SPRINGER PROCEEDINGS IN PHYSICS 118

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106 **Modern Trends in Geomechanics**
Editors: W. Wu and H.S. Yu

107 **Microscopy of Semiconducting Materials**
Proceedings of the 14th Conference, April 11–14, 2005, Oxford, UK
Editors: A.G. Cullis and J.L. Hutchison

108 **Hadron Collider Physics 2005**
Proceedings of the 1st Hadron Collider Physics Symposium, Les Diablerets, Switzerland, July 4–9, 2005
Editors: M. Campanelli, A. Clark, and X. Wu

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Proceedings of the iTi Conference in Turbulence 2005
Editors: M. Oberlack, G. Khujadze, S. Guenther, T. Weller, M. Frewer, J. Peinke, S. Barth

110 **Nonequilibrium Carrier Dynamics in Semiconductors**
Proceedings of the 14th International Conference, July 25–29, 2005, Chicago, USA
Editors: M. Saraniti, U. Ravaioli

111 **Vibration Problems ICOVP 2005**
Editors: E. Inan, A. Kiris

112 **Experimental Unsaturated Soil Mechanics**
Editor: T. Schanz

113 **Theoretical and Numerical Unsaturated Soil Mechanics**
Editor: T. Schanz

114 **Advances in Medical Engineering**
Editor: T.M. Buzug

115 **X-Ray Lasers 2006**
Proceedings of the 10th International Conference, August 20–25, 2006, Berlin, Germany
Editors: P.V. Nickles, K.A. Janulewicz

116 **Lasers in the Conservation of Artworks**
LACONA VI Proceedings, Vienna, Austria, Sept. 21–25, 2005
Editors: J. Nimmrichter, W. Kautek, M. Schreiner

117 **Advances in Turbulence XI**
Proceedings of the 11th EUROMECH European Turbulence Conference, June 25–28, 2007, Porto, Portugal
Editors: J.M.L.M. Palma and A. Silva Lopes

118 **The Standard Model and Beyond**
Proceedings of the 2nd International Summer School in High Energy Physics, Muğla, 25–30 September 2006
Editors: T. Aliev, N.K. Pak, and M. Serin

Volumes 81–105 are listed at the end of the book.

T. Aliev N.K. Pak M. Serin
(Eds.)

The Standard Model and Beyond

Proceedings of the 2nd
International Summer School
in High Energy Physics,
Muğla, 25–30 September 2006

With 50 Figures

Springer

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Preface

This is the Proceedings of the Second International Summer School in High Energy Physics, focusing on "Standard Model and Beyond", we have held in Mḡla, Turkey, on 25–30 September 2006.

The participants were mostly graduate and post-graduate students (about 50) working on the experimental and phenomenological problems. The students were mostly Turkish, and there were some from Azerbaijan.

In addition to excellent tutorial lectures on the Standard Model and the Grand Unification, there were exposures on the current developments on heavy flavor physics, the physics at LHC, recent experimental data coming from BABAR and BELLE, as well as neutrino oscillations data, which were presented by the senior spoke-persons of TEXONO Collaboration.

Encouraged by the success of this year's event, as well as the preceding one, we are planning to make this school an annual event, with substantial international student participation, and with special focus to those from central Asian countries.

We think we owe a lot to the venue where the event has been held. The special location of the convention center where the event has been held, Gökova, within half an hour driving distance from Ephesus and Milethos, has been superbly praised by the international participants. These great heritages from antique Ionia are the home of some of our founding fathers of western civilization and science, from Thales to Heraklitos. There is another event which already gained the status of a world event, namely the Gökova School on Geometry and Topology which uses the same venue. This success story plays a weighted role in our encouragement. This very special spot of the world deserves to be paid special tribute when it comes to looking for a special place for intellectual gatherings in Europe.

Clearly it is not an easy matter to elevate such events into a world status, like Erice, Les Houches, Cargese, etc. Continuity and publicity are two most crucial factors among others. There is another factor, which is a prerequisite of the first factor, namely the commitment of the sponsors.

For domestic administrative publicity purpose, as well as for reference purposes for graduate and post-graduate students, we have decided to publish the proceedings in the book form. Another reason to publish the proceedings in the book form is to draw the attention of the international peer community, to this wonderful spot, one of the cradles of western civilizations. We are sure that this particular volume will help us to achieve the goals we are aiming at concerning the future faith of our conference.

The event was partially supported by funds from Turkish National Agencies, Turkish Atomic Energy Authority (TAEK) and The Scientific and Technical Research Council of Turkey (TÜBİTAK), which we thankfully acknowledge. This proceeding could not have been published without the financial support of TAEK. We are very grateful to Okay Çakıroğlu, the President of TAEK, for providing this support.

The Conference would not have been possible without the generous services provided by the Muḡla University. We thank the Administration of the University as well as many volunteers from the Physics Department that helped us to organize social events and the excursions to the antique cities of Ephesus and Miletos.

