Springer Proceedings in Mathematics & Statistics

Colin C. Adams
Cameron McA. Gordon
Vaughan F. R. Jones
Louis H. Kauffman
Sofia Lambropoulou
Kenneth C. Millett
Jozef H. Przytycki
Renzo Ricca
Radmila Sazdanovic *Editors*

Knots, Low-Dimensional Topology and Applications

Knots in Hellas, International Olympic Academy, Greece, July 2016

Springer

Springer Proceedings in Mathematics & Statistics

Volume 284

Springer Proceedings in Mathematics & Statistics

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including operation research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

More information about this series at http://www.springer.com/series/10533

Colin C. Adams · Cameron McA. Gordon ·
Vaughan F. R. Jones · Louis H. Kauffman ·
Sofia Lambropoulou · Kenneth C. Millett ·
Jozef H. Przytycki · Renzo Ricca ·
Radmila Sazdanovic
Editors

Knots, Low-Dimensional Topology and Applications

Knots in Hellas, International Olympic Academy, Greece, July 2016

Springer

Editors
Colin C. Adams
Department of Mathematics
Williams College
Williamstown, MA, USA

Cameron McA. Gordon
Department of Mathematics
University of Texas at Austin
Austin, TX, USA

Vaughan F. R. Jones
Department of Mathematics
Vanderbilt University
Nashville, TN, USA

Louis H. Kauffman
Department of Mathematics, Statistics
and Computer Science
University of Illinois at Chicago
Chicago, IL, USA

Sofia Lambropoulou
School of Applied Mathematical
and Physical Sciences
National Technical University of Athens
Athens, Greece

Kenneth C. Millett
Department of Mathematics
University of California, Santa Barbara
Santa Barbara, CA, USA

Jozef H. Przytycki
Department of Mathematics, Columbian
College of Arts & Sciences
George Washington University
Washington, DC, USA

University of Gdańsk
Gdańsk, Poland

Renzo Ricca
Department of Mathematics
and Applications
University of Milano-Bicocca
Milan, Italy

Radmila Sazdanovic
Department of Mathematics
North Carolina State University
Raleigh, NC, USA

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-3-030-16030-2 ISBN 978-3-030-16031-9 (eBook)
https://doi.org/10.1007/978-3-030-16031-9

Library of Congress Control Number: 2019935549

Mathematics Subject Classification (2010): 90B05, 90B06, 90B10, 90B15, 90B18, 90B20, 90B22, 90B25, 90B30, 90B35, 90B36, 90B40, 90B50, 90B60, 90B70, 90B80, 90B85, 90B90, 90B99

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This collection of papers originates from the conference: *International Conference on Knots, Low-Dimensional Topology and Applications—Knots in Hellas 2016*. The conference was held at the International Olympic Academy, Ancient Olympia, Greece from July 17–23, 2016. The conference was an occasion to celebrate the 70th birthday of Louis H. Kauffman.

The website for the conference is: https://toce27.wixsite.com/knotsinhellas2016.

The link includes detailed information on the organization of the conference, such as copies of talks given at the conference, photos and videos, as well as the members of the International and Local Committees, to all of whom, as well as to the staff of the International Olympic Academy, we are indebted for their work toward the realization and the success of the Conference.

The goal of this international cross-disciplinary conference was to enable exchange of methods and ideas as well as exploration of fundamental research problems in the fields of knot theory and low-dimensional topology, from theory to applications in sciences like biology and physics, and to provide high-quality interactions across fields and generations of researchers, from graduate students to the most senior researchers. In this sense, this volume is one of the few published books covering and combining these topics.

This volume features cutting-edge research papers written by conference participants. The authors were asked to include illuminating state-of-the-art surveys and overviews of their research fields and of the topics they presented in the conference. The book is expected to be most useful for researchers who wish to expand their research to new directions, to learn about new tools and methods in the area, and need to find relevant and recent bibliography.

The focal topics include the wide range of classical and contemporary invariants of knots and links and related topics such as three- and four-dimensional manifolds, braids, virtual knot theory, quantum invariants, braids, skein modules and knot algebras, link homology, quandles, and their homology; hyperbolic knots and geometric structures of three-dimensional manifolds; and the mechanism of topological surgery in physical processes, knots in nature in the sense of physical knots with applications to polymers, DNA enzyme mechanisms, and protein structure and function.

We proceed now to give summaries of the chapters.

The chapter "A Survey of Hyperbolic Knot Theory" by David Futer, Efstratia Kalfagianni, and Jessica S. Purcell surveys tools and techniques for determining geometric properties of a link complement from a link diagram. In particular, it examines the tools used to estimate geometric invariants in terms of basic diagrammatic link invariants. The focus is on determining when a link is hyperbolic, estimating its volume, and bounding its cusp shape and cusp area. Sample applications are given, and open questions and conjectures are discussed.

The chapter "Spanning Surfaces for Hyperbolic Knots in the 3-Sphere" by Colin Adams studies surfaces with boundary a given knot in the 3-sphere. The paper considers such surfaces, both embedded and singular, for hyperbolic knots and discusses how the hyperbolic invariants affect the surfaces and how the surfaces affect the hyperbolic invariants.

The chapter "On the Construction of Knots and Links from Thompson's Groups" by Vaughan F. R. Jones reviews recent developments in the theory of Thompson group representations related to knot theory. It is a readable introduction to the topology of these new relationships.

The chapter "Virtual Knot Theory and Virtual Knot Cobordism" by Louis H. Kauffman is an introduction to virtual knot theory and virtual knot cobordism. Nontrivial examples of virtual slice knots are given and determinations of the four-ball genus of positive virtual knots are explained in relation to joint work with Dye and Kaestner. The paper studies the affine index polynomial, proves that it is a concordance invariant, shows that it is invariant also under certain forms of labeled cobordism, and studies a number of examples in relation to these phenomena. In particular, the paper shows how a mod-2 version of the affine index polynomial is a concordance invariant of flat virtual knots and links, and explores a number of examples in this domain.

The chapter "Knot Theory: From Fox 3-Colorings of Links to Yang–Baxter Homology and Khovanov Homology" by Józef H. Przytycki is an introduction to knot theory from the historical perspective. The chapter describes how the work of Ralph H. Fox was generalized to distributive colorings (rack and quandle) and eventually in the work of Jones and Turaev to link invariants via Yang-Baxter operators. By analogy to Khovanov homology, the paper builds homology of distributive structures (including homology of Fox colorings) and generalizes it to homology of Yang-Baxter operators.

The chapter "Algebraic and Computational Aspects of Quandle 2-Cocycle Invariant" by W. Edwin Clark and Masahico Saito studies quandle homology theories. These theories have been developed and their cocycles have been used to construct invariants in state-sum form for knots using colorings of knot diagrams by quandles. In this chapter, recent developments in these matters, as well as computational aspects of the invariants, are reviewed. Problems and conjectures pertinent to the subject are discussed.

The chapter "A Survey of Quantum Enhancements" by Sam Nelson is a survey article that summarizes the current state of the art in the nascent field of quantum enhancements, a type of knot invariant defined by collecting values of quantum invariants of knots with colorings by various algebraic objects over the set of such colorings. This class of invariants includes classical skein invariants and quandle and biquandle cocycle invariants as well as new invariants.

The chapter "From Alternating to Quasi-Alternating Links" by Nafaa Chbili introduces the class of quasi-alternating links and reviews some of their basic properties. In particular, the paper discusses the obstruction criteria for link quasi-alternateness introduced recently in terms of quantum link invariants.

