector space $\mathbb{C}[\widetilde{X}_n]$, dual of $\mathcal{A}_1(SO_n)$, is therefore an $H(SO_n)$ -module in a natural way. For instance, if p is a fixed prime, it is an exercise to see that the endomorphism of $\mathbb{C}[\widetilde{X}_n]$ mapping the class of a lattice to the sum of the classes of its p -neighbors is induced by an element of $H_p(SO_n)$, which we denote by T_p . The determination of this endomorphism is exactly the problem considered at the beginning of the introduction.⁴ Let us add that the spaces $\mathcal{A}_U(SO_n)$, with U arbitrary (but necessarily finite-dimensional), have similar interpretations as spaces of $SO_n(\mathbb{R})$ -equivariant functions $\mathcal{L}_n \rightarrow U^*$; many such spaces will play a role in this book.

LANGLANDS' FUNCTORIALITY PRINCIPLE

Let us describe, rather briefly, Langlands' general conjectures in the case of level 1 automorphic forms. A starting point is the notion of *dual* group, introduced by

⁴ Actually, we will mostly consider the analogous, only slightly simpler, problem in which SO_n is replaced by $O_n := O_{E_n}$, whose only flaw is that it does not quite fit the conventions adopted here because O_n is not connected, but this slight difference is inessential.

Langlands. If G is reductive over \mathbb{Z} , its dual in the sense of Langlands is simply “the” complex linear algebraic reductive group, denoted by \widehat{G} , whose *based root datum* is *dual* (or *inverse*) to that of $G_{\mathbb{C}}$:

$G_{\mathbb{C}}$	GL_n	PGL_n	Sp_{2g}	PGSp_{2g}	SO_{2n+1}	SO_{2n}	PGSO_{2n}
\widehat{G}	GL_n	SL_n	SO_{2g+1}	Spin_{2g+1}	Sp_{2n}	SO_{2n}	Spin_{2n}

This group first allows Langlands to parametrize the elements of $\Pi(G)$. He observes that the *Satake isomorphism* provides a canonical bijection, for each prime p , between the set of ring homomorphisms $H_p(G) \rightarrow \mathbb{C}$ and the set of semisimple conjugacy classes in $\widehat{G}(\mathbb{C})$. In a similar way, he interprets the *infinitesimal character* (in the sense of Harish-Chandra) of a unitary representation of $G(\mathbb{R})$ as a semisimple conjugacy class in the Lie algebra of \widehat{G} . Finally, with each element π of $\Pi(G)$ is associated a collection of conjugacy classes

$$c(\pi) = (c_{\infty}(\pi), c_2(\pi), c_3(\pi), \dots)$$

that uniquely determines π_p for each prime p , as well as the infinitesimal character of π_{∞} , which only leaves finitely many possibilities for π_{∞} . These parametrizations, recalled in Chap. 6, have some very concrete aspects. For example, we will see that for π in $\Pi(\mathrm{SO}_n)$, we have the relation

$$\pi_p(\Gamma_p) = p^{n/2-1} \mathrm{trace} c_p(\pi). \tag{2}$$

Let G and G' be two reductive groups over \mathbb{Z} , and consider a morphism of algebraic groups $r: \widehat{G} \rightarrow \widehat{G}'$. Langlands’ *functoriality principle* predicts, for each *constituent* π of $\mathcal{A}^2(G)$, the existence of a constituent π' of $\mathcal{A}^2(G')$ that *corresponds* to π , in the sense that we have an equality of conjugacy classes $r(c_v(\pi)) = c_v(\pi')$ for each v in the set $V := \{\infty, 2, 3, 5, \dots\}$ of all places of \mathbb{Q} . It is only a principle, rather than a conjecture, as it is not quite accurate as stated, even if we give a reasonable sense to the term “constituent.” In what follows, we propose to make the statement of the functoriality principle precise in the important case $G' = \mathrm{GL}_n$, in which r is nothing but an n -dimensional representation of the algebraic group \widehat{G} . We will later refer to this statement as the *Arthur–Langlands conjecture*.

THE LANGLANDS GROUP OF \mathbb{Z}

Langlands observed that the formulation of his conjectures is enlightened if one assumes the existence of a certain group, which we will denote here by⁵ $\mathbf{L}_{\mathbb{Z}}$, whose

⁵ To be completely honest, Langlands considers a group that applies to all automorphic forms, rather than to level 1 forms only, of which our $\mathbf{L}_{\mathbb{Z}}$ would merely be a quotient [139, Sect. 2]. Moreover, following Arthur in [9, Sect. 8], we adopt Kottwitz’s point of view [129, Sect. 12] on the

\widehat{G} -valued representations parametrize the automorphic representations of G in an appropriate sense. We may think of this group as being an extension of the absolute Galois group of \mathbb{Z} (. . . trivial according to Minkowski!). For our needs in this preface, we only assume that $\mathbf{L}_{\mathbb{Z}}$ is a compact Hausdorff topological group (hence an inverse limit of compact Lie groups) satisfying the axioms denoted by (L1), (L2), and (L3) that we introduce below.

For every prime p , $\mathbf{L}_{\mathbb{Z}}$ is endowed with a conjugacy class Frob_p . Moreover, the complex pro-Lie-algebra of $\mathbf{L}_{\mathbb{Z}}$ is endowed with a semisimple conjugacy class Frob_{∞} . (L1)

Let G be reductive over \mathbb{Z} . Following Arthur and Langlands, we denote by $\Psi(G)$ the set of $\widehat{G}(\mathbb{C})$ -conjugacy classes of continuous group homomorphisms

$$\psi: \mathbf{L}_{\mathbb{Z}} \times \text{SL}_2(\mathbb{C}) \longrightarrow \widehat{G}(\mathbb{C}) \tag{3}$$

that are polynomial on the $\text{SL}_2(\mathbb{C})$ -factor. Such a ψ is called *discrete* if the centralizer C_{ψ} of $\text{Im } \psi$ in $\widehat{G}(\mathbb{C})$ is finite modulo the center $Z(\widehat{G})$ of $\widehat{G}(\mathbb{C})$. For example, if G is GL_n , in which case we also have $\widehat{G} = \text{GL}_n$ and ψ is nothing but an n -dimensional representation of $\mathbf{L}_{\mathbb{Z}} \times \text{SL}_2(\mathbb{C})$, then ψ is discrete if and only if it is an irreducible representation. We denote by $\Psi_{\text{disc}}(G) \subset \Psi(G)$ the subset of classes of discrete morphisms.