We thank B. Ögel, the vice president of Middle East Technical University, and the administration of the Physics Department for a wide spectrum of support, from printing services to providing conference stationary.

We are most grateful to our speakers for their excellent talks and friendly cooperation to prepare their manuscripts under a pressing schedule, and the Springer Verlag publishers for their very positive, and cooperative approach.

We thank the members of organizing committee for their valuable contributions in shaping the scientific program and various organizational services. We also thank the young researchers who interactively contributed to the running of discussion hours and tutorials. There are contributed talks which we could not unfortunately include in these proceeding due to some internal constraints; we acknowledge these sincerely. We further thank all participants for their interest and commitment to the school.

We thank last but not the least, to our students, Ç Özkan, S. Sekmen and H. Gamsızkan for their untiring assistances.

Ankara,
May 2007

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Contents

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Introduction to Chiral Perturbation Theory

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A brief introduction to chiral perturbation theory, the effective field theory of quantum chromodynamics at low energies, is given.

1 Introduction

The strong interactions are described by quantum chromodynamics (QCD), a local non-abelian gauge theory. The QCD Lagrangian comprises quark and gluon fields which carry *color* charges and interact with coupling strength g. The renormalized coupling g depends on the momentum at which the measurement is performed and decreases as the momentum scale Q is increased. This behavior is referred to as *running* of the strong coupling constant $\alpha_s(Q) = g^2(Q)/(4\pi)$. The coupling α_s decreases for large momenta and the theory becomes *asymptotically free* with quasi-free quarks and gluons [1].

In this regime of QCD perturbation theory in α_s converges. For small momenta, on the other hand, α_s is large so that quarks and gluons arrange themselves in strongly bound clusters to form hadrons, e.g., protons, neutrons, pions, kaons, etc. In order to describe the physics of hadrons at low energies, perturbation theory is not useful because α_s is large. This is illustrated for $\pi\pi$ scattering in Fig. 1 where both sample diagrams—along with infinitely many other contributions—are equally important, although the right diagram appears at a much higher order in the perturbative series in α_s.

Alternative model-independent approaches are required in the non-perturbative regime of QCD. These are provided either by QCD lattice simulations which are a numerical solution to QCD or—at low energies—by chiral perturbation theory, the effective field theory of QCD. In the first case the QCD path integral in Euclidean space-time is evaluated numerically via Monte Carlo sampling, see e.g. [2]. In the latter case, one makes use of the fact that at low energies the relevant, effective degrees of freedom are hadrons rather than quarks and gluons which are not observed as free particles.

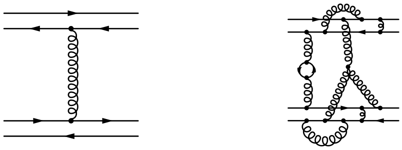

Fig. 1. Two sample diagrams which contribute to $\pi\pi$ scattering. Solid and curly lines denote quarks and gluons, respectively.

It is thus convenient to replace in the low-energy limit the QCD Lagrangian by an effective Lagrangian which is formulated in terms of the effective degrees of freedom, i.e. pions, kaons, eta, etc. The corresponding field theoretical formalism is called chiral perturbation theory (ChPT) [3–5].

In these lectures a brief introduction to ChPT is presented emphasizing some basic principles and a few simple applications. It is not intended to provide a detailed review of ChPT, in particular we restrict ourselves to the purely mesonic sector and do not consider baryons. For more comprehensive reviews the reader is referred to [6].

This paper is organized as follows. In the next section some well-known examples and basic principles of effective field theories in general are presented. Section 3 describes the construction principles for the chiral effective Lagrangian. Higher orders and loops are discussed in Sec. 4.

2 Effective Field Theories

The basic idea of an effective field theory is to treat the active, light particles as relevant degrees of freedom, while the heavy particles are frozen and reduced to static sources. The dynamics are described by an effective Lagrangian which is formulated in terms of the light particles and incorporates all important symmetries and symmetry-breaking patterns of the underlying fundamental theory.

2.1 Scattering of Light by Light in QED at Very Low Energies

The Lagrangian of quantum electrodynamics (QED) is given by

$$\mathcal{L}_{\mathrm{QED}} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{int}} \tag{1}$$

with the free part

$$\mathcal{L}_0 = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2}$$

and the interaction piece

$$\mathcal{L}_{\mathrm{int}} = -e\bar{\psi}A\!\!\!/\psi \ . \tag{3}$$

Fermion and photon fields are denoted by ψ and A_μ, respectively, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, and a gauge fixing term has been omitted for brevity.