The chapter "Hoste's Conjecture and Roots of the Alexander Polynomial" by Alexander Stoimenov studies the Alexander polynomial. The Alexander polynomial remains one of the most fundamental invariants of knots and links in 3-space. Its topological understanding has led a long time ago to a complete understanding about what (Laurent) polynomials can occur as the Alexander polynomial of an arbitrary knot. Ironically, the question to characterize the Alexander polynomials of alternating knots turns out to be far more difficult, even though in general alternating knots are much better understood. Hoste, based on computer verification, made the following conjecture about 15 years ago: If z is a complex root of the Alexander polynomial of an alternating knot, then $Re\, z \geq -1$. This paper discusses some results toward this conjecture, about 2-bridge (rational) knots or links, 3-braid alternating links, and Montesinos knots.

The chapter "A Survey of Grid Diagrams and a Proof of Alexander's Theorem" by Nancy Scherich studies grid diagrams in relation to classical knot theory and computer coding of knots and links. Grid diagrams are a representation of knot projections that are particularly useful as a format for algorithmic implementation on a computer. This paper gives an introduction to grid diagrams and demonstrates their programmable viability in an algorithmic proof of Alexander's theorem. Throughout, there are detailed comments on how to program a computer to encode the diagrams and algorithms.

The chapter "Extending the Classical Skein" by Louis H. Kauffman and Sofia Lambropoulou summarizes the skein-theoretic and combinatorial approaches to the new generalizations of skein polynomials for links. The first one of these generalizations, the invariant Θ that generalizes the HOMFLYPT polynomial, was discovered by Chlouveraki, Juyumaya, Karvounis, and the second author, and it has its roots in the Yokonuma–Hecke algebra of type A and a Markov trace defined on this algebra. The authors gave a skein-theoretic proof of the existence of Θ, while W.B. R. Lickorish gave a closed combinatorial formula for Θ. The authors also extend the Kauffman (Dubrovnik) polynomial to a new skein invariant for links and provide a Lickorish-type closed formula for this extension.

The chapter "From the Framisation of the Temperley–Lieb Algebra to the Jones Polynomial: An Algebraic Approach" by Maria Chlouveraki proves that the Framisation of the Temperley–Lieb algebra is isomorphic to a direct sum of matrix algebras over tensor products of classical Temperley–Lieb algebras. This result is used to obtain a closed combinatorial formula for the invariant for classical links obtained from a Markov trace on the Framisation of the Temperley–Lieb algebra. For a given link L, this formula involves the Jones polynomials of all sublinks of L, as well as linking numbers.

The chapter "A Note on $\mathfrak{gl}_{\mathrm{m}|\mathrm{n}}$ Link Invariants and the HOMFLY–PT Polynomial" by Hoel Queffelec and Antonio Sartori presents a short and unified representation-theoretical treatment of type A link invariants (that is, the HOMFLY-PT polynomials, the Jones polynomial, the Alexander polynomial, and, more generally, the $gl_{m|n}$ quantum invariants) as link invariants with values in the quantized oriented Brauer category.

The chapter "On the Geometry of Some Braid Group Representations" by Mauro Spera reports on recent differential geometric constructions that can produce representations of braid groups, together with applications in different domains of mathematical physics. The classical Kohno construction for the 3- and 4-strand pure braid groups is explicitly implemented by resorting to the Chen–Hain–Tavares nilpotent connections and to hyperlogarithmic calculus, yielding unipotent representations able to detect Brunnian and nested Brunnian phenomena. Physically motivated unitary representations of Riemann surface braid groups are then described, relying on Bellingeri's presentation and on the geometry of Hermitian–Einstein holomorphic vector bundles on Jacobians, via representations of Weyl–Heisenberg groups.

The chapter "Towards a Version of Markov's Theorem for Ribbon Torus-Links in $\mathbb{R}^4$" by Celeste Damiani studies ribbon torus-links embedded in $\mathbb{R}^4$. In classical knot theory, Markov's theorem gives a way of describing all braids with isotopic closures as links in $\mathbb{R}^3$. This paper presents a version of Markov's theorem for extended loop braids with closure in $B^3 \times S^1$, as a first step toward a Markov's theorem for extended loop braids and ribbon torus-links in $\mathbb{R}^4$.

The chapter "An Alternative Basis for the Kauffman Bracket Skein Module of the Solid Torus via Braids" by Ioannis Diamantis gives an alternative basis for the Kauffman bracket skein module of the solid torus. The new basis is obtained with the use of the Temperley–Lieb algebra of type B and it is appropriate for computing the Kauffman bracket skein module of the lens spaces $L(p, q)$ via braids.

The chapter "Knot Invariants in Lens Spaces" by Bostjan Gabrovsek and Eva Horvat summarizes results regarding the Kauffman bracket skein module, the HOMFLYPT skein module, and the Alexander polynomial of links in lens spaces, represented as mixed link diagrams. These invariants generalize the classical Kauffman bracket, the HOMFLYPT, and the Alexander polynomials, respectively. We compare the invariants by means of their ability to distinguish between some difficult cases of knots with certain symmetries.

The chapter "Identity Theorem for Pro-p-groups" by Andrey M. Mikhovich studies algebra related to knot theory and combinatorial group theory. The concept of schematization consists in replacing simplicial groups by simplicial affine group schemes. A schematic approach makes it possible to consider the problems of pro-p-group theory through the prism of Tannaka duality, concentrating on the category of representations.

The chapter "A Survey on Knotoids, Braidoids and Their Applications" by Neslihan Gügümcü, Louis H. Kauffman, and Sofia Lambropoulou is a survey of knotoids and braidoids, their theory and invariants, as well as their applications in the study of proteins. Knotoids were introduced by Turaev and they are represented by knot diagrams with ends such that the ends can inhabit different regions in the diagram. Equivalence is generated by Reidemeister moves that do not slide arcs across these free ends. New invariants of knotoids are constructed using the virtual closure and corresponding invariants in virtual knot theory. A version of the theory of braids is formulated for knotoids and applications of these structures to the study of proteins are described.

The chapter "Regulation of DNA Topology by Topoisomerases: Mathematics at the Molecular Level" by Rachel E. Ashley and Neil Osheroff studies the topology of DNA. Even though genetic information is encoded in a one-dimensional array of nucleic acid bases, three-dimensional relationships within DNA play a major role in how this information is accessed and utilized by living organisms. Because of the intertwined nature of the DNA 2-braid and its extreme length and compaction in the cell, some of the most important three-dimensional relationships in DNA are topological in nature. This article reviews the mathematics of DNA topology, describes the different classes of topoisomerases, and discusses the mechanistic basis for their actions in both biological and mathematical terms. It also discusses how topoisomerases recognize the topological states of their DNA substrates and products and how some of these enzymes distinguish supercoil handedness during catalysis and DNA cleavage.

The chapter "Topological Entanglement and Its Relation to Polymer Material Properties" by Eleni Panagiotou reviews recent results that show how measures of topological entanglement can be used to provide information relevant to dynamics and mechanics of polymers. The paper uses molecular dynamics simulations of coarse-grained models of polymer melts and solutions of linear chains in different settings. The paper applies the writhe to give estimates of the entanglement length and to study the disentanglement of polymer melts in an elongational flow.