In parallel with what has been done for $\Pi(G)$, Arthur and Langlands associate with each ψ in $\Psi(G)$ a collection of conjugacy classes $c(\psi) = (c_v(\psi))_{v \in V}$ defined by $c_{\infty}(\psi) = \psi(\text{Frob}_{\infty}, e_{\infty})$ and $c_p(\psi) = \psi(\text{Frob}_p, e_p)$, where the e_v are the elements of $\mathfrak{sl}_2(\mathbb{C})$ for $v = \infty$ and of $\text{SL}_2(\mathbb{C})$ for $v = p$, defined by

$$e_{\infty} = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \text{and} \quad e_p = \begin{bmatrix} p^{-1/2} & 0 \\ 0 & p^{1/2} \end{bmatrix} .$$

For every integer $n \geq 1$, there is a unique bijection

$$\Pi_{\text{disc}}(\text{GL}_n) \xrightarrow{\sim} \Psi_{\text{disc}}(\text{GL}_n) , \quad \pi \mapsto \psi_{\pi} \tag{L2}$$

such that we have $c(\pi) = c(\psi_{\pi})$ for all $\pi \in \Pi_{\text{disc}}(\text{GL}_n)$. Moreover, ψ_{π} is trivial on $\text{SL}_2(\mathbb{C})$ if and only if we have $\pi \in \Pi_{\text{cusp}}(\text{GL}_n)$.

This axiom, together with the compactness of $\mathbf{L}_{\mathbb{Z}}$, implies that for any π in $\Pi_{\text{cusp}}(\text{GL}_n)$ and for any prime p , the eigenvalues of the conjugacy class $c_p(\pi)$ all have absolute value 1: this is the so-called *generalized Ramanujan conjecture*. It also shows⁶ $|\mathbf{L}_{\mathbb{Z}}^{\text{ab}}| = \dim \mathcal{A}(\text{GL}_1) = 1$.

Langlands group, which amounts to viewing it as a topological group rather than a pro-algebraic one as Langlands does. See [11] for another reference on the Langlands group.

⁶ We will prove that (L2) also implies that $\mathbf{L}_{\mathbb{Z}}$ is connected; see Proposition 9.3.4.

For every G reductive over \mathbb{Z} , there exists a decomposition

$$\mathcal{A}_{\text{disc}}(G) = \bigoplus_{\psi \in \Psi_{\text{disc}}(G)}^{\perp} \mathcal{A}_{\psi}(G), \tag{L3}$$

stable under $G(\mathbb{R})$ and $H(G)$ and satisfying the following property: if $\pi \in \Pi(G)$ appears in $\mathcal{A}_{\psi}(G)$, then we have $c(\pi) = c(\psi)$.

In particular, if a representation $\pi \in \Pi(G)$ appears in a summand $\mathcal{A}_{\psi}(G)$ as in (L3), then π satisfies the Ramanujan conjecture (in the sense that for every prime p , $c_p(\pi)$ is the conjugacy class of a “compact element” of $\widehat{G}(\mathbb{C})$) if and only if ψ is trivial on $1 \times \text{SL}_2(\mathbb{C})$. It is Arthur’s idea that the failure of Ramanujan’s conjecture may, in general, be entirely explained by the presence of $\text{SL}_2(\mathbb{C})$ in the definition of $\Psi(G)$ (formula (3)).

Arthur and Langlands strengthen the axiom (L3) by adding a converse statement, called the *multiplicity formula*, whose formulation, however, requires the introduction of more technical ingredients. Let us simply say that if $\psi \in \Psi_{\text{disc}}(G)$ and $\pi \in \Pi(G)$ satisfy $c(\pi) = c(\psi)$, this formula expresses the multiplicity of π in the subspace $\mathcal{A}_{\psi}(G)$ as the scalar product of two “explicit”⁷ characters of the finite group $C_{\psi}/Z(\widehat{G})$.

THE ARTHUR–LANGLANDS CONJECTURE

Let us go back to the statement of the Arthur–Langlands conjecture alluded to above. In order to state it, let us first assume the existence of a compact group $\mathbf{L}_{\mathbb{Z}}$ satisfying the axioms (L1), (L2), and (L3). Let G be reductive over \mathbb{Z} , π in $\Pi_{\text{disc}}(G)$, and $r: \widehat{G} \rightarrow \text{GL}_n$ a representation. Let $\psi \in \Psi_{\text{disc}}(G)$ be such that π appears in $\mathcal{A}_{\psi}(G)$; such a ψ exists by Axiom (L3). The decomposition into irreducibles of the representation $r \circ \psi$ of the direct product $\mathbf{L}_{\mathbb{Z}} \times \text{SL}_2(\mathbb{C})$ can be written as $\bigoplus_i r_i \otimes \text{Sym}^{d_i-1} \mathbb{C}^2$ for some irreducible representations r_i of dimension n_i of $\mathbf{L}_{\mathbb{Z}}$ and certain integers $d_i \geq 1$. By Axiom (L2), we have $r_i \simeq \psi_{\pi_i}$ for a unique π_i in $\Pi_{\text{cusp}}(\text{GL}_{n_i})$. In particular, for every $v \in V$, we have the identity between conjugacy classes

$$r(c_v(\pi)) = \bigoplus_i c_v(\pi_i) \otimes \text{Sym}^{d_i-1}(e_v) \tag{4}$$

(the reader will have no trouble deciphering the meaning of the right-hand side of this equality).

⁷ The definition of these characters is very delicate. One of them is a group homomorphism $C_{\psi}/Z(\widehat{G}) \rightarrow \mathbb{C}^{\times}$ defined by Arthur in [9, p. 55] with the help of the ε -factors of certain L -functions associated with ψ . The other one depends on the definition of a certain finite subset of irreducible unitary representations of $G(\mathbb{R})$ associated with ψ , denoted by $\Pi_{\infty}(\psi)$, nowadays usually called an *Arthur packet* [9, Sect. 4]. This character is nonzero if and only if π_{∞} belongs to $\Pi_{\infty}(\psi)$. In the important special case $C_{\psi} = Z(\widehat{G})$, the multiplicity of π in $\mathcal{A}_{\psi}(G)$ is thus nonzero if and only if we have $\pi_{\infty} \in \Pi_{\infty}(\psi)$.

As a consequence of this analysis, we have shown that the existence of a compact group $\mathbf{L}_{\mathbb{Z}}$ satisfying the axioms (L1), (L2) and (L3) implies the following statement, whose formulation does not involve $\mathbf{L}_{\mathbb{Z}}$: *For every reductive group G over \mathbb{Z} , every π in $\Pi_{\text{disc}}(G)$, and every representation $r: \widehat{G} \rightarrow \text{GL}_n$, there exists a unique collection of triples (d_i, n_i, π_i) , with $d_i, n_i \geq 1$ integers satisfying $n = \sum_i d_i n_i$ and π_i a representation in $\Pi_{\text{cusp}}(\text{GL}_{n_i})$, such that Equality (4) holds.* This is the precise form of the Arthur–Langlands conjecture that had been promised.

