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Reflections on the Foundations of Mathematics

Univalent Foundations, Set Theory and General Thoughts

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Introduction

Stefania Centrone, Deborah Kant, and Deniz Sarikaya

The present volume originates from the conference *Foundations of Mathematics: Univalent Foundations and Set Theory (FOMUS)*, which was held at the Center for Interdisciplinary Research of Bielefeld University from the 18th to the 23rd of July 2016. Within this framework approximately 80 graduate students, junior researchers and leading experts gathered to investigate and discuss suitable foundations for mathematics and their qualifying criteria, with an emphasis on homotopy type theory (HoTT) and univalent foundations (UF) as well as set theory. This interdisciplinary workshop, conceived of as a hybrid between summer school and research conference, was aimed at students and researchers from the fields of mathematics, computer science and philosophy.

A collected volume represents, it goes without saying, an excellent opportunity to pursuing and deepening the lively discussions of a conference. This volume, however, is not a conference proceedings in the narrow sense since it contains also contributions from authors who were not present at FOMUS. Specifically, 6 from the 19 contributions have been developed from presentations at the conference and only 9 from 24 authors were present at FOMUS.

As to the conference, the concomitant consideration of different foundational theories for mathematics is an ambitious goal. This volume integrates both univalent foundations and set theory and aims to bring some novelty in the discussion on the foundations of mathematics. Indeed, a comparative study of foundational

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frameworks with an eye to the current needs of mathematical practice is, even to this day, a *desideratum*.

The FOMUS conference was organized with the generous support of the Association for Symbolic Logic (ASL), the German Mathematical Society (DMV), the Berlin Mathematical School (BMS), the Center of Interdisciplinary Research (ZiF), the Deutsche Vereinigung für Mathematische Logik und für Grundlagenforschung der Exakten Wissenschaften (DVMLG), the German Academic Scholarship Foundation (Stipendiaten machen Programm), the Fachbereich Grundlagen der Informatik of the German Informatics Society (GI) and the German Society for Analytic Philosophy (GAP).

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Very special thanks go to the above-mentioned associations for the support of the conference as well as of the editors. Without this support the volume as it stands would not have been possible. The opinions in the volume do not necessarily match with those of the agencies.

The editors warmly thank Lukas Kühne and Balthasar Grabmayr for their help in organizing the conference. The encouragement at a decisive moment and the friendly advice from *Synthese Library*'s editor-in-chief, Otávio Bueno and from Springer's project coordinator, Palani Murugesan, were truly invaluable. A very special thanks goes to the authors of the contributions and to all anonymous referees who reviewed each single contribution.

The Topic

Set theory is widely assumed to serve as a suitable framework for foundational issues in mathematics. However, an increasing number of researchers are currently investigating Univalent Foundations as an alternative framework for foundational issues. This relatively young approach is based on HoTT. It links Martin-Löf's intuitionistic type theory and homotopy theory from topology. Such developments show the necessity of a novel discussion on the foundation of mathematics, or so we believe. The volume pursues two complementary goals:

1. To provide a systematic framework for an interdisciplinary discussion among philosophers, computer scientists and mathematicians
2. To encourage systematic thought on criteria for a suitable foundation

General criteria for foundations of mathematics can be drawn from the single contributions. Some candidates thereof are:

- Naturalness with regard to mathematical practice

- Applicability in mathematical practice
- Expressive power
- Possibility of extending the theory by justified new axioms
- Possibility of implementing the theory into formal proof systems
- Interpretability of non-classical (e.g. constructive) approaches
- Plausibility of the ontological implications

As far as set theory is concerned, the research literature is rich in perspectives and argumentations on foundational criteria. Roughly, set theory is seen as a theory with much expressive power, in which almost every other mathematical theory can be interpreted and which also may very well serve as ontological foundational framework for mathematics. However, it is not applicable in all mathematical areas and is not easily implemented in formal proof systems.