The chapter "Topological Surgery in the Small and in the Large" by Stathis Antoniou, Louis H. Kauffman, and Sofia Lambropoulou directly connects topological changes that can occur in mathematical three-space via surgery, with black hole formation, the formation of wormholes, and new generalizations of these phenomena. This work enhances the bridge between topology and natural sciences and creates a new platform for exploring geometrical physics.

We hope the reader finds in this small collection of excellent papers a sense of the spirit of our conference and of the creativity of this topological subject.

Williamstown, USA	Colin C. Adams
Austin, USA	Cameron McA. Gordon
Nashville, USA	Vaughan F. R. Jones
Chicago, USA	Louis H. Kauffman
Athens, Greece	Sofia Lambropoulou
Santa Barbara, USA	Kenneth C. Millett
Washington, USA	Jozef H. Przytycki
Milan, Italy	Renzo Ricca
Raleigh, USA	Radmila Sazdanovic

Contents

A Survey of Hyperbolic Knot Theory

David Futer, Efstratia Kalfagianni and Jessica S. Purcell

Abstract We survey some tools and techniques for determining geometric properties of a link complement from a link diagram. In particular, we survey the tools used to estimate geometric invariants in terms of basic diagrammatic link invariants. We focus on determining when a link is hyperbolic, estimating its volume, and bounding its cusp shape and cusp area. We give sample applications and state some open questions and conjectures.

Keywords Hyperbolic knot · Hyperbolic link · Volume · Slope length · Cusp shape · Dehn filling

2010 Mathematics Subject Classification 57M25 · 57M27 · 57M50

1 Introduction

Every link $L \subset S^3$ defines a compact, orientable 3-manifold boundary consisting of tori; namely, the link exterior $X(L) = S^3 \backslash N(L)$, where $N(L)$ denotes an open regular neighborhood. The interior of $X(L)$ is homeomorphic to the link complement $S^3 \backslash L$. Around 1980, Thurston proved that link complements decompose into pieces that admit locally homogeneous geometric structures. In the most interesting scenario, the entire link complement has a hyperbolic structure, that is a metric of constant curvature -1. By Mostow–Prasad rigidity, this hyperbolic structure is

D. Futer
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA
e-mail: dfuter@temple.edu

E. Kalfagianni (✉)
Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
e-mail: kalfagia@math.msu.edu

J. S. Purcell
School of Mathematical Sciences, Monash University, Clayton, VIC 3800, Australia
e-mail: jessica.purcell@monash.edu

C. C. Adams et al. (eds.), *Knots, Low-Dimensional Topology and Applications*, Springer Proceedings in Mathematics & Statistics 284,
https://doi.org/10.1007/978-3-030-16031-9_1

unique up to isometry, hence geometric invariants of $S^3 \setminus L$ give topological invariants of L that provide a wealth of information about L to aid in its classification.

An important and difficult problem is to determine the geometry of a link complement directly from link diagrams, and to estimate geometric invariants such as volume and the lengths of geodesics in terms of basic diagrammatic invariants of L. This problem often goes by the names *WYSIWYG topology*[1] or *effective geometrization* [60]. Our purpose in this paper is to survey some results that effectively predict geometry in terms of diagrams, and to state some open questions. In the process, we also summarize some of the most commonly used tools and techniques that have been employed to study this problem.

1.1 Scope and Aims

This survey is primarily devoted to three main topics: determining when a knot or link is hyperbolic, bounding its volume, and estimating its cusp geometry. Our main goal is to focus on the methods, techniques, and tools of the field, in the hopes that this paper will lead to more research, rather than strictly listing previous results.

This focus overlaps significantly with the list of topics in Adams' survey article *Hyperbolic knots* [2]. That survey, written in 2003 and published in 2005, came out just as the pursuit of effective geometrization was starting to mature. Thus, although the topics are quite similar, both the results and the underlying techniques have advanced to a considerable extent. This is especially visible in efforts to predict hyperbolic volume (Sect. 4), where only a handful of the results that we list were known by 2003. The same pattern asserts itself throughout.

As with all survey articles, the list of results and open problems that we can address is necessarily incomplete. We are not addressing the very interesting questions on the geometry of embedded surfaces, lengths and isotopy classes of geodesics, exceptional Dehn fillings, or geometric properties of other knot and link invariants. Some of the results and techniques we have been unable to cover will appear in a forthcoming book in preparation by Purcell [76].

1.2 Originality, or Lack Thereof

With one exception, all of the results presented in this survey have appeared elsewhere in the literature. For all of these results, we point to references rather than giving rigorous proofs. However, we often include quick sketches of arguments to convey a sense of the methods that have been employed.

The one exception to this rule is Theorem 4.11, which has not previously appeared in writing. Even this result cannot be described as truly original, since the proof works by assembling a number of published theorems. We include the proof to indicate how to assemble the ingredients.

[1] *WYSIWYG* stands for "what you see is what you get".

1.3 Organization

We organize this survey as follows: Sect. 2 introduces terminology and background that we will use throughout. Section 3 is concerned with the problem of determining whether a given link is hyperbolic. We summarize some of the most commonly used methods used for this problem, and provide examples. In Sects. 4 and 5, we address the problem of estimating important geometric invariants of hyperbolic link complements in terms of diagrammatic quantities. In Sect. 4, we discuss methods for obtaining two sided combinatorial bounds on the hyperbolic volume of link complements. In Sect. 5, we address the analogous questions for cusp shapes and for lengths of curves on cusp tori.

1.4 Acknowledgements

Futer is supported in part by NSF grant DMS–1408682. Kalfagianni is supported in part by NSF grants DMS–1404754 and DMS–1708249. Purcell is supported in part by the Australian Research Council. All three authors acknowledge support from NSF grants DMS–1107452, 1107263, 1107367, “RNMS: Geometric Structures and Representation Varieties” (the GEAR Network).

2 Definitions

In this section, we gather many of the key definitions that will be used throughout the paper. Most of these definitions can be found (and are better motivated) in standard textbooks on knots and links, and on 3–manifolds and hyperbolic geometry. We list them briefly for ease of reference.

2.1 Diagrams of Knots and Links

Some of the initial study of knots and links, such as the work of Tait in the late 1800s, was a study of *diagrams*: projections of a knot or link onto a plane $\mathbb{R}^2 \subset \mathbb{R}^3$, which can be compactified to $S^2 \subset S^3$. We call the surface of projection the *plane of projection* for the diagram. We may assume that a link has a diagram that is a 4-valent graph on S^2, with over-under crossing information at each vertex. When studying a knot via diagrams, there are obvious moves that one can make to the diagram that do not affect the equivalence class of knot; for example these include *flypes* studied by Tait, shown in Fig. 1, and Reidemeister moves studied in the 1930s. Without going

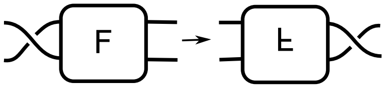

Fig. 1 A flype

into details on these moves, we do want our diagrams to be "sufficiently reduced," in ways that are indicated by the following definitions.

Definition 2.1 A diagram of a link is *prime* if for any simple closed curve $\gamma \subset S^2$, intersecting the diagram transversely in exactly two points in the interior of edges edges, γ bounds a disk $D^2 \subset S^2$ that intersects the diagram in a single arc with no crossings.

Two non-prime diagrams are shown in Fig. 2, left. The first diagram can be simplified by removing a crossing. The second diagram cannot be reduced in the same way, because the knot is composite; it can be thought of as composed of two simpler prime diagrams by joining them along unknotted arcs. Prime diagrams are seen as building blocks of all knots and links, and so we restrict to them.