In his work mentioned earlier, Arthur proved the following special cases of this conjecture: $G_{\mathbb{Q}}$ is either the symplectic group Sp_{2g} of a symplectic space over \mathbb{Q} of dimension $2g$, or the special orthogonal group of a quadratic space of dimension $2n$ or $2n+1$ over \mathbb{Q} that has a totally isotropic subspace of dimension n , $\pi \in \Pi_{\text{disc}}(G)$ is arbitrary, and r is the natural representation of \widehat{G} , called the *standard representation*, whose dimension is $2g+1$, $2n$, and $2n$, respectively. For such groups, Arthur also proves a version of the multiplicity formula to which we alluded during the discussion of Axiom (L3). We will state more precise forms of Arthur’s results in Chap. 8. However, let us stress that we will not say anything about Arthur’s proofs; they go far beyond the scope of this work.

GALOIS REPRESENTATIONS AND MOTIVES

The group $\mathbf{L}_{\mathbb{Z}}$ is subject to several other conjectures. A most tempting one is that it satisfies the *Sato–Tate property*: the Frob_p are equidistributed in the set of conjugacy classes of $\mathbf{L}_{\mathbb{Z}}$, endowed with its invariant probability measure.⁸ In this section, we will instead discuss the conjectural relation between $\mathbf{L}_{\mathbb{Z}}$, Grothendieck motives, and Galois representations.

These links will only concern the quotient of $\mathbf{L}_{\mathbb{Z}}$, whose irreducible representations parametrize, in the sense of Axiom (L2), the representations π in $\Pi_{\text{cusp}}(\text{GL}_n)$ that are *algebraic*. Here, this adjective means that if we denote by λ_i the eigenvalues of the conjugacy class $c_{\infty}(\pi) \subset M_n(\mathbb{C})$, we have $\lambda_i - \lambda_j \in \mathbb{Z}$ for all i, j . We then denote by $w(\pi)$ the maximum of the differences $\lambda_i - \lambda_j$, and call it the *motivic weight* of π .

Denote by $\overline{\mathbb{Q}} \subset \mathbb{C}$ the subfield of algebraic numbers. Fix a prime ℓ , an algebraic closure $\overline{\mathbb{Q}}_{\ell}$ of the field of ℓ -adic numbers, and an embedding $\iota: \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_{\ell}$. Thanks to the works of a number of mathematicians (including Clozel, Deligne, Fontaine, Grothendieck, Langlands, Mazur, Serre, Shimura, Taniyama, Tate, Weil . . .), one conjectures the existence of a natural bijection $\pi \mapsto \rho_{\pi, \iota}$ between the set of algebraic π in $\Pi_{\text{cusp}}(\text{GL}_n)$ and the set of isomorphism classes of irreducible continuous representations $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_{\ell})$ that are *unramified* at each prime $p \neq \ell$ and *crystalline at ℓ* in the sense of Fontaine, with lowest *Hodge–Tate weight* 0. In particular, one requires this bijection to satisfy the equality⁹

⁸ Given the connectedness of $\mathbf{L}_{\mathbb{Z}}$, it would be easy to see, for instance, that this property implies the usual Sato–Tate conjecture for modular forms for $\text{SL}_2(\mathbb{Z})$.

⁹ This equality makes sense because we also conjecture that we have $\det(t - c_p(\pi)) \in \overline{\mathbb{Q}}[t]$ if π is algebraic.

$$\det(t - \rho_{\pi, \iota}(\text{Frob}_p)) = \iota(\det(t - p^{w(\pi)/2} c_p(\pi)))$$

for each prime $p \neq \ell$, which determines it uniquely.

This conjecture may readily be seen as an “algebraic” analog of Axiom (L2). Many difficult and important special cases of it are known. According to Fontaine and Mazur, one expects that the Galois representations above are exactly those appearing in the ℓ -adic realizations of pure motives over \mathbb{Q} with *everywhere good reduction*.

CONCLUSION

Let G be reductive over \mathbb{Z} and r a representation of \widehat{G} . As we have seen, the Arthur–Langlands conjecture predicts that for every π in $\Pi_{\text{disc}}(G)$, the collection of conjugacy classes $r(c(\pi))$ can be expressed in a very precise way in terms of *building blocks* that are elements π_i of $\Pi_{\text{cusp}}(\text{GL}_{n_i})$ and integers d_i with $\dim r = \sum_i n_i d_i$. Here are some questions that arise naturally: Assuming that a representation π in $\Pi_{\text{disc}}(G)$ is given, for instance such that π_f appears concretely in a specific $\mathcal{A}_U(G)$, can we determine the associated representations π_i and integers d_i ? Is it easier to determine them rather than π itself?

A first obstacle we encounter when trying to illustrate these questions is to have at our disposal examples of groups G and of irreducible unitary representations U of $G(\mathbb{R})$ for which we know how to determine whether $\mathcal{A}_U(G)$ is nonzero, or even better determine its dimension. When U is a discrete series representation, this is an accessible but notoriously difficult problem: for example, when we have $G = \text{Sp}_{2g}$, it contains the question of determining¹⁰ the dimension of spaces of Siegel modular cusp forms for $\text{Sp}_{2g}(\mathbb{Z})$. When U is not in the discrete series, it seems hopeless to obtain a formula for $\dim \mathcal{A}_U(G)$, as is shown by the example $G = \text{PGL}_2$.

The special case where $G(\mathbb{R})$ is compact, for which all the irreducible representations are in the discrete series, has the peculiar feature that the question of determining $\dim \mathcal{A}_U(G)$ is significantly more elementary. We will give many such examples with $G = \text{SO}_n$. The case $G = \text{SO}_{24}$ is especially interesting from this point of view, as it is one of the groups of highest rank for which $\dim \mathcal{A}_U(G)$ can be computed for at least *one* U (and with $\mathcal{A}_U(G) \neq 0$). Indeed, we have $\dim \mathcal{A}_1(G) = |\widetilde{X}_{24}|$, and this cardinality is 25 because the Leech lattice is the only one, among the 24 Niemeier lattices, not to admit any improper isometry. We are forced to ask ourselves the following question.

Question 1. Let r be the standard representation of $\widehat{\text{SO}}_{24}$ and π in $\Pi_{\text{disc}}(\text{SO}_{24})$ with $\pi_\infty = 1$; can we determine the collection of representations π_i and the integers d_i corresponding to π and r according to the Arthur–Langlands conjecture?¹¹

¹⁰ This determination is classical for $g = 1$, due to Igusa (in the scalar-valued case) and Tsushima (in the general case) for $g = 2$, to Tsuyumine for $g = 3$ (again in the scalar-valued case), and has been solved only very recently by Taïbi in general for $g \leq 7$.

¹¹ Observe that Arthur’s results do not immediately apply here because SO_n is not (quasi-)split over \mathbb{Q} . Nevertheless, we will prove that the Arthur–Langlands conjecture is satisfied when π and

This is the question at the origin of this work. Formulas (4) and (2) show that a positive answer to this question gives decisive information about the p -neighbor problem in dimension 24.