Consider light by light scattering at very low photon energies $\omega \ll m$. In this instance, electrons (and positrons) cannot be produced in the final state, but contribute instead via virtual processes. The calculation of the lowest order diagram which is given by a single electron loop, Fig. 2, is straightforward but cumbersome. However, at very low energies the amplitude for light by light scattering is equally reproduced by the effective Lagrangian [7, 8]

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^4}{1440\,\pi^2 m^4} \left[(F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{16} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 \right] + \dots \quad (4)$$

which only contains the field strength tensor $F_{\mu\nu}$ and its dual counterpart $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ as explicit degress of freedom. The ellipsis denotes corrections to this Lagrangian involving more derivatives which arise from the energy expansion of the original one-loop diagram in powers of ω/m. Moreover, the coefficients of the operators in the effective Lagrangian receive corrections of higher orders in e^2 through multiloop diagrams.

It is instructive to illustrate the conversion to the effective field theory with Feynman diagrams. By treating at very low photon energies the electrons as heavy static sources the electron propagators of the electron loop in QED "shrink" to a single point. This gives rise to 4-photon contact interactions which correspond to the vertices of the effective Lagrangian, see Fig. 3.

A significant property is that the U(1) gauge symmetry of the underlying QED Lagrangian is maintained by the effective Lagrangian, Eq. (4), since the building blocks $F_{\mu\nu} F^{\mu\nu}$ and $F_{\mu\nu} \tilde{F}^{\mu\nu}$ are both gauge invariant. As we will see below, invariance under the relevant symmetries is an important constraint in constructing effective Lagrangians.

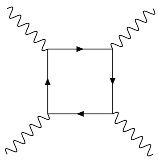

Fig. 2. Light by light scattering to lowest order. The wavy and solid lines denote the photons and electrons, respectively.

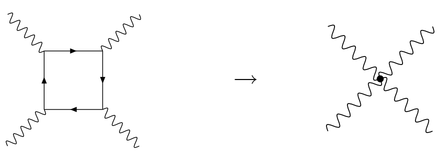

Fig. 3. The one-loop diagram of QED is replaced in the effective theory by 4-photon contact interactions.

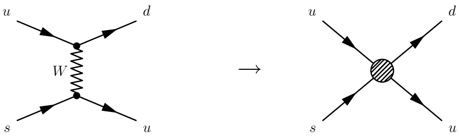

Fig. 4. At low energies the single W boson exchange reduces to a four-quark contact interaction.

2.2 Weak Interactions at Very Low Energies

A second well-known example of an effective field theory is encountered in weak interactions. Consider the amplitude for the flavor changing weak process at lowest order from single W boson exchange

$$\mathcal{A} = \left(\frac{ig}{\sqrt{2}}\right)^2 V_{us}V_{ud}^* \left(\bar{u}\gamma^\mu \tfrac{1-\gamma_5}{2} s\right) \left(\bar{d}\gamma^\nu \tfrac{1-\gamma_5}{2} u\right) \left(\frac{-ig_{\mu\nu}}{p^2 - M_W^2}\right) , \tag{5}$$

where V_{ij} are elements of the Kobayashi-Maskawa mixing matrix and the W propagator is given in Feynman gauge. In the limit of small momentum transfer, $p^2 \ll M_W^2$, the W propagator can be expanded in p^2/M_W^2 such that the amplitude is approximated by the local interaction

$$\mathcal{A} = \frac{i}{M_W^2} \left(\frac{ig}{\sqrt{2}}\right)^2 V_{us}V_{ud}^* \left(\bar{u}\gamma^\mu \tfrac{1-\gamma_5}{2} s\right) \left(\bar{d}\gamma_\mu \tfrac{1-\gamma_5}{2} u\right) + \mathcal{O}\left(\frac{p^2}{M_W^4}\right) . \tag{6}$$

Diagrammatically this approximation is illustrated in Fig. 4, where the contact interaction arises from the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = -2\sqrt{2} G_F V_{us}V_{ud}^* \left(\bar{u}\gamma^\mu \tfrac{1-\gamma_5}{2} s\right) \left(\bar{d}\gamma_\mu \tfrac{1-\gamma_5}{2} u\right) \tag{7}$$

with the Fermi constant $G_F = g^2/\left(4\sqrt{2}M_W^2\right)$.

2.3 Chiral Symmetry in QCD

As mentioned above, the relevant symmetries of the underlying theory must also be maintained by the effective field theory. In this section, we will study the (approximate) chiral symmetry of QCD. The QCD Lagrangian reads in compact notation

$$\mathcal{L}_{\text{QCD}} = \bar{q}\left(i\gamma^\mu D_\mu - m_q\right) q - \frac{1}{2}\text{Tr}_c\left(G^{\mu\nu}G_{\mu\nu}\right) , \tag{8}$$

where $q^T = (u, d, s, c, b, t)$ comprises the six quark flavors, $D_\mu = \partial_\mu - igG_\mu$ is the covariant derivative, G_μ the gluon fields, and $G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - ig\left[G_\mu, G_\nu\right]$ the gluon field strength tensor. Tr_c denotes the trace in color space. The Dirac field q is a 72-component object; each of the 6 quark flavors appears in 3 different colors and has 4 spinor components.