Definition 2.2 Suppose K is a knot or link with diagram D. The *crossing number* of the diagram, denoted $c(D)$, is the number of crossings in D. The *crossing number* of K, denoted $c(K)$, is defined to be the minimal number of crossings in any diagram of K.

Removing a crossing as on the left of Fig. 2 gives a diagram that is more reduced. The following definition gives another way to reduce diagrams.

Definition 2.3 Let K be a knot or link with diagram D. The diagram is said to be *twist reduced* if whenever γ is a simple closed curve in the plane of projection intersecting the diagram exactly twice in two crossings, running directly through the crossing, then γ bounds a disk containing only a string of alternating bigon regions in the diagram. See Fig. 2, right.

Any diagram can be modified to be twist reduced by performing a sequence of flypes and removing unnecessary crossings.

Definition 2.4 Two crossings in a diagram D are called *twist equivalent* if they are connected by a string of bigons, as in the far right of Fig. 2. A *twist region* in D is

Fig. 2 Left: two diagrams that are not prime. Right: a twist reduced diagram

an equivalence class. We always require twist regions to be alternating (otherwise, D can be simplified by removing crossings).

The number of twist regions in a prime, twist reduced diagram is the *twist number* of the diagram, and is denoted $t(D)$. The minimum of $t(D)$ over all diagrams of K is denoted $t(K)$.

2.2 The Link Complement

Rather than study knots exclusively via diagrams and graphs, we typically consider the *knot complement*, namely the 3–manifold $S^3 \backslash K$. This is homeomorphic to the interior of the compact manifold $X(K) := S^3 \backslash N(K)$, called the *knot exterior*, where $N(K)$ is a regular neighborhood of K. When we consider knot complements and knot exteriors, we are able to apply results in 3–manifold topology, and consider curves and surfaces embedded in them. The following definitions apply to such surfaces.

Definition 2.5 An orientable surface S properly embedded in a compact orientable 3–manifold $\overline{M}$ is *incompressible* if whenever $E \subset \overline{M}$ is a disk with $\partial E \subset S$, there exists a disk $E' \subset S$ with $\partial E' = \partial E$. S is *∂-incompressible* if whenever $E \subset \overline{M}$ is a disk whose boundary is made up of an arc α on S and an arc on $\partial \overline{M}$, there exists a disk $E' \subset S$ whose boundary is made up of the arc α on S and an arc on ∂S.

Definition 2.6 Let $\overline{M}$ be a compact orientable 3–manifold. A two–sphere $S \subset M$ is called *essential* if it does not bound a 3–ball.

Consider a (possibly non-orientable) properly embedded surface $S \subset \overline{M}$. Let $\widetilde{S}$ be the boundary of a regular neighborhood $N(S) \subset \overline{M}$. If $S \neq S^2$, it is said to be *essential* if $\widetilde{S}$ is incompressible and ∂-incompressible.

We will say that $\overline{M}$ is *Haken* if it is irreducible and contains an essential surface S. In this case, we also say the interior M is Haken.

Finally, we will sometimes consider knot complements that are fibered, in the following sense.

Definition 2.7 A 3–manifold M is said to be *fibered* if it can be written as a fiber bundle over S^1, with fiber a surface. Equivalently, M is the mapping torus of a self-homeomorphism f of a (possibly punctured) surface S. That is, there exists $f : S \to S$ such that

$$M = S \times I / (x, 0) \sim (f(x), 1).$$

The map f is called the *monodromy* of the fibration.

2.3 Hyperbolic Geometry Notions

The knot and link complements that we address in this article also admit geometric structures, as in the following definition.

Definition 2.8 A knot or link K is said to be *hyperbolic* if its complement admits a complete metric of constant curvature -1. Equivalently, it is hyperbolic $S^3 \backslash K = \mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is hyperbolic 3–space and Γ is a discrete, torsion-free group of isometries, isomorphic to $\pi_1(S^3 \backslash K)$.

Thurston showed that a prime knot in S^3 is either hyperbolic, or it is a *torus knot* (can be embedded on an unknotted torus in S^3), or it is a *satellite knot* (can be embedded in the regular neighborhood of a non-trivial knot) [81]. This article is concerned with hyperbolic knots and links.

Definition 2.9 Suppose $\overline{M}$ is a compact orientable 3–manifold with ∂M a collection of tori, and suppose the interior $M \subset \overline{M}$ admits a complete hyperbolic structure. We say M is a *cusped manifold*.

Moreover, M has ends of the form $T^2 \times [1, \infty)$. Under the covering projection $\rho : \mathbb{H}^3 \to M$, each end is geometrically realized as the image of a horoball $H_i \subset \mathbb{H}^3$. The preimage $\rho^{-1}(\rho(H_i))$ is a collection of horoballs. By shrinking H_i if necessary, we can ensure that these horoballs have disjoint interiors in $\mathbb{H}^3$. For such a choice of H_i, $\rho(H_i) = C_i$ is said to be a *horoball neighborhood* of the *cusp* C_i, or *horocusp* in M.

Definition 2.10 The boundary of a horocusp inherits a Euclidean structure from the hyperbolic structure on M. This Euclidean structure is well defined up to similarity. The similarity class is called the *cusp shape*.

Definition 2.11 For each cusp of M there is an 1–parameter family of horoball neighborhoods obtained by expanding the horoball H_i while keeping the same limiting point on the sphere at infinity. In the preimage, expanding H_i expands all horoballs in the collection $\rho^{-1}(C_i)$. Expand each cusp until the collection of horoballs $\rho^{-1}(\cup C_i)$ become tangent, and cannot be expanded further while keeping their interiors disjoint. This is a choice of *maximal cusps*. The choice depends on the order of expansion of cusps $C_1, \ldots, C_n$. If M has a single end C_1 then there is a unique choice of expansion, giving a unique maximal cusp referred to as the *the maximal cusp* of M.

Definition 2.12 For a fixed set of embedded horoball neighborhoods $C_1, \ldots, C_n$ of the cusps of a cusped hyperbolic 3–manifold M, we have noted that the torus ∂C_i inherits a Euclidean metric. Any isotopy class of simple closed curves on the torus is called a *slope*. The *length of a slope* s, denoted $\ell(s)$, is defined to be the length of a geodesic representative of s on the Euclidean torus ∂C_i.

3 Determining Hyperbolicity

Given a combinatorial description of a knot or link, such as a diagram or braid presentation, one of the first things we would often like to ascertain is whether the link complement admits a hyperbolic structure. In this section, we describe the currently available tools to check this and give examples of knots to which they apply.

There are three main tools used to prove a link or family of links is hyperbolic. The first is direct calculation, for example using gluing and completeness equations, often with the help of a computer. The second is Thurston's geometrization theorem for Haken manifolds, which says that the only obstruction to $X(K)$ being hyperbolic consists of surfaces with non-negative Euler characteristic. The third is to perform a long Dehn filling on a manifold that is already known to be hyperbolic, for instance by one of the previous two methods.

3.1 Computing Hyperbolicity Directly

From Riemannian geometry, a manifold M admits a hyperbolic structure if and only if $M = \mathbb{H}^3/\Gamma$, where $\Gamma \cong \pi_1(M)$ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{PSL}(2, \mathbb{C})$. See Definition 2.8.