As far as HoTT is concerned, discussion on specific foundational criteria has not yet been properly carried out. Many scholars defend the thesis that HoTT is a very well applicable theory, easily implemented in formal proof systems (such as Coq and Agda) that can serve as a good foundational framework by dwelling rather on intensional than on extensional features of mathematical objects, at a variance with set theory.

Not least, the question about alternatives to the standard set-theoretic foundation of mathematics seems to be relevant also in view of the last developments of formal mathematics that appears to develop more and more in the direction of a mathematical practice focusing on the use of automated theorem provers.

Historical Background

Foundational disputes are not new in the history of mathematics. The *Grundlagen-streit* at the beginning of the twentieth century is but one ultimate example of a controversy, sometimes acrimonious, between various schools of thought opposed to each other.

Among the questions that mathematics poses to philosophical reflection on mathematics, those concerning the nature of mathematical knowledge and the ontological status of mathematical objects are central, when it comes to the foundations of mathematics. Is mathematics a *science* with an own content or is it a *language*, or rather, a *language-schema* that admits different interpretations? Does mathematical activity describe objects that are there, or does it constitute them? How do we explain our knowledge of mathematical objects?

If we believe that mathematics is a science with an own content, we have to say which objects mathematics is talking about. If these objects are mind-independent, how do we have access to them? If we constitute them, how do we explain the fact that the *same* mind-dependent objects are grasped by different subjects?

Philosophy of mathematics generally distinguishes three different views as to the foundations of mathematics: Logicism, Formalism and Intuitionism.

Logicism. The *descriptive view* at the turn of the century was represented by the *Logicism* of Gottlob Frege (1848–1925). Even the word "descriptive" hints at the fact that we are confronted with a version of the traditional standpoint of *platonism*. Numbers and numerical relations are abstract logical objects. Number systems are well-determined mathematical realities. The task of the knowing subject is to discover and to describe such realities that subsist independently of him *via* true propositions about such objects. The latter, organized systematically, make up the theory of that mathematical reality. Platonism is most often associated with an eminently *non-epistemic* conception of truth: the truth value of a proposition is independent of its being known.

The programme of logicism was to ground finite arithmetic on logic: the basic concepts of arithmetic (natural number, successor, order relation, etc.) had to be defined in purely logical terms, and arithmetical true propositions had to be *ideographically* derived from logical principles.

The arithmetization of analysis initiated by Karl Weierstrass (1815–1897) had concluded with the simultaneous publication in 1872 of the foundations of the system of real numbers by Richard Dedekind (1831–1916)[1] and Georg Cantor (1845–1918)[2]. Since, before then, it was well known how to define rational numbers in terms of integers and the latter in terms of natural numbers, the last question to be answered was how to lead back natural numbers to logic. To answer such question, Frege formulated in his *Grundgesetze der Arithmetik* (1901-1903) a system of principles from which the axioms of finite arithmetic should have been derived. Just as the second volume of the *Grundgesetze* was getting into print, Frege received a letter from Bertrand Russell (1872–1970), who called his attention on an antinomy arising by an indiscriminate use of the principle of unlimited comprehension, that is, the assumption that *each* concept has an extension. Frege could only recognize the mistake in a postscript.

Russell's Antinomy. Russell's antinomy, we recall it, says that class of all classes that are not elements of themselves is and is not element of itself. Indeed, if it is, it is not, since it is the class of all classes that do not have themselves as elements. If it is not, it is, for the same reason. The antinomy turns out to be relevant associated with Basic Law V of Frege's *Grundgesetze* that applies to functions and their course of values as well as to concepts (which for Frege are a special kind of functions) and their extensions. The gap is caused by the fact that Frege takes *courses of values* as well as *extensions of concepts* to be objects and takes Basic Law V as an *identity criterium* for such objects. Basic Law V stipulates that the extensions of two concepts (more generally the courses of values of two functions) are the same, if and only if the concepts apply exactly to the same objects (or the two functions have the same input-output behaviour).

[1] Hereto see Dedekind 1872.

[2] Hereto see Cantor 1872.