The quarks can be grouped into light and heavy flavors according to their masses: the u, d, s quarks are substantially lighter than the c, b, t quarks [9]. Hence, the limit of massless light quarks, $m_u = m_d = m_s = 0$, the so-called chiral limit, seems to be a reasonable approximation and can be improved by treating the light quark masses as perturbations. The c, b, t quarks, on the other hand, can be treated at low energies as infinitely heavy and the only active degrees of freedom are those associated with the light u, d, s quarks.

It is straightforward to see that in the chiral limit the QCD Lagrangian has an extra symmetry. In this limit, the relevant part of $\mathcal{L}_{\rm QCD}$ is (we use the same notation for simplicity)

$$\mathcal{L}_{\rm QCD} = \sum_{q=u,d,s} \bar{q} i \gamma_\mu D^\mu q - \frac{1}{2} \mathrm{Tr}_c (G_{\mu\nu} G^{\mu\nu}) \ . \tag{9}$$

Here, q represents a one-flavor quark field. By introducing right- and left-handed quark fields

$$q_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\, q \tag{10}$$

one arrives at

$$\mathcal{L}_{\rm QCD} = \sum_{q=u,d,s} (\bar{q}_L i \gamma_\mu D^\mu q_L + \bar{q}_R i \gamma_\mu D^\mu q_R) - \frac{1}{2} \mathrm{Tr}_c (G_{\mu\nu} G^{\mu\nu}) \ . \tag{11}$$

Independent transformations of the right- and left-handed quark fields

$$q_R \to R\, q_R \ , \qquad\qquad q_L \to L\, q_L \tag{12}$$

with $R \in \mathrm{SU}(3)_R$, $L \in \mathrm{SU}(3)_L$ leave the massless QCD Lagrangian invariant. This invariance is referred to as $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ chiral symmetry of massless QCD. One observes that the gluon interactions do not change the helicity of quarks but the quark mass term does.

Due to Noether's theorem an immediate consequence of a continuous symmetry of a Lagrangian is the existence of a conserved current J_μ with $\partial_\mu J^\mu = 0$. The corresponding charge

$$Q(t) = \int d^3x \ J_0(t, \mathbf{x}) \tag{13}$$

is time-independent, i.e. $dQ/dt = 0$. Familiar examples are the invariance of the Lagrangian with regard to translations in time and space and rotations which imply, respectively, conservation of energy, momentum and angular momentum. At the operator level, the conserved charges commute with the Hamiltonian.

In the chiral limit of QCD the conserved currents of chiral symmetry are

$$L^a_\mu = \sum_{q=u,d,s} \bar{q}_L \gamma_\mu \frac{\lambda^a}{2} q_L \ , \qquad R^a_\mu = \sum_{q=u,d,s} \bar{q}_R \gamma_\mu \frac{\lambda^a}{2} q_R \tag{14}$$

with the Gell-Mann matrices λ^a. The invariant charges Q_L^a, Q_R^a generate the algebra of SU(3)$_L$ and SU(3)$_R$, respectively. It is useful to define the combinations

$$Q_V^a = Q_R^a + Q_L^a \ ; \qquad Q_A^a = Q_R^a - Q_L^a \tag{15}$$

which have a different behavior under parity

$$Q_V^a \to Q_V^a \ ; \qquad Q_A^a \to -Q_A^a \ . \tag{16}$$

Consider an eigenstate $|\psi\rangle$ of $H_{\rm QCD}$ (in the chiral limit)

$$H_{\rm QCD}|\psi\rangle = E|\psi\rangle \ . \tag{17}$$

The states $Q_V^a|\psi\rangle$ and $Q_A^a|\psi\rangle$ have the same energy E but opposite parity. Thus for each positive parity state there should be a negative parity state with equal mass. This pattern is, however, not observed in the particle spectrum [9]. For example, the light pseudoscalar ($J^P = 0^-$) mesons, (π, K, η), have a considerably lower mass than the scalar ($J^P = 0^+$) mesons.

The solution to this paradoxon is provided by the Nambu-Goldstone realization of chiral symmetry [10] which asserts that the QCD vacuum, $|0\rangle$, is not invariant under the action of the axial charges

$$Q_V^a|0\rangle = 0 \qquad Q_A^a|0\rangle \neq 0 \ . \tag{18}$$

The chiral $SU(3)_L \times SU(3)_R$ symmetry of the QCD Hamiltonian is said to be *spontaneously* broken down to $SU(3)_V$. Spontaneous breakdown of a symmetry takes place if the full symmetry group of the Hamiltonian is not shared by the vacuum.