Therefore one way to find a hyperbolic structure on a link complement is to find a discrete faithful representation of its fundamental group into $\mathrm{PSL}(2, \mathbb{C})$. This is usually impractical to do directly. However, note that if a manifold M can be decomposed into simply connected pieces, for example a triangulation by tetrahedra, then these lift to the universal cover. If this cover is isometric to $\mathbb{H}^3$, then the lifted tetrahedra will be well-behaved in hyperbolic 3–space. Conversely, if the lifted tetrahedra fit together coherently in $\mathbb{H}^3$, in a group–equivariant fashion, one can glue the metrics on those tetrahedra to obtain a hyperbolic metric on M. This gives a condition for determining hyperbolicity, which is often implemented in practice.

Gluing and Completeness Equations for Triangulations

The first method for finding a hyperbolic structure is direct, and is used most frequently by computer, such as in the software SnapPy that computes hyperbolic structures directly from diagrams [29]. The method is to first decompose the knot or link complement into ideal tetrahedra, as in Definition 3.1, and then to solve a system of equations on the tetrahedra to obtain a hyperbolic structure. See Theorem 3.6.

This method is most useful for a single example, or for a finite collection of examples. For example, it was used by Hoste, Thistlethwaite, and Weeks to classify all prime knots with up to 16 crossings [55]. Of the 1, 701, 903 distinct prime knots with at most 16 crossings, all but 32 are hyperbolic.

We will give a brief description of the method. For further details, there are several good references, including notes of Thurston [80] where these ideas first appeared, and papers by Neumann and Zagier [71], and Futer and Guéritaud [35]. Purcell is developing a book with full details and examples [76].

Definition 3.1 An *ideal tetrahedron* is a tetrahedron whose vertices have been removed. When a knot or link complement is decomposed into ideal tetrahedra, all ideal vertices lie on the link, hence have been removed.

There are algorithms for decomposing knot and link complements into ideal tetrahedra. For example, Thurston decomposes the figure–8 knot complement into two ideal tetrahedra [80]. Menasco generalizes this, describing how to decompose a link complement into two ideal polyhedra, which can then be subdivided into tetrahedra [67]. Weeks uses a different algorithm in his computer software SnapPea [84].

Assuming we have a decomposition of a knot or link complement into ideal tetrahedra, we now describe how to turn this into a complete hyperbolic structure. The idea is to associate a complex number to each ideal edge of each tetrahedron encoding the hyperbolic structure of the ideal tetrahedron. The triangulation gives a complete hyperbolic structure if and only if these complex numbers satisfy certain equations: the *edge gluing* and *completeness* equations.

Consider $\mathbb{H}^3$ in the upper half space model, $\mathbb{H}^3 \cong \mathbb{C} \times (0, \infty)$. An ideal tetrahedron $\Delta \subset \mathbb{H}^3$ can be moved by isometry so that three of its vertices are placed at 0, 1, and ∞ in $\partial\mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$. The fourth vertex lies at a point $z \in \mathbb{C}\backslash\{0, 1\}$. The edges between these vertices are hyperbolic geodesics.

Definition 3.2 The parameter $z \in \mathbb{C}$ described above is called the *edge parameter* associated with the edge from 0 to ∞. It determines Δ up to isometry.

Notice if z is real, then the ideal tetrahedron is flat, with no volume. We will prefer to work with z with positive imaginary part. Such a tetrahedron Δ is said to be *geometric*, or positively oriented. If z has negative imaginary part, the tetrahedron Δ is negatively oriented.

Given a hyperbolic ideal tetrahedron embedded in $\mathbb{H}^3$ as above, we can apply (orientation–preserving) isometries of $\mathbb{H}^3$ taking different vertices to $0, 1, \infty$. By taking each edge to the geodesic from 0 to ∞, we assign edge parameters to all six edges of the ideal tetrahedron. This leads to the following relations between edge parameters:

Lemma 3.3 *Suppose Δ is a hyperbolic ideal tetrahedron with vertices at 0, 1, ∞, and z. Then the edge parameters of the six edges of Δ are as follows:*

- *Edges* $[0, \infty]$ *and* $[1, z]$ *have edge parameter* z.
- *Edges* $[1, \infty]$ *and* $[0, z]$ *have edge parameter* $1/(1 - z)$.
- *Edges* $[z, \infty]$ *and* $[0, 1]$ *have edge parameter* $(z - 1)/z$.

In particular, opposite edges in the tetrahedron have the same edge parameter.

Suppose an ideal tetrahedron Δ with vertices at $0, 1, \infty$ and z is glued along the triangle face with vertices at $0, \infty$, and z to another tetrahedron Δ'. Then Δ' will have vertices at $0, \infty, z$ and at the point zw, where w is the edge parameter of Δ' along the edge $[0, \infty]$. When we glue all tetrahedra in $\mathbb{H}^3$ around an ideal edge of the triangulation, if the result is hyperbolic then the product of all edge parameters must be 1 with arguments summing to 2π. More precisely, the sum of the logs of the edge parameters must be $0 + 2\pi\, i$.

Definition 3.4 (*Gluing equations*) Let e be an ideal edge of a triangulation of a 3–manifold M, for example a knot or link complement. Let $z_1, \ldots, z_k$ be the edge parameters of the edge of the tetrahedra identified to e. The *gluing equation* associated with the edge e is:

$$\prod_{i=1}^{k} z_i = 1 \quad \text{and} \quad \sum_{i=1}^{k} \arg(z_i) = 2\pi. \tag{1}$$

Writing this in terms of logarithms, (1) is equivalent to:

$$\sum_{i=1}^{k} \log(z_i) = 2\pi\, i. \tag{2}$$

A triangulation may satisfy all gluing equations at all its edges, and yet fail to give a complete hyperbolic structure. To ensure the structure is complete, an additional condition must be satisfied for each torus boundary component.

Definition 3.5 (*Completeness equations*) Let T be a torus boundary component of a 3–manifold M whose interior admits an ideal triangulation.

Truncate the tips of all tetrahedra to obtain a triangulation of T. Let μ be an oriented simple closed curve on T, isotoped to meet edges of the triangulation transversely, and to avoid vertices. Each segment of μ in a triangle cuts off a single vertex of the triangle, which comes from an edge of the ideal triangulation and so has an associated edge parameter z_i. If the vertex lies to the right of μ, let $\varepsilon_i = +1$; otherwise let $\varepsilon_i = -1$. The *completeness equation* associated to μ is:

$$\sum_i \varepsilon_i \, \log(z_i) = 0, \quad \text{which implies} \quad \prod_i z_i^{\varepsilon_i} = 1. \tag{3}$$

With these definitions, we may state the main theorem.

Theorem 3.6 *Suppose $\overline{M}$ is a 3-manifold with torus boundary, equipped with an ideal triangulation. Suppose for some choice of positively oriented edge parameters $\{z_1, \ldots, z_n\}$, the gluing equations are satisfied for each edge, and the completeness equations are satisfied for homology generators μ, λ on each component of $\partial\overline{M}$. Then the interior of $\overline{M}$, denoted by M, admits a complete hyperbolic structure. Furthermore, the unique hyperbolic metric on M is given by the geometric tetrahedra determined by the edge parameters.*

In fact, the hypotheses of Theorem 3.6 are stronger than necessary. If $\overline{M}$ has k torus boundary components, then only $n - k$ of the n gluing equations are necessary (see [71] or [28]). In addition, only one of μ or λ is required from each boundary component [28].