Before saying more about Question 1, let us add that the π_i that appear in its statement are not arbitrary: they are algebraic. More generally, if G is reductive over \mathbb{Z} and if π is in $\Pi_{\text{disc}}(G)$ with π_∞ a discrete series representation, then the eigenvalues of $c_\infty(\pi)$ in the adjoint representation of $\text{Lie } \widehat{G}$ are in \mathbb{Z} (Harish-Chandra); it follows that if r is an arbitrary representation of \widehat{G} , then the representations π_i associated with π and r by the Arthur–Langlands conjecture are necessarily algebraic. As a consequence, those π are related to motives and Galois representations, which makes them even more interesting. Those links are deep. We will show, for example, that Arthur’s multiplicity formula suggests that if π in $\Pi_{\text{cusp}}(\text{GL}_{8k})$ is algebraic, isomorphic to its *dual*, and if the eigenvalues of $c(\pi_\infty)$ are distinct integers, then there exists a π' in $\Pi_{\text{disc}}(\text{SO}_{8k})$ satisfying $r(c(\pi')) = c(\pi)$. These unexpected relations between Galois representations and even unimodular lattices clearly show the interest of studying $\Pi_{\text{disc}}(\text{SO}_n)$ for the number-theorist.

Let us return to Question 1. An obstacle we immediately faced, at least when we started working on this question, is that very few results were known about $\Pi_{\text{cusp}}(\text{GL}_n)$ with $n > 2$, even if we restrict ourselves to algebraic representations.¹² For instance, assuming that there exists a representation π in $\Pi_{\text{disc}}(\text{SO}_{24})$ satisfying $\pi_\infty = 1$ and such that one of the associated π_i is in $\Pi_{\text{cusp}}(\text{GL}_{n_i})$ with n_i big, it is very likely that we would never be able to say anything interesting either about this π , or about the p -neighbor problem in dimension 24. Note that we always have $n_i \leq 24$, but also $w(\pi_i) \leq 22$, as can be seen by considering $c_\infty(\pi)$.

One of our main results will be the proof in Sect. 9.3 of a classification of the automorphic representations π in $\Pi_{\text{cusp}}(\text{GL}_n)$, with $n \geq 1$ arbitrary, that are algebraic of motivic weight $w(\pi) \leq 22$. We will see that there are only 11 such representations and that they all appear (as π_i) in the answer to Question 1. We furthermore have $n \leq 4$ in all cases, with exactly four of the representations in $\Pi_{\text{cusp}}(\text{GL}_4)$. These four, which actually *come* from certain vector-valued Siegel cusp form of genus 2, will play an important role in this book.

The scope of the classification above is broader: for arbitrary G , the Arthur–Langlands conjecture suggests that every representation π in $\Pi_{\text{disc}}(G)$ with π_∞ in the discrete series and such that $c_\infty(\pi)$ is “small enough,” is *built* from the 11 automorphic representations mentioned above. For example, we will see how to use this approach to determine the dimension of the space of Siegel modular cusp forms of weight at most 12 for $\text{Sp}_{2g}(\mathbb{Z})$.

It seems reasonable to end this preface here, and to leave to the reader the pleasure of immersing themselves in the actual introduction of the book.

r are as in the statement of Question 1, by applying Arthur’s results to Sp_{2g} and using some theta series arguments.

¹² The situation is very different by now, thanks to the works [55] and [195]. Note that although these works were published before the present book, they were actually entirely motivated by it. Many important questions remain; for example, we do not know the number of algebraic π in $\Pi_{\text{cusp}}(\text{GL}_3)$ with a given Archimedean component π_∞ .

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Chapter 1

Introduction

1.1 Even Unimodular Lattices

Let $n \geq 1$ be an integer, and consider the Euclidean space \mathbb{R}^n endowed with the standard inner product $(x_i) \cdot (y_i) = \sum_i x_i y_i$. An even unimodular lattice of rank n is a lattice $L \subset \mathbb{R}^n$ with covolume 1 such that $x \cdot x$ is an even integer for all x in L . The set \mathcal{L}_n of these lattices is endowed with an action of the orthogonal group $O(\mathbb{R}^n)$; we denote the set of isometry classes of even unimodular lattices of rank n by

$$X_n := O(\mathbb{R}^n) \backslash \mathcal{L}_n .$$

To each L in \mathcal{L}_n , there corresponds a quadratic form

$$q_L : L \rightarrow \mathbb{Z} , \quad x \mapsto \frac{x \cdot x}{2} ,$$

whose associated bilinear form $x \cdot y$ has determinant 1. The map $L \mapsto q_L$ then induces a bijection between X_n and the set of isomorphism classes of positive definite quadratic forms over \mathbb{Z} of rank n and determinant 1.

As is well known, the set X_n is finite. It is nonempty if and only if $n \equiv 0 \pmod 8$. A standard example of an element of \mathcal{L}_n is the lattice

$$E_n := D_n + \mathbb{Z}e ,$$

where $D_n = \{(x_i) \in \mathbb{Z}^n ; \sum_i x_i \equiv 0 \pmod 2\}$, $e = \frac{1}{2}(1, 1, \dots, 1)$, and $n \equiv 0 \pmod 8$. Let us explain this notation. With each element L of \mathcal{L}_n is associated a root system

$$R(L) := \{x \in L ; x \cdot x = 2\}$$

of rank at most n . Its irreducible components are of type A_n , D_n , E_6 , E_7 , or E_8 ; we will say that it is “of type ADE.” The root system $R(E_8)$ is, for example, of type E_8 and generates the lattice E_8 over \mathbb{Z} . For $n > 8$, the root system $R(E_n)$ is of type D_n and generates D_n .

The set X_n has only been determined in dimension $n \leq 24$. Mordell and Witt proved, respectively,

$$X_8 = \{E_8\} \text{ and } X_{16} = \{E_{16}, E_8 \oplus E_8\}.$$

The two lattices E_{16} and $E_8 \oplus E_8$ will play an important role in this book. They are both easy and difficult to distinguish from each other: their root systems are different, but they represent each integer exactly the same number of times. This last, well-known, property leads, for example, to Milnor's isospectral tori.

The elements of X_{24} were classified by Niemeier [158], who proved, in particular, $|X_{24}| = 24$. Before saying more about these lattices, let us mention that for $n \geq 32$, the Minkowski–Siegel–Smith mass formula shows that the size of X_n explodes. For example, we have $|X_{32}| > 8 \cdot 10^6$ [177]; in fact, X_{32} even has more than 10^9 elements, as shown by King [118].

An element of \mathcal{L}_{24} is called a *Niemeier lattice*; the most famous one is the Leech lattice. Up to isometry, it is the only element L of \mathcal{L}_{24} with $R(L) = \emptyset$ (Conway). A remarkable fact is that if L is a Niemeier lattice that is not isomorphic to the Leech lattice, then $R(L)$ has rank 24 and all its irreducible components have the same Coxeter number. A simple proof of this was given by Venkov [201]. The miracle is then that the map $L \mapsto R(L)$ induces a bijection between $X_{24} - \{\text{Leech}\}$ and the set of isomorphism classes of root systems R of rank 24 and type ADE with components all having the same Coxeter number $h(R)$. The proof is a rather tedious case-by-case verification.