Another example of spontaneous symmetry breakdown occurs in ferromagnets. For temperatures above the Curie temperature, $T > T_c$, the magnetic dipoles are randomly oriented. As soon as the temperature falls below the Curie temperature T_c spontaneous magnetization occurs and the dipoles are aligned in some arbitrary direction. Spontaneous symmetry breakdown takes also place for the SU(2)$_L\times$U(1) symmetry of the electroweak interactions.

In general, spontaneous breakdown of a continuous symmetry has important consequences. Goldstone's theorem states that a spontaneously broken continuous symmetry implies massless spinless particles: the Goldstone bosons. In the case of massless QCD, the eight axial charges Q_A^a create states $|\phi\rangle = Q_A|0\rangle$ which are energetically degenerate with the vacuum $|0\rangle$ since

$$H|\phi\rangle = HQ_A|0\rangle = Q_A H|0\rangle = 0 \ . \tag{19}$$

This gives rise to eight massless pseudoscalar mesons. The axial charges Q_A^a acting on any particle state generate Goldstone bosons, e.g. an energy eigenstate $|\psi\rangle$ is degenerate with the multi-particle state $Q_A^a|\psi\rangle$ which resolves the paradoxon from above.

The eight lightest hadrons are indeed the pseudoscalars $\pi^\pm$,π^0,$K^\pm$,K^0,$\bar{K}^0$, η with masses $m_\pi \approx 138$ MeV, $m_K \approx 495$ MeV and $m_\eta \approx 547$ MeV [9]. Since the nonzero masses of the light quarks break chiral symmetry explicitly the Goldstone bosons are not exactly massless. However, the explicit breaking can be considered to be small and treated perturbatively. In the limit of vanishing quark masses, $m_u, m_d, m_s \to 0$, the Goldstone boson masses approach zero, $m_\pi, m_K, m_\eta \to 0$, while all other hadrons remain massive in the chiral limit and are separated from the ground state roughly by a characteristic gap

$$\Delta \sim M_{\text{proton}} \sim 1\,\text{GeV} . \tag{20}$$

In the remainder of this section, it is demonstrated that only the spontaneous breakdown of a *continuous* symmetry gives rise to Goldstone bosons, whereas in the case of a discrete symmetry Goldstone bosons are not generated.

Discrete symmetry case

Consider the Lagrangian density with a scalar field ϕ

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4 . \tag{21}$$

The Lagrangian is invariant under the discrete symmetry of reflections, $\phi \to -\phi$. The corresponding potential is given by

$$V(\phi^2) = -\frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 , \tag{22}$$

and since the energy must be bound from below the coupling λ is positive. The coefficient m^2, on the other hand, is not constrained. There are two possible cases depending on the sign of m^2 as illustrated in Fig. 5. For $m^2 < 0$ there is a unique minimum at $\phi = 0$, but for $m^2 > 0$ the potential $V(\phi^2)$ is minimized by two possible ground state fields $\phi = \pm\sqrt{m^2/\lambda}$. In the quantum field theoretical language this implies that the field ϕ develops a vacuum expectation value

$$\langle 0|\phi|0\rangle = \pm\sqrt{\frac{m^2}{\lambda}} . \tag{23}$$

Hence, there are two possible vacua but each vacuum is not invariant under reflection symmetry, i.e. the theory is spontaneously broken. Massless Goldstone bosons, however, do not appear.

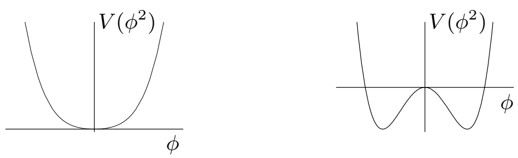

Fig. 5. Potential $V(\phi^2)$, Eq. (22), for $m^2 < 0$ (left) and $m^2 > 0$ (right).

Continuous symmetry case

Consider now the Lagrangian with two scalar fields σ and π

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi)^2 - V(\sigma^2 + \pi^2) \tag{24}$$

with V defined as in Eq. (22). It exhibits an O(2) symmetry; continuous transformations of the type

$$\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix} \tag{25}$$

leave the Lagrangian invariant. The extrema of the corresponding potential V are determined by the equations

$$\begin{aligned} \frac{dV}{d\sigma} &= \sigma[-m^2 + \lambda(\sigma^2 + \pi^2)] = 0 \ , \\ \frac{dV}{d\pi} &= \pi[-m^2 + \lambda(\sigma^2 + \pi^2)] = 0 \ . \end{aligned} \tag{26}$$

For $m^2 > 0$ the minima are at $\sigma^2 + \pi^2 = m^2/\lambda$ and related to each other through O(2) rotations. Any point on the circle of minima may be chosen to be the true vacuum $|0\rangle$. One may take, e.g.,

$$\langle 0|\sigma|0\rangle = \sqrt{\frac{m^2}{\lambda}} \ ; \qquad \langle 0|\pi|0\rangle = 0 \ . \tag{27}$$