Some classes of 3–manifolds that can be shown to be hyperbolic using Theorem 3.6 include the classes of once-punctured torus bundles, 4-punctured sphere bundles, and 2–bridge link complements [49]. (In each class, some low-complexity

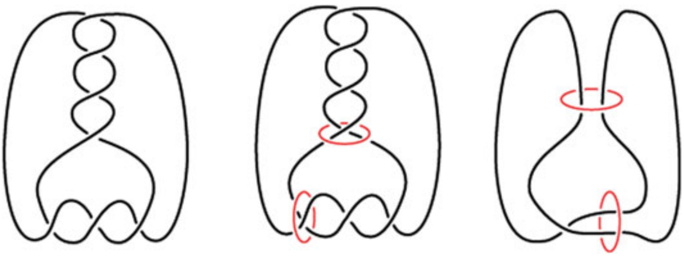

Fig. 3 Left: a diagram of a knot K. Center: adding a crossing circles for each twist region of K produces a link J. Right: removing full twists produces a *fully augmented link* L with the property that $S^3 \setminus J$ is homeomorphic to $S^3 \setminus L$

examples must be excluded to ensure hyperbolicity.) These manifolds have natural ideal triangulations guided by combinatorics. In the case of 2–bridge knot and link complements, the triangulation is also naturally adapted to a planar diagram of the link [78]. Once certain low-complexity cases (such as $(2, q)$ torus links) have been excluded, one can show that the gluing equations for these triangulations have a solution. This gives a direct proof that the manifolds are hyperbolic.

Circle Packings and Right Angled Polyhedra

Certain link complements have very special geometric properties that allow us to compute their hyperbolic structure directly, but with less work than solving nonlinear gluing and completeness equations as above. These include the Whitehead link, which can be obtained from a regular ideal octahedron with face-identifications [80]. They also include an important and fairly general family of link complements called *fully augmented links*, which we now describe.

Starting with any knot or link diagram, identify *twist regions*, as in Definition 2.4. The left of Fig. 3 shows a knot diagram with two twist regions. Now, to each twist region, add a simple unknotted closed curve encircling the two strands of the twist region, as shown in the middle of Fig. 3. This is called a *crossing circle*. Because each crossing circle is an unknot, we may perform a full twist along a disk bounded by that unknot without changing the homeomorphism type of the link complement.

This allows us to remove as many pairs of crossings as possible from twist regions. An example is shown on the right of Fig. 3. The result is the diagram of a fully augmented link.

Provided the original link diagram before adding crossing circles is sufficiently reduced (prime and twist reduced; see Definitions 2.1 and 2.3), the resulting fully augmented link will be hyperbolic, and its hyperbolic structure can be completely determined by a circle packing. The procedure is as follows.

Replace the diagram of the fully augmented link with a trivalent graph by replacing each neighborhood of a crossing circle (with or without a bounded crossing) by a single edge running between knot strands, closing the knot strands. See Fig. 4, left. Now take the dual of this trivalent graph; this is a triangulation of S^2. Provided the original diagram was reduced, there will be a circle packing whose nerve is this

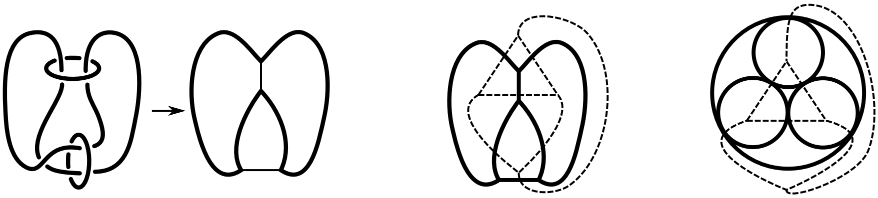

Fig. 4 Left: Obtain a 3-valent graph by replacing crossing circles with edges. Middle: The dual is a triangulation of S^2. Right: The nerve of the triangulation defines a circle packing that cuts out a polyhedron in $\mathbb{H}^3$. Two such polyhedra glue to form $S^3 \setminus L$

triangulation of S^2. The circle packing and its orthogonal circles cut out a right angled ideal polyhedron in $\mathbb{H}^3$. The hyperbolic structure on the complement of the fully augmented link is obtained by gluing two copies of this right angled ideal polyhedron. More details are in [44, 75].

3.2 Geometrization of Haken Manifolds

The methods of the previous section have several drawbacks. While solving gluing and completeness equations works well for examples, it is difficult to use the methods to find hyperbolic structures for infinite classes of examples. The method that has been most useful to show infinite examples of knots and links are hyperbolic is to apply Thurston's geometrization theorem for Haken manifolds, which takes the following form for manifolds with torus boundary components.

Theorem 3.7 (Geometrization of Haken manifolds) *Let M be the interior of a compact manifold $\overline{M}$, such that $\partial\overline{M}$ is a non-empty union of tori. Then exactly one of the following holds:*

- *$\overline{M}$ admits an essential torus, annulus, sphere, or disk, or*
- *M admits a complete hyperbolic metric.*

Thus the method to prove M is hyperbolic following Theorem 3.7 is to show $\overline{M}$ cannot admit embedded essential surfaces of nonnegative Euler characteristic. Arguments ruling out such surfaces are typically topological or combinatorial in nature.

Some sample applications of this method are as follows. Menasco used the method to prove any alternating knot or link, aside from a $(2, q)$-torus link, is hyperbolic [68]. Adams and his students generalized Menasco's argument to show that almost alternating and toroidally alternating links are hyperbolic [8, 9]. There are many other generalizations, e.g. [43].

Menasco's idea was to subdivide an alternating link complement into two balls, above and below the plane of projection, and crossing balls lying in a small neighborhood of each crossing, with equator along the plane of projection. An essential surface can be shown to intersect the balls above and below the plane of projection in disks only, and to intersect crossing balls in what are called *saddles*. These saddles act as fat vertices on the surface, and can be used to obtain a bound on the Euler characteristic of an embedded essential surface. Combinatorial arguments, using properties of alternating diagrams, then rule out surfaces with non-negative Euler characteristic.

More generally, classes of knots and links can be subdivided into simpler pieces, whose intersection with essential surfaces is then examined. Typically, surfaces with nonnegative Euler characteristic can be restricted to lie in just one or two pieces, and then eliminated.

Thurston's Theorem 3.7 can also be used to show that manifolds with certain properties are hyperbolic. For example, consider again the gluing equations. This gives a complicated nonlinear system of equations. If we consider only the imaginary part of the logarithmic gluing equation (2), the system becomes linear: the sums of dihedral angles around each edge must be 2π. It is much easier to solve such a system of equations.

Definition 3.8 Suppose M is the interior of a compact manifold with torus boundary, with an ideal triangulation. A solution to the imaginary part of the (logarithmic) gluing equations (2) for the triangulation is called a *generalized angle structure* on M. If all angles lie strictly between 0 and π, the solution is called an *angle structure*. See [35, 66] for background on (generalized) angle structures.

Theorem 3.9 (Angle structures and hyperbolicity) *If M admits an angle structure, then M also admits a hyperbolic metric.*

The proof has been attributed to Casson, and appears in Lackenby [62]. The idea is to consider how essential surfaces intersect each tetrahedron of the triangulation. These surfaces can be isotoped into *normal form*. A surface without boundary in normal form intersects tetrahedra only in triangles and in quads. The angle structure on M can be used to define a combinatorial area on a normal surface. An adaptation of the Gauss–Bonnet theorem implies that the Euler characteristic is a negative multiple of the combinatorial area. Then one shows that the combinatorial area of an essential surface must always be strictly positive, hence Euler characteristic is strictly negative. Then Theorem 3.7 gives the result.