Table 1.1 The 23 roots systems of type ADE and rank 24 with components all having the same Coxeter number

R	D_{24}	$D_{16}E_8$	$3E_8$	A_{24}	$2D_{12}$	$A_{17}E_7$	$D_{10}2E_7$	$A_{15}D_9$
$h(R)$	46	30	30	25	22	18	18	16
R	$3D_8$	$2A_{12}$	$A_{11}D_7E_6$	$4E_6$	$2A_9D_6$	$4D_6$	$3A_8$	$2A_72D_5$
$h(R)$	14	13	12	12	10	10	9	8
R	$4A_6$	$4A_5D_4$	$6D_4$	$6A_4$	$8A_3$	$12A_2$	$24A_1$	
$h(R)$	7	6	6	5	4	3	2	

The results mentioned in this section are discussed in Chap. 2, which is mainly devoted to recalling classical results. We first develop prerequisites from bilinear and quadratic algebra necessary to understand the constructions of the quadratic forms to which we have alluded above, as well as others we will need. In particular, we recall Venkov's theory and explain the construction of certain Niemeier lattices. We also take the opportunity to recall some basic facts on classical group schemes over \mathbb{Z} that will be used later. Appendix B contains, among other things, a variant of the results of Chap. 2: in it, we study the even lattices in \mathbb{R}^n of determinant 2 as well as the corresponding theory of quadratic forms over \mathbb{Z} (in odd dimensions).

1.2 Kneser Neighbors

Let p be a prime. The notion of p -neighbors was introduced by M. Kneser; it can be viewed as a tool for constructing a set of even unimodular lattices from a given lattice and the prime p . In Chap. 3, we study several variations on this notion and give many examples.

Kneser defines two lattices L and M in \mathcal{L}_n to be p -neighbors if $L \cap M$ has index p in L (and therefore in M). It is easy to construct all p -neighbors of a given lattice L . Indeed, with any isotropic line ℓ in $L \otimes \mathbb{F}_p$, say generated by an element x of L satisfying $q_L(x) \equiv 0 \pmod{p^2}$, we can associate the even unimodular lattice¹

$$\text{vois}_p(L; \ell) := H + \mathbb{Z} \frac{x}{p},$$

where $H = \{y \in L; x \cdot y \equiv 0 \pmod{p}\}$ (the lattice above depends only on the choice of ℓ). The map $\ell \mapsto \text{vois}_p(L; \ell)$ induces a bijection between $C_L(\mathbb{F}_p)$ and the set of p -neighbors of L , where C_L denotes the projective (and smooth) quadric over \mathbb{Z} defined by $q_L = 0$. This quadric turns out to be hyperbolic over \mathbb{F}_p for every prime p , so the number of p -neighbors of L is exactly $|C_L(\mathbb{F}_p)| = 1 + p + p^2 + \dots + p^{n-2} + p^{n/2-1}$, which we will denote by $c_n(p)$.

Consider, for example, the element $\rho = (0, 1, 2, \dots, 23)$ of E_{24} . It generates an isotropic line in $E_{24} \otimes \mathbb{F}_{47}$ because of the congruence $\sum_{i=1}^{23} i^2 \equiv 0 \pmod{47}$. It is not very difficult to verify that $\text{vois}_{47}(E_{24}; \rho)$ does not have any roots, so that we have an isometry

$$\text{vois}_{47}(E_{24}; \rho) \simeq \text{Leech}.$$

This particularly simple construction of the Leech lattice is attributed to Thompson in [68]; we will return to it later. It illustrates the saying that many constructions of lattices are special cases of constructions of neighbors.

Returning to the general setting, for every L in \mathcal{L}_n , we have a partition of the quadric $C_L(\mathbb{F}_p)$ given by the isometry class of the associated p -neighbor. One of the aims of this book is to study this partition in dimension $n \leq 24$. For example, can we determine the number $N_p(L, M)$ of p -neighbors of L isometric to a given $M \in \mathcal{L}_n$? The first interesting case is, of course, that of dimension $n = 16$. To state the result, we introduce the linear map $T_p : \mathbb{Z}[X_n] \rightarrow \mathbb{Z}[X_n]$ defined by $T_p[L] = \sum[M]$, where the sum is taken over the p -neighbors M of L .

Theorem A. *In the basis $E_8 \oplus E_8$, E_{16} , the matrix of T_p is*

$$c_{16}(p) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1 + p + p^2 + p^3) \frac{1 + p^{11} - \tau(p)}{691} \begin{bmatrix} -405 & 286 \\ 405 & -286 \end{bmatrix},$$

where τ is Ramanujan's function defined by $q \prod_{m \geq 1} (1 - q^m)^{24} = \sum_{n \geq 1} \tau(n) q^n$.

¹ The notation *vois* comes from the French word *voisin* for *neighbor*.

For example, this theorem asserts that for every prime p , we have $N_p(E_8 \oplus E_8, E_{16}) = (405/691)(1 + p^{11} - \tau(p))(p^4 - 1)/(p - 1)$. This theorem is probably known to specialists, but we have not been able to find it stated this way in the literature. We will give several proofs of it further on. In view of the theory of theta series and modular forms for $SL_2(\mathbb{Z})$, the presence of $\tau(n)$ in the statement seems, at first sight, rather classical. For example, if we set $r_L(n) = |\{x \in L; x \cdot x = 2n\}|$, then we can easily show that we have $r_{\text{Leech}}(p) = (65520/691)(1 + p^{11} - \tau(p))$ for every prime p , a formula that resembles that of the theorem. Nevertheless, the presence of the term $\tau(p)(p^4 - 1)(p - 1)^{-1}$ in the formula for $N_p(E_8 \oplus E_8, E_{16})$ given above appears to be much more subtle; it will turn out to be equivalent to a nontrivial case of the Arthur–Langlands functoriality conjecture.²

Our main theorem is similar to Theorem A but concerns Niemeier lattices. We can state it in the same style as Theorem A, namely as an explicit formula for the matrix of T_p on $\mathbb{Z}[X_{24}]$, but the result is very hard to digest. This explicit formula involves rational coefficients with such large denominators that it appears quite exceptional that $N_p(L, M)$ is an integer! We will state a more conceptual (and equivalent) version of our result in Sect. 1.4 (Theorem E). A remarkable feature is that the statement involves all cuspidal modular forms of weight $k \leq 22$ for the group $SL_2(\mathbb{Z})$, as well as four vector-valued Siegel modular forms for $Sp_4(\mathbb{Z})$. Let us already discuss a number of consequences concerning the Niemeier lattices that follow from our formulas.