Clearly, the O(2) symmetry of the Lagrangian is spontaneously broken by the vacuum state. Small oscillations around this vacuum state can be described by shifting the σ field

$$\sigma' \equiv \sigma - \sqrt{\frac{m^2}{\lambda}} \tag{28}$$

so that the Lagrangian reads in terms of the new fields (up to an irrelevant constant)

$$\begin{aligned} \mathcal{L} = &\ \frac{1}{2}\left(\partial_\mu \sigma'\right)^2 + \frac{1}{2}\left(\partial_\mu \pi\right)^2 - m^2\sigma'^2 \\ &-\lambda\sqrt{\frac{m^2}{\lambda}}\,\sigma'\,\left(\sigma'^2 + \pi^2\right) - \frac{1}{4}\lambda\,\left(\sigma'^2 + \pi^2\right)^2 \ . \end{aligned} \tag{29}$$

With this choice of coordinates the mass term for the π field has disappeared and the π becomes massless. The π field is then interpreted as a polar angle oscillation around the vacuum which does not cost any energy. A Goldstone boson has been created through spontaneous breakdown of the continuous O(2) symmetry in the original Lagrangian.

3 Construction of the Chiral Effective Lagrangian

In this section we outline the construction principles for the chiral effective Lagrangian. The chiral SU(3) Lagrangian is in general a function of the Goldstone boson (GB) fields $(\pi^0, \pi^\pm, K^\pm, K^0, \bar{K}^0, \eta)$. In order to construct the effective Lagrangian, we must first know the interaction between the GBs.

To this aim, we recall from the previous section that the eight axial charges Q_A^a do not annihilate the vacuum, $Q_A^a|0\rangle \neq 0$. The states $Q_A^a|0\rangle \neq 0$ are associated with the GBs $\phi^a = (\pi, K, \eta)$. This implies non-vanishing matrix elements of the axial vector current A_μ^a

$$\langle 0|A_\mu^a(x)|\phi^b(p)\rangle = ie^{-ip\cdot x}\, p_\mu\, \delta^{ab}\, f_a \tag{30}$$

(no summation over a). The decay constant f_a measures the strength with which the Goldstone boson ϕ^a decays via the axial vector current A_μ^a into the hadronic vacuum. The decay constants are extracted experimentally from weak decays of the GBs, e.g., $\pi^+ \to l^+\nu_l$ yields $f_\pi = 92.4$ MeV [11].

Taking the divergence of Eq. (30) leads to

$$\langle 0|\partial^\mu A_\mu^a(0)|\phi^b(p)\rangle = \delta^{ab} m_a^2 f_a \ . \tag{31}$$

In the chiral limit, the axial vector current is conserved, $\partial^\mu A_\mu^a = 0$, so that $m_a^2 = 0$ as required by Goldstone's theorem. In the real world, however, chiral $SU(3)_L \times SU(3)_R$ symmetry is explicitly broken by the finite quark masses m_u, m_d, m_s and the axial vector current is not conserved. One introduces the GB field operators Φ^a with the normalization $\langle 0|\Phi^a(0)|\phi^b(p)\rangle = \delta^{ab}$. Eq. (31) can then be rewritten as

$$\langle 0|\partial^\mu A_\mu^a(0)|\phi^b(p)\rangle = m_a^2 f_a \langle 0|\Phi^a(0)|\phi^b(p)\rangle \ . \tag{32}$$

At the operator level, this is the hypothesis of the partially conserved axial vector current (PCAC)

$$\partial^\mu A_\mu^a = m_a^2 f_a \Phi^a \ . \tag{33}$$

The axial currents can thus be employed as interpolating fields for the Goldstone bosons and identity (33) implies a vanishing interaction between the GBs at zero momentum. Consider to this end, e.g., the matrix element (suppressing flavor indices)

$$\mathcal{M}_\mu(p_1, p_2, p_3) = \langle \phi(p_2)\phi(p_3)|A_\mu(0)|\phi(p_1)\rangle \ . \tag{34}$$

The amplitude $\mathcal{M}_\mu$ contains two parts: contributions with no GB poles and contributions where the axial current generates a GB pole, see Fig. 6. In the chiral limit, the matrix element has the decomposition

$$\mathcal{M}_\mu(p_1, p_2, p_3) = \frac{f q_\mu}{q^2} T(p_1, p_2, p_3, q) + R_\mu \ , \tag{35}$$

Fig. 6. Contributions to $\mathcal{M}_\mu$ in Eq. (34) with (left) and without (right) a GB pole. The wavy and solid lines denote the axial vector current and the Goldstone bosons, respectively.

where $q = -p_1 - p_2 - p_3$, T is the GB-GB scattering matrix element, f the decay constant in the chiral limit, and R_μ is non-singular as $q_\mu \to 0$ by definition. Contracting both sides with q^μ yields

$$0 = q^\mu \mathcal{M}_\mu(p_1, p_2, p_3) = f T(p_1, p_2, p_3, q) + q^\mu R_\mu \ . \tag{36}$$

In the limit $q_\mu \to 0$ one obtains

$$T(p_1, p_2, p_3, q) = 0 \ . \tag{37}$$

The GBs do not interact at vanishing momenta.