Knots and links that can be shown to be hyperbolic using the tools of Theorem 3.9 include arborescent links, apart from three enumerated families of non-hyperbolic exceptions. This can be shown by constructing an ideal triangulation (or a slightly more general ideal decomposition) of the complement of an arborescent link, and endowing it with an angle structure [34].

Conversely, every hyperbolic knot or link complement in S^3 admits *some* ideal triangulation with an angle structure [52]. However, this triangulation is not explicitly constructed, and need not have any relation to the combinatorics of a diagram.

3.3 Hyperbolic Dehn Filling

Another method for proving that classes of knots or links are hyperbolic is to use Dehn filling. Thurston showed that all but finitely many Dehn fillings on a hyperbolic manifold with a single cusp yield a closed hyperbolic 3–manifold [80].

More effective versions of Thurston's theorem have been exploited to show hyperbolicity for all but a bounded number of Dehn fillings. Results in this vein include the 2π–theorem that yields negatively curved metrics [21], and geometric deformation theorems of Hodgson and Kerckhoff [51]. The sharpest result along these lines is the 6–Theorem, due independently to Agol [11] and Lackenby [62]. (The statement below assumes the geometrization conjecture, proved by Perelman shortly after the papers [11, 62] were published.)

Theorem 3.10 (6–Theorem) *Suppose M is a hyperbolic 3–manifold homeomorphic to the interior of a compact manifold $\overline{M}$ with torus boundary components $T_1, \dots, T_k$. Suppose $s_1, \dots, s_k$ are slopes, with $s_i \subset T_i$. Suppose there exists a choice of disjoint horoball neighborhoods of the cusps of M such that in the induced Euclidean metric on T_i, the slope s_i has length strictly greater than 6, for all i. Then the manifold obtained by Dehn filling along $s_1, \dots, s_k$, denoted $M(s_1, \dots, s_k)$, is hyperbolic.*

Theorem 3.10 can be used to prove that a knot or link is hyperbolic, as follows. First, show the knot complement $S^3 \setminus K$ is obtained by Dehn filling a manifold Y that is known to be hyperbolic. Then, prove that the slopes used to obtain $S^3 \setminus K$ from Y have length greater than 6 on a horoball neighborhood of the cusps of Y. See also Sect. 5 for ways to prove that slopes are long.

Some examples of links to which this theorem has been applied include *highly twisted links*, which have diagrams with 6 or more crossings in every twist region. (See Definition 2.4.) These links can be obtained by surgery, as follows. Start with a fully augmented link as described above, for instance the example shown in Fig. 3. Performing a Dehn filling along the slope $1/n$ on a crossing circle adds $2n$ crossings to the twist region encircled by that crossing circle, and removes the crossing circle from the diagram. When $|n| \geq 3$, the result of such Dehn filling on each crossing circle is highly twisted.

Using the explicit geometry of fully augmented links obtained from the circle packing, we may give a lower bound on the lengths of the slopes $1/n_i$ on crossing circles. Then Theorem 3.10 shows that the resulting knots and links must be hyperbolic [44].

Other examples can also be obtained in this manner. For example, Baker showed that infinite families of Berge knots are hyperbolic by showing they are Dehn fillings of minimally twisted chain link complements, which are known to be hyperbolic, along sequences of slopes that are known to grow in length [18].

The 6–Theorem is sharp. This was shown by Agol [11], and by Adams and his students for a knot complement [5]. The pretzel knot $P(n, n, n)$, which has 3 twist regions, and the same number of crossings in each twist region, has a toroidal Dehn filling along a slope with length exactly 6.

3.4 Fibered Knots and High Distance Knots

We finish this section with a few remarks about other ways to prove manifolds are hyperbolic, and give references for further information. However, these methods seem less directly applicable to knots in S^3 than those discussed above, and the full details are beyond the scope of this paper.

Recall Definition 2.7 of a fibered knot. When the monodromy is pseudo-Anosov, the knot complement is known to be hyperbolic [82]. The figure–8 knot complement can be shown to be hyperbolic in this way; see for example [80, p. 70]. Certain links obtained as the complement of closed braids and their braid axis have also been shown to be hyperbolic using these methods [50]. It seems difficult to apply these methods directly to knots, however.

Another method is to consider bridge surfaces of a knot. Briefly, there is a notion of distance that measures the complexity of the bridge splitting of a knot. Bachman and Schleimer proved that any knot whose bridge distance is at least 3 must be hyperbolic [17]. It seems difficult to bound bridge distance for classes of examples directly from a knot diagram. Recent work of Johnson and Moriah is the first that we know to obtain such bounds [61].

4 Volumes

As mentioned in the introduction, the goal of effective geometrization is to determine or estimate geometric invariants directly from a diagram. As volume is the first and most natural invariant of a hyperbolic manifold, the problem of estimating volume from a diagram has received considerable attention. In this section, we survey some of the results and techniques on both upper and lower bounds on volume.

4.1 Upper Bounds on Volume

Many bounds in this section involve constants with geometric meaning. In particular, we define

$$v_{\mathrm{tet}} = \text{volume of a regular ideal tetrahedron in } \mathbb{H}^3 = 1.0149\ldots$$

and

$$v_{\mathrm{oct}} = \text{volume of a regular ideal octahedron in } \mathbb{H}^3 = 3.6638\ldots$$

These constants are useful in combinatorial upper bounds on volume because every geodesic tetrahedron in $\mathbb{H}^3$ has volume at most v_{tet}, and every geodesic octahedron has volume at most v_{oct}. See e.g. Benedetti and Petronio [19].

Bounds in Terms of Crossing Number

The first volume bounds for hyperbolic knots are due to Adams [1]. He showed that, if $D = D(K)$ is a diagram of a hyperbolic knot or link with $c \geq 5$ crossings, then

$$\mathrm{vol}\,(S^3 \setminus K) \leq 4(c(D) - 4)v_{\mathrm{tet}}. \tag{4}$$

Adams' method of proof was to use the knot diagram to divide $S^3 \setminus K$ into tetrahedra with a mix of ideal and material vertices, and to count the tetrahedra. Since the subdivision contains at most $4(c(D) - 4)$ tetrahedra, and each tetrahedron has volume at most v_{tet}, the bound follows.

In a more recent paper [3], Adams improved the upper bound of (4):

Theorem 4.1 *Let $D = D(K)$ be a diagram of a hyperbolic link K, with at least* 5 *crossings. Then*

$$\mathrm{vol}\,(S^3 \setminus K) \leq (c(D) - 5)v_{\mathrm{oct}} + 4v_{\mathrm{tet}}.$$

Again, the method is to divide the link complement into a mixture of tetrahedra and octahedra, and to bound the volume of each polyhedron by v_{tet} or v_{oct} respectively. The subdivision into octahedra was originally described by D. Thurston.