Consider the graph $K_n(p)$ with set of vertices X_n , where the classes of two nonisomorphic lattices L and M are joined by an edge if and only if $N_p(L, M) \neq 0$. Kneser proved that $K_n(p)$ is connected for all n and p , as a consequence of his famous strong approximation theorem. This nice result shows that we can, theoretically, reconstruct X_n from the lattice E_n and a prime p . Niemeier used this to compute X_{24} using 2-neighborhoods.

The graph $K_{16}(p)$ is the connected graph with 2 vertices (Kneser). This is, of course, compatible with the bound $|\tau(p)| < 2p^{11/2}$ (Deligne–Ramanujan) and the formula for $N_p(E_8 \oplus E_8, E_{16})$ given by Theorem A. On the other hand, the graph $K_{24}(2)$, determined by Borcherds [68], is not at all trivial. It has diameter 5 and the Wikipedia page http://en.wikipedia.org/wiki/Niemeier_lattice gives a nice representation of it, also due to Borcherds. Our results allow us, for example, to determine $K_{24}(p)$ for every prime p (Sect. 10.2).³

Theorem B. (i) *Let L be a Niemeier lattice with roots. Then L is a p -neighbor of the Leech lattice if and only if $p \geq h(\mathbb{R}(L))$.*
(ii) *The graph $K_{24}(p)$ is complete if and only if $p \geq 47$.*

² The comparison of Theorem A with the formula for $r_{\text{Leech}}(p)$ given above leads to the “purely quadratic” relation $N_p(E_8 \oplus E_8, E_{16}) = (9/1456) \cdot r_{\text{Leech}}(p) \cdot (p^4 - 1)(p - 1)^{-1}$, which we do not know how to prove directly.

³ A list of these graphs can be found at <http://gaetan.chenevier.perso.math.cnrs.fr/niemeier/niemeier.html>.

Statement (i) of this theorem concerns the constructions of the Leech lattice as a p -neighbor of a Niemeier lattice with roots. For example, on the Borcherds graph $K_{24}(2)$, we see that the distance from the Leech lattice to E_{24} is 5 and that the Leech lattice is linked only to the lattice with root system $24A_1$. (The latter is the Niemeier lattice with roots that is the most delicate to construct, as it needs the Golay code; see Sect. 2.3.) This last property is, in fact, quite easy to understand: if the Leech lattice is a 2-neighbor of a Niemeier lattice L (with roots), then L has an index 2 subgroup without any roots. In particular, $R(L)$ has the property that the sum of two roots is not a root, so that its irreducible components are of rank 1, which implies that $R(L) = 24A_1$. Of the root systems in Table 1.1, this is also the only one with Coxeter number 2, in accordance with statement (i).

The most elementary part of Theorem B, which is proved in Sect. 3.4 and generalizes the observation above, consists in verifying that $p \geq h(R(L))$ if the Leech lattice is a p -neighbor of L . This is a formal analog of a result of Kostant [128] asserting that the minimal order of a regular element of finite order in a connected, compact, adjoint Lie group coincides with the Coxeter number of its root system. The proof of the other statements, on the other hand, requires Theorem E as well as a number of Ramanujan-type inequalities. It will be completed only in Chap. 10 (Sects. 10.2 and 10.3).

In Chap. 3, we also study the limit cases of assertion (i) of Theorem B (with direct arguments, that is, without using Theorem E). For this, we carry out a detailed analysis of the elements c of $C_L(\mathbb{F}_p)$ satisfying $\text{vois}_p(L; c) \simeq \text{Leech}$, where L is a Niemeier lattice with nonempty root system $R = R(L)$. For the relevance of the statements, we need to study, more generally, the d -neighbors of L , where $d \geq 1$ is an integer that need not be prime (Sect. 3.1). We prove that if ρ is a Weyl vector of R and we set $h = h(R)$, then we have isometries (Theorem 3.4.2.10)

$$\text{vois}_h(L; \rho) \simeq \text{vois}_{h+1}(L; \rho) \simeq \text{Leech} . \quad (1.2.1)$$

This is well defined because $\rho \in L$ (Borcherds) and $q_L(\rho) = h(h+1)$ (Venkov). This statement contains, for example, the observation by Thompson mentioned earlier. In fact, these 23 (or 46) constructions of the Leech lattice are none other than the famous *holy constructions* of Conway and Sloane [67]. We, however, give a new proof of the isometries (1.2.1) using the theory of neighbors and show the identities

$$N_h(L, \text{Leech}) = \frac{|W|}{\varphi(h)g} \quad \text{and} \quad N_{h+1}(L, \text{Leech}) = \frac{|W|}{\varphi(h+1)} , \quad (1.2.2)$$

where W denotes the Weyl group of R and g^2 its index of connection in the sense of Bourbaki. We conclude Chap. 3 with an analysis of $\text{vois}_2(L; \rho)$ inspired by results of Borcherds (Fig. 3.1).

1.3 Theta Series and Siegel Modular Forms

Let us return to the determination of the operator T_p on $\mathbb{Z}[X_n]$. We begin with a few simple observations. The T_p commute and are self-adjoint for a suitable inner product on $\mathbb{R}[X_n]$ [156] (Sect. 3.2). We must therefore determine a basis of common eigenvectors of the T_p , as well as the corresponding sets of eigenvalues. There is only one obvious stable line, generated by $\sum_{L \in X_n} [L]/|O(L)|$, on which the operator T_p has “trivial” eigenvalue $c_n(p)$.

As hinted at in the preface, we are in fact dealing with a disguised problem belonging to the spectral theory of automorphic forms. Indeed, if $G = O_n$ denotes the orthogonal group scheme over \mathbb{Z} defined by the quadratic form q_{E_n} and \mathbb{A} the adèle ring of \mathbb{Q} , then arguments from genus theory lead to an isomorphism of $G(\mathbb{R})$ -sets $\mathcal{L}_n \simeq G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\widehat{\mathbb{Z}})$ (Sects. 2.2 and 4.1). Consequently, the dual of $\mathbb{R}[X_n]$ can be identified with the space of real-valued functions on $G(\mathbb{Q}) \backslash G(\mathbb{A})$ that are invariant under the right action of $G(\mathbb{R}) \times G(\widehat{\mathbb{Z}})$ by translation. In this description, the operator T_p is induced by a specific element of the ring $H(G)$ of Hecke operators of G .