At low but finite energies, the interaction between GBs can be expanded in powers of small momenta. Consider for example the GB-GB scattering matrix T which can be written as a function of the three Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$ and $u = (p_1 + q)^2$. Its low energy expansion reads

$$T(s, t, u) = f_1 s + g_1 t + h_1 u + \ldots \tag{38}$$

with momentum-independent expansion coefficients f_i, g_i, h_i. The chiral effective Lagrangian is also ordered according to the low energy expansion. Powers of GB momenta in the amplitude correspond to powers of derivatives on GB fields in the Lagrangian. The ordering of the effective Lagrangian in increasing powers of derivatives is called *chiral ordering* or *chiral power counting.*

Next, we would like to investigate how GB fields are represented in the chiral Lagrangian. To this aim, we shall study the transformation properties of the GBs under chiral transformations.

Let G be the group of chiral $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ transformations. For a given representation of G the GB fields transform according to

$$\phi \to \phi' = F(g, \phi) \ , \qquad g \in G \tag{39}$$

with the representation property

$$F(g_1, F(g_2, \phi)) = F(g_1 g_2, \phi) \ . \tag{40}$$

Consider group elements $h \in G$ which leave the "origin", i.e. the vacuum, invariant, $F(h, 0) = 0$. Obviously, these elements form a subgroup H: for

$h_1, h_2 \in H$ it follows that $h_1 h_2 \in H$. H is equivalent to the subgroup $\mathrm{SU}(3)_V$ which leaves the vacuum invariant.

The function

$$g \to F(g, 0) = F(gh, 0) \qquad h \in H \tag{41}$$

maps the coset space G/H onto the space of GB fields. This mapping is invertible since $F(g_1, 0) = F(g_2, 0)$ implies $g_1^{-1} g_2 \in H$. As the dimension of the coset space is equal to the number of Goldstone boson fields, the GBs can be identified with elements of G/H. The Goldstone boson fields are said to *live* on the coset space $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R/\mathrm{SU}(3)_V$.

Any $g \in G$ can be decomposed as $g = qh$ with $q \in G/H$ and $h \in H$. The choice of representatives in the coset space G/H is arbitrary. Possible choices are for example

$$g = (g_L, g_R) = (1, g_R g_L^{-1})(g_L, g_L) \equiv qh \tag{42}$$

or

$$g = (g_L, g_R) = (g_L g_R^{-1}, 1)(g_R, g_R) \equiv q'h' \ . \tag{43}$$

If we pick, e.g., the latter choice then the action of G on G/H is given by

$$(L, R)(g_L g_R^{-1}, 1) = (L g_L g_R^{-1}, R) = (L g_L g_R^{-1} R^{-1}, 1)(R, R) \ . \tag{44}$$

The Goldstone bosons are then summarized by the matrix-valued field $U = g_L g_R^{-1}$ which transforms under chiral transformations as

$$U(x) \to U'(x) = LU(x)R^{-1} = LU(x)R^{\dagger} \tag{45}$$

for $L/R \in \mathrm{SU}(3)_{L/R}$. The exponential representation is convenient for $U \in \mathrm{SU}(3)$

$$U = \exp\left(\frac{i}{f}\phi^a \lambda^a\right) \ , \tag{46}$$

where λ^a are the generators of SU(3)

$$\phi = \phi^a \lambda^a = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} \ . \tag{47}$$

The chiral effective Lagrangian for QCD is written in terms of the GB fields which are collected in the matrix-valued field U

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots) \ . \tag{48}$$

The effective Lagrangian shares the same symmetries with QCD: C, P, T, Lorentz invariance and, in particular, chiral $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ symmetry. As outlined above, the chiral Lagrangian is expanded in chiral powers which are related (in the chiral limit) to the number of derivatives acting on the GB fields. The chiral power counting of the Lagrangian reads

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^{(0)} + \mathcal{L}_{\text{eff}}^{(2)} + \mathcal{L}_{\text{eff}}^{(4)} + \dots \ . \tag{49}$$

Only even chiral powers arise since the Lagrangian is a Lorentz scalar which implies that tensor indices of derivatives appear in pairs. At each chiral order the effective Lagrangian must be invariant under chiral $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ transformations. At zeroth chiral order this invariance implies that $\mathcal{L}_{\text{eff}}^{(0)}$ can only be a function of $UU^\dagger = 1$. This amounts to an irrelevant constant in the Lagrangian which can be dropped.