The upper bound of Theorem 4.1 is known to be asymptotically sharp, in the sense that there exist diagrams of knots and links K_n with $\mathrm{vol}\,(S^3 \setminus K_n)/c(K_n) \to v_{\mathrm{oct}}$ as $n \to \infty$; see [26]. On the other hand, this upper bound can be arbitrarily far from sharp. A useful example is the sequence of twist knots K_n depicted in Fig. 5. Since the number of crossings is $n + 2$, the upper bound of Theorem 4.1 is linear in n. However, the volumes of K_n are universally bounded and only increasing to an asymptotic limit:

$$\mathrm{vol}\,(S^3 \setminus K_n) < v_{\mathrm{oct}}, \qquad \lim_{n \to \infty} \mathrm{vol}\,(S^3 \setminus K_n) = v_{\mathrm{oct}}$$

This holds as a consequence of the following theorem of Gromov and Thurston [80, Theorem 6.5.6].

Theorem 4.2 *Let M be a finite volume hyperbolic manifold with cusps. Let $N = M(s_1, \ldots, s_n)$ be a Dehn filling of some cusps of M. Then* $\mathrm{vol}\,(N) < \mathrm{vol}\,(M)$.

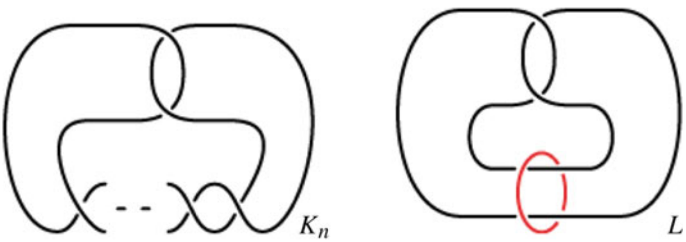

Fig. 5 Every twist knot K_n has two twist regions, consisting of 2 and n crossings. Every K_n can be obtained by Dehn filling the red component of the Whitehead link L, depicted on the right

Returning to the case of twist knots, every K_n can be obtained by Dehn filling on one component of the Whitehead link L, depicted in Fig. 5, right. Thus Theorem 4.2 implies $\mathrm{vol}\,(S^3 \setminus K_n) < \mathrm{vol}\,(S^3 \setminus L) = v_{\mathrm{oct}}$.

Bounds in Terms of Twist Number

Following the example of twist knots in Fig. 5, it makes sense to seek upper bounds on volume in terms of the twist number $t(K)$ of a knot K (see Definition 2.4), rather than the crossing number alone.

The following result combines the work of Lackenby [63] with an improvement by Agol and D. Thurston [63, Appendix].

Theorem 4.3 *Let $D(K)$ be a diagram of a hyperbolic link K. Then*

$$\mathrm{vol}\,(S^3 \setminus K) \leq 10(t(D) - 1)v_{\mathrm{tet}}.$$

Furthermore, this bound is asymptotically sharp, in the sense that there exist knot diagrams $D_n = D(K_n)$ with $\mathrm{vol}\,(S^3 \setminus K_n)/t(D_n) \to 10v_{\mathrm{tet}}$.

The method of proof is as follows. First, one constructs a *fully augmented link* L, by adding an extra component for each twist region of $D(K)$ (see Fig. 3). As described in Sect. 3.1, the link complement $S^3 \setminus L$ has simple and explicit combinatorics, making it relatively easy to bound $\mathrm{vol}\,(S^3 \setminus L)$ by counting tetrahedra. Then, Theorem 4.2 implies that the same upper bound on volume applies to $S^3 \setminus K$.

As a counterpart to the asymptotic sharpness of Theorem 4.3, there exist sequences of knots where $t(K_n) \to \infty$ but $\mathrm{vol}\,(S^3 \setminus K_n)$ is universally bounded. One family of such examples is the *double coil knots* studied by the authors [40].

Subsequent refinements or interpolations between Theorems 4.1 and 4.3 have been found by Dasbach and Tsvietkova [30, 31] and Adams [4]. These refinements produce a smaller upper bound compared to that of Theorem 4.3 when the diagram $D(K)$ has both twist regions with many crossings and with few crossings. However, the worst case scenario for the multiplicative constant does not improve due to the asymptotic sharpness of Theorems 4.1 and 4.3.

4.2 Lower Bounds on Volume

By results of Jorgensen and Thurston [80], the volumes of hyperbolic 3–manifolds are well-ordered. It follows that every family of hyperbolic 3–manifolds (e.g. link complements; fibered knot complements, knot complements of genus 3, etc.) contains finitely many members realizing the lowest volume. Gabai, Meyerhoff, and Milley [46] showed that the three knot complements of lowest volume are the figure-8 knot, the 5_2 knot, and the $(-2, 3, 7)$ pretzel, whose volumes are

$$\mathrm{vol}\,(4_1) = 2v_{\mathrm{tet}} = 2.0298\ldots, \qquad \mathrm{vol}\,(5_2) = \mathrm{vol}\,(P(-2, 3, 7)) = 2.8281\ldots. \tag{5}$$

Agol [12] showed that the two multi-component links of lowest volume are the Whitehead link and the $(-2, 3, 8)$ pretzel link, both of which have volume $v_{\mathrm{oct}} = 3.6638\ldots$. Yoshida [86] has identified the smallest volume link of 4 components, with volume $2v_{\mathrm{oct}}$. Beyond these entries, lower bounds applicable to *all* knots (or *all* links) become scarce. Not even the lowest volume link of 3 components is known to date.

Nevertheless, there are several practical methods of obtaining diagrammatic lower bounds on the volume of a knot or link, each of which applies to an infinite family of links, and each of which produces *scalable* lower bounds that become larger as the complexity of a diagram becomes larger. We survey these methods below.

Angle Structures

Suppose that $S^3 \setminus K$ has an ideal triangulation τ supporting an angle structure θ. (Recall Definition 3.8.) Every ideal tetrahedron of τ, supplied with angles via θ, has an associated volume. As a consequence, one may naturally define a volume $\mathrm{vol}\,(\theta)$ by summing the volumes of the individual tetrahedra.

Conjecture 4.4 (Casson) *Let τ be an ideal triangulation of a hyperbolic manifold M, which supports an angle structure θ. Then*

$$\mathrm{vol}\,(\theta) \leq \mathrm{vol}\,(M),$$

with equality if and only if θ solves the gluing equations and gives the complete hyperbolic structure on M.

While Conjecture 4.4 is open in general, it is known to hold if the triangulation τ is *geometric*, meaning that some (possibly different) angle structure θ' solves the gluing equations on τ. In this case, a theorem of Casson and Rivin [35, 77] says that θ' uniquely maximizes volume over all angle structures on τ, implying in particular that $\mathrm{vol}\,(\theta) \leq \mathrm{vol}\,(\theta') = \mathrm{vol}\,(M)$.

In particular, the known case of Conjecture 4.4 has been applied to the family of 2–bridge links. In this case, the link complement has a natural angled triangulation whose combinatorics is closely governed by the link diagram [49, Appendix]. It follows that, for a sufficiently reduced diagram D of a 2–bridge link K,

$$2v_{\mathrm{tet}}t(D) - 2.7066 \leq \mathrm{vol}\,(S^3 \setminus K) \leq 2v_{\mathrm{oct}}(t(D) - 1), \tag{6}$$

which both sharpens the upper bound of Theorem 4.3 and proves a comparable lower bound.

There are rather few other families where this method has been successfully applied. One is the weaving knots studied by Champanerkar, Kofman, and Purcell [27].

In the spirit of open problems, we mention the family of fibered knots and links. Agol showed that these link complements admit combinatorially natural *veering triangulations* [13], which have angle structures with nice properties [36, 53]. A proof of Conjecture 4.4, even for this special family, would drastically expand the list