These classical observations are recalled in Chap. 4. Although we are mainly interested in the automorphic forms for the \mathbb{Z} -group O_n , our statements and proofs will require the introduction of several variants (automorphic forms for SO_n , PGO_n and $PGSO_n$), as well as modular forms for $SL_2(\mathbb{Z})$, vector-valued Siegel modular forms for $Sp_{2g}(\mathbb{Z})$, and even, through Arthur’s results, automorphic forms for PGL_n . Therefore, from the beginning, we need to adopt a sufficiently general point of view embracing all these objects (Sect. 4.3). In Sects. 4.1 and 4.2, the reader can find an elementary exposition on Hecke operators. The emphasis is on the examples provided by the classical groups and their variants (Hecke, Satake, Shimura); these lead to a wider overview of p -neighbors and their generalizations. Sections 4.4 and 4.5 are devoted to recalling some properties of automorphic forms for O_n and Siegel modular forms. Let us emphasize that this chapter is intended for nonspecialists and does not pretend to any originality.

One approach to studying the $H(O_n)$ -module $\mathbb{Z}[X_n]$ is to examine the Siegel theta series $\vartheta_g(L)$ of each genus $g \geq 1$ of the elements L of \mathcal{L}_n . For every $n \equiv 0 \pmod 8$ and $g \geq 1$, they allow us to define a linear map

$$\vartheta_g : \mathbb{C}[X_n] \rightarrow M_{n/2}(Sp_{2g}(\mathbb{Z})) , \quad [L] \mapsto \vartheta_g(L) ,$$

where $M_k(Sp_{2g}(\mathbb{Z}))$ denotes the space of Siegel modular forms of weight $k \in \mathbb{Z}$ for $Sp_{2g}(\mathbb{Z})$ (Sect. 5.1). The relevance of this map for our problem comes from the generalized Eichler commutation relations; they assert that ϑ_g intertwines each element of $H(O_n)$ with an “explicit” element of $H(Sp_{2g})$ (Eichler, Freitag, Yoshida, Andrianov, Sect. 5.1). The map ϑ_g is trivially injective for $g \geq n$. It seems, however, quite difficult to determine the structure of the $H(Sp_{2g})$ -module $M_k(Sp_{2g}(\mathbb{Z}))$, especially for large g . Nevertheless, in Chap. 9, we develop a strategy that allows us to solve new cases of this problem. Our strategy relies, among other things, on results of Arthur [13].

The map ϑ_g has been widely studied. Its kernel, which shrinks when g increases, describes the linear relations between the theta series of genus g of the elements of \mathcal{L}_n , and determining its image is an example of Eichler's famous *basis problem*. More precisely, ϑ_g induces an injective map

$$\text{Ker } \vartheta_{g-1} / \text{Ker } \vartheta_g \longrightarrow S_{n/2}(\text{Sp}_{2g}(\mathbb{Z})), \quad (1.3.1)$$

where $S_k(\text{Sp}_{2g}(\mathbb{Z})) \subset M_k(\text{Sp}_{2g}(\mathbb{Z}))$ denotes the subspace of cusp forms (see Sect. 5.1 or footnote 4 below for the convention on ϑ_0), and Eichler asks whether this map is surjective. An important result of Böcherer [27] gives a necessary and sufficient condition for an eigenform for $H(\text{Sp}_{2g})$ to be in the image of (1.3.1), in terms of the vanishing of an associated L-function at the integer $n/2 - g$ (Sect. 7.2).

THE CASE $n = 16$

The case $n = 16$ is the subject of a famous story, recalled in Sect. 5.2. Indeed, a classical result of Witt and Igusa asserts that we have

$$\vartheta_g(\mathbb{E}_8 \oplus \mathbb{E}_8) = \vartheta_g(\mathbb{E}_{16}) \quad \text{if } g \leq 3. \quad (1.3.2)$$

These remarkable identities mean that $\mathbb{E}_8 \oplus \mathbb{E}_8$ and \mathbb{E}_{16} represent each positive integral quadratic form of rank at most 3 exactly the same number of times. This is well known in genus $g = 1$, as a consequence of the vanishing $S_8(\text{SL}_2(\mathbb{Z})) = 0$ (and leads to the isospectral tori of Milnor mentioned earlier). This, incidentally, shows⁴ that “the” nontrivial eigenvector of $\mathbb{Z}[X_{16}]$ is $[\mathbb{E}_{16}] - [\mathbb{E}_8 \oplus \mathbb{E}_8]$. The difficulty in genera 2 and 3 is that the vanishing of $S_8(\text{Sp}_{2g}(\mathbb{Z}))$, though still true, is more difficult to prove. In Appendix A, we will give another proof of the identities (1.3.2) which does not rely on any such vanishing results (that ingenious proof is due to Kneser).

The form $J = \vartheta_4(\mathbb{E}_8 \oplus \mathbb{E}_8) - \vartheta_4(\mathbb{E}_{16})$, which is nothing but the famous *Schottky form*, is easily shown to be nonzero. By results of Poor and Yuen [167], we even know that it generates $S_8(\text{Sp}_8(\mathbb{Z}))$. Theorem A then follows from the resolution by Ikeda [108] of the Duke–Imamoğlu conjecture [40]. Indeed, when applied to Jacobi's modular form Δ in $S_{12}(\text{SL}_2(\mathbb{Z}))$, Ikeda's theorem shows the existence of a nonzero Siegel modular form in $S_8(\text{Sp}_8(\mathbb{Z}))$ that is an eigenform for $H(\text{Sp}_8)$, with Hecke eigenvalues explicitly determined by the $\tau(p)$. Ikeda's proof is quite difficult; one of the main contributions of this book to Theorem A is to give another, very different, proof of Ikeda's result in this specific case.

Our main result is the following. For any map $f: \mathcal{L}_n \rightarrow \mathbb{C}$, we define $T_p(f): \mathcal{L}_n \rightarrow \mathbb{C}$ by setting $T_p(f)(L) = \sum_M f(M)$ for every $L \in \mathcal{L}_n$, where the sum is taken over the p -neighbors M of L . If $1 \leq g \leq n/2$, we denote by $H_{d,g}(\mathbb{R}^n)$ the space of polynomials $(\mathbb{R}^n)^g \rightarrow \mathbb{C}$ that are harmonic for the Euclidean

⁴ This assertion can be proved much more directly. Indeed, if ϑ_0 denotes the linear map $\mathbb{C}[X_n] \rightarrow \mathbb{C}$ that sends the class of any element of \mathcal{L}_n to 1, then we have $\vartheta_0 \circ T_p = c_n(p) \vartheta_0$, so that $\text{Ker } \vartheta_0$ is stable by T_p .

Laplace operator on $(\mathbb{R}^n)^g$ and satisfy $P \circ \gamma = (\det \gamma)^d P$ for all $\gamma \in \mathrm{GL}_g(\mathbb{C})$ (Sect. 5.4). This space is endowed with a linear action of $\mathrm{O}(\mathbb{R}^n)$.