At second order, the chiral invariant terms with two derivatives are

$$\mathcal{L}_{\text{eff}}^{(2)} = c_1 \langle \partial_\mu U^\dagger \partial^\mu U \rangle + c_2 \langle U^\dagger \Box U \rangle \ , \tag{50}$$

where $\langle \dots \rangle$ is the trace in flavor space. The second term can be reduced to the first one by partial integration; only one term remains at second chiral order

$$\mathcal{L}_{\text{eff}}^{(2)} = c_1 \langle \partial_\mu U^\dagger \partial^\mu U \rangle \ . \tag{51}$$

Since terms of zeroth chiral order have been dropped, the second chiral order is effectively the leading order (LO). We note the appearance of a coupling constant c_1, a so-called low-energy constant (LEC). It is fixed by expanding the matrix-valued field U in the GB fields ϕ

$$U = \exp\left(\frac{i}{f}\phi\right) = 1 + \frac{i}{f}\phi - \frac{1}{2f^2}\phi^2 + \mathcal{O}(\phi^3) \tag{52}$$

and requiring the standard kinetic term

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a + \mathcal{O}(\phi^4) \tag{53}$$

which yields $c_1 = f^2/4$.

Therefore, the effective Lagrangian at LO reads

$$\mathcal{L}_{\text{eff}}^{(2)} = \frac{f^2}{4} \langle \partial_\mu U^\dagger \partial^\mu U \rangle \ . \tag{54}$$

At leading chiral order there is only one LEC (in the chiral limit) and chiral symmetry constrains all vertices with increasing number of GB fields in the LO Lagrangian.

The interpretation of the LEC f can be directly inferred by considering the Noether axial current of chiral symmetry for $\mathcal{L}_{\text{eff}}^{(2)}$

$$A_\mu^a = i\frac{f^2}{4} \langle \lambda^a \{\partial_\mu U, U^\dagger\} \rangle \ . \tag{55}$$

Upon comparison with the PCAC hypothesis

$$\langle 0|A_\mu^a(0)|\phi^b(p)\rangle = ip_\mu \delta^{ab} f_a \tag{56}$$

one confirms that f is the GB decay constant in the chiral limit.

As a first application we are now in a position to predict, e.g., $\pi\pi$ scattering at leading chiral order. The scattering amplitude has the decomposition

$$\begin{aligned}&\mathcal{M}(\pi^a(p_a)\,\pi^b(p_b) \to \pi^c(p_c)\,\pi^d(p_d))\\ &\quad = \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,s,u) + \delta^{ad}\delta^{bc}A(u,t,s)\ ,\end{aligned} \tag{57}$$

where a, b, c, d are flavor indices and $s = (p_a + p_b)^2$, $t = (p_a - p_c)^2$, $u = (p_a - p_d)^2$. Employing $\mathcal{L}_{\text{eff}}^{(2)}$ one calculates

$$A(s,t,u) = \frac{s}{f^2}\ . \tag{58}$$

Up to now, we have worked in chiral limit $m_u, m_d, m_s = 0$ where chiral symmetry is exact. In the real world the quark masses do not vanish and introduce an explicit breaking of chiral symmetry in $\mathcal{L}_{\text{QCD}}$

$$\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}^0 - \bar{q}\mathcal{M}q = \mathcal{L}_{\text{QCD}}^0 - \bar{q}_R\mathcal{M}q_L - \bar{q}_L\mathcal{M}q_R\ , \tag{59}$$

where $\mathcal{L}_{\text{QCD}}^0$ is the massless QCD Lagrangian and $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$ the light quark mass matrix. The chiral symmetry breaking patterns induced by the light quark masses must be reproduced at the level of the effective field theory. To this end, we interpret the quark mass matrix as an external scalar source s

$$\bar{q}\mathcal{M}q = \bar{q}_L\mathcal{M}q_R + \bar{q}_R\mathcal{M}^\dagger q_L \to \bar{q}_L s q_R + \bar{q}_R s^\dagger q_L\ . \tag{60}$$

The external scalar source s is required to transform under chiral rotations as

$$s \to L s R^\dagger\ . \tag{61}$$

Obviously, this leaves the QCD Lagrangian invariant under chiral rotations and implies that the effective Lagrangian must also remain invariant in the presence of s. Hence, the chiral invariant effective Lagrangian is extended with s as an additional building block

$$\mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots) \to \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots; s)\ . \tag{62}$$

Once the effective Lagrangian is constructed one can go back to the real world by setting $s = \mathcal{M}$. In (standard) chiral perturbation theory[1] the chiral counting rule is $s = \mathcal{M} = \mathcal{O}(p^2)$, i.e. the quark masses are booked as second

[1] We do not consider here the framework of generalized ChPT wherein $\mathcal{M} = \mathcal{O}(p)$ [12].