Theorem C. *Let $q + \sum_{n>2} a_n q^n$ be a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ that is an eigenform for the Hecke operators, and let $d = k/2 - 2$. There exists a map $f: \mathcal{L}_8 \rightarrow \mathbb{C}$ such that*

- (i) *for every prime p , we have $T_p(f) = p^{-d} (p^4 - 1)(p - 1)^{-1} a_p f$;*
- (ii) *under the action of $\mathrm{O}(\mathbb{R}^8)$, the function f generates a representation isomorphic to $\mathrm{H}_{d,4}(\mathbb{R}^8)$.*

Section 5.4 is mainly devoted to proving a specific case of this theorem when $k = 12$, which leads to a complete and relatively elementary proof of Theorem A. The general case will be addressed and made more precise in Sect. 7.2.

Let us sketch the proof. We begin by realizing the initial modular form as a theta series $\sum_{x \in \mathbb{E}_8} P(x) q^{x \cdot x/2}$, where $P: \mathbb{R}^8 \rightarrow \mathbb{C}$ is a suitable harmonic polynomial. In the case of Δ , we verify that any nonzero harmonic polynomial of degree 8 which is invariant under the Weyl group $W(\mathbb{E}_8)$ does the trick, and in general we invoke a result of Waldspurger [205]. This construction defines a subspace of the functions $\mathcal{L}_8 \rightarrow \mathbb{C}$ with the following two properties: First, they are eigenvectors for the Hecke operators in $\mathrm{H}(\mathrm{O}_8)$, with eigenvalues related to the a_p by the Eichler commutation relations. Second, they generate a representation isomorphic to $\mathrm{H}_{8,1}(\mathbb{R}^8)$ under the action of $\mathrm{O}(\mathbb{R}^8)$. The main idea then consists in applying, at the source, an automorphism of \mathcal{L}_8 of order 3 arising from triality. Such an automorphism is constructed from a structure of Coxeter octonions on the lattice \mathbb{E}_8 and from an isomorphism $\mathcal{L}_8 \simeq G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\widehat{\mathbb{Z}})$, where $G = \mathrm{PGSO}_8$. The resulting functions satisfy the conditions of the theorem: we refer to Sect. 5.4 for the details.

Condition (ii) of Theorem C implies that the function f generates a Siegel theta series of genus 4 (with “pluriharmonic” coefficients). When nonzero, this theta series is a substitute for the Ikeda lift of genus 4 of the initial modular form. We show that it is nonzero when $k = 12$; Theorem A easily follows.

Finally, let us mention that we will prove the vanishing of $\mathrm{S}_8(\mathrm{Sp}_{2g}(\mathbb{Z}))$ for all $g \neq 4$ and $g \leq 8$ further on (Theorem 9.5.9). For $g = 5, 6$, it had already been obtained by Poor and Yuen [168] by different methods. Consequently, the map $\vartheta_g: \mathbb{C}[\mathrm{X}_{16}] \rightarrow \mathrm{M}_8(\mathrm{Sp}_{2g}(\mathbb{Z}))$ is surjective for every genus $1 \leq g \leq 8$.

THE CASE $n = 24$

This case is the subject of remarkable work by Erokhin [80], Borcherds–Freitag–Weissauer [31], and Nebe–Venkov [156] (Sect. 5.3). Erokhin showed that $\mathrm{Ker} \vartheta_{12} = 0$, and the three authors of [31] proved that $\mathrm{Ker} \vartheta_{11}$ has dimension 1. In [156], Nebe and Venkov undertook a detailed study of the entire filtration of $\mathbb{Z}[\mathrm{X}_{24}]$ given by the sequence of the $\mathrm{Ker} \vartheta_g$ for $g \geq 1$. Their starting point is an explicit expression for the operator T_2 on $\mathbb{Z}[\mathrm{X}_{24}]$, which they deduce from results of Borcherds (Sect. 3.3). They observe that the eigenvalues of T_2 are distinct integers, which allows them to

give an explicit basis of $\mathbb{Q}[X_{24}]$ consisting of common eigenvectors for all of the T_p . They also state a conjecture on the dimension of the image of the map (1.3.1) for every integer $1 \leq g \leq 12$, which they prove in many, but not all, cases. We establish their conjecture and even show that the Eichler basis problem admits a positive solution in dimension $n = 24$ for every genus $1 \leq g \leq 12$ (Theorem 9.5.2 and Corollary 9.5.6).

Theorem D. *The map $\vartheta_g : \mathbb{C}[X_{24}] \rightarrow M_{12}(\mathrm{Sp}_{2g}(\mathbb{Z}))$ is surjective and induces an isomorphism $\mathrm{Ker} \vartheta_{g-1} / \mathrm{Ker} \vartheta_g \xrightarrow{\sim} S_{12}(\mathrm{Sp}_{2g}(\mathbb{Z}))$ for every integer $g \leq 12$. The dimension of $S_{12}(\mathrm{Sp}_{2g}(\mathbb{Z}))$ for $g \leq 12$ is given by the following table:*

g	1	2	3	4	5	6	7	8	9	10	11	12
$\dim S_{12}(\mathrm{Sp}_{2g}(\mathbb{Z}))$	1	1	1	2	2	3	3	4	2	2	1	1

We will sketch the proof in Sect. 1.6; the most difficult part is the first assertion. The theorem leads to a complete description of the filtration $(\mathrm{Ker} \vartheta_g)_{g \geq 1}$ on $\mathbb{Z}[X_{24}]$. We note that the Eichler basis problem has a negative answer in dimension $n = 32$ and genus $g = 14$, as stated in Corollary 7.3.5.

1.4 Automorphic Forms for the Classical Groups

Siegel modular forms, as well as the automorphic forms for O_n , can be studied from the perspective of recent work by Arthur [13], published in 2013. However, in order to state the results, we first need to recall some basic features of the Langlands point of view on the theory of automorphic forms [135, 33], which we gather in Chap. 5. The main aspects of this point of view have already been touched upon in the preface. We briefly recall it.

Let G be a semisimple group scheme⁵ over \mathbb{Z} . We denote by $\Pi_{\mathrm{disc}}(G)$ the set of topologically irreducible subrepresentations of the space of square-integrable functions on $G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\widehat{\mathbb{Z}})$ for the natural actions of $G(\mathbb{R})$ and of the commutative ring $H(G)$ of Hecke operators of G (Sect. 4.3). The Satake isomorphism associates with each $\pi \in \Pi_{\mathrm{disc}}(G)$ and each prime p a semisimple conjugacy class $c_p(\pi)$ in $\widehat{G}(\mathbb{C})$, where \widehat{G} denotes the complex semisimple algebraic group that is dual to $G_{\mathbb{C}}$ in the sense of Langlands (Sects. 6.1 and 6.2). In Sect. 6.2.8, we make this enlightening point of view on the eigenvalues of the Hecke operators due to Langlands explicit in the case of the classical groups and Hecke operators we are interested in; we follow Gross' article [97]. Furthermore, we recall how the Harish-Chandra isomorphism

⁵ The discussion that follows does not apply verbatim to certain nonconnected group schemes that naturally occur here, such as O_n or PGO_n . We will, when necessary, indicate any modifications needed to include them, but in this introduction we will ignore this detail.