Basic Law V for concepts runs as follows:

$$Ext(P) = Ext(Q) \leftrightarrow \forall x(P(x) \leftrightarrow Q(x))$$

while its more general version for functions runs as follows:

$$\varepsilon' f(\varepsilon) = \varepsilon' g\,(\varepsilon) \leftrightarrow \forall x(f(x) = g(x))\,.$$

Russell's antinomy can be presented in Frege's system thus:
Let "R(x)" stand for the predicate "x is Russellian":

$$(*)\;\; R(x) \leftrightarrow \exists Y(\, x = Ext(Y) \wedge \neg Y(x))\,.$$

That is, x is Russellian iff x is the extension of a concept that does not apply to x.

Let "r" be short for "*Ext(R)*", the extension of the concept R. Then the contradiction

$$R(r) \leftrightarrow \neg R(r).$$

is easily derived as follows.

(1) ¬R(r) → R(r). Assume ¬R(r); then, according to the definition of r:

$$r = Ext(R) \text{ and } \neg R(r).$$

By existential quantifier introduction it follows thereof:

$$(**)\, \exists Y.\;\; r = Ext(Y) \wedge \neg Y(r),$$

and so, by (*), we get $R(r)$.

(2) $R(r) \rightarrow \neg R(r)$. Assume $R(r)$; then by (*)

$$\exists Y(\, r = Ext(Y) \wedge \neg Y(r))\,.$$

Let then (by ekthesis) Y be such that $r = Ext(Y)$ and $\neg Y(r)$.

So $r = Ext(R)$ and $r = Ext(Y)$, hence $Ext(Y) = Ext(R)$. At this point Basic Law V comes into play, yielding

$$\forall z(Y(z) \leftrightarrow R(z))\,.$$

Therefore from $\neg Y(r)$ we get $\neg R(r)$.

Thus, in conclusion, r is at once *Russellian* and *not-Russellian*, and Frege's system is, at least without amendments, inconsistent.

Formalism: Hilbert's Program. At the turn of the century Hilbert was trying to establish in Germany an interdisciplinary area of research for mathematicians, logicians and philosophers modelled on the kind of cooperation that was in bloom at Cambridge around Bertrand Russell. Hilbert held Russell's and Whitehead's work in high esteem, and he was "convinced that the combination of mathematics, philosophy and logic ... should play a greater role in science".[3] Hilbert's Program undergoes many phases. The initial phase of his reflections on foundations spans from 1898 to ca. 1901. Hilbert aimed at an axiomatic foundation not only of mathematics but also of physics and other sciences through the formal-axiomatic method. As to mathematics he took to be possible to reduce the axiomatic foundation of all mathematics to that of the arithmetic of real numbers and set theory. More specifically, Hilbert took the complete formalization of concrete mathematics, i.e. of the theory of natural numbers as well as analysis and set theory, to be possible. He took Peano-Arithmetik (**PA**), Peano-Arithmetik at the second order ($\mathbf{PA_2}$) and Zermelo-Fraenkel set theory (**ZF**) to be formal counterparts of the concrete mathematical theories. Formalization, however, was only the first step of Hilbert's Program. For, once the concrete theories were formalized, the proper task was to prove the consistency of the formalized theories. In this way, Hilbert thought to avoid antinomies like the Russellian one.

Hilbert addressed the problem of a proof of consistency for arithmetic in a conference, entitled "Mathematical Problems (*Mathematische Probleme*)", held to the second International Congress of Mathematicians in Paris in 1900. As he put it: "But above all I wish to designate the following as the most important among the numerous questions which can be asked with regard to the axioms: To prove that they are not contradictory, that is, that a definite number of logical steps based upon them can never lead to contradictory results."

A kind of criticism is often raised against formalism, namely, that the latter reduces mathematics to a meaningless symbolic game by investigating the logical consequences of axiom systems set up arbitrarily. However, formalization was not the primary goal of formalism, but was conceived of as a necessary condition for proving the consistency of mathematics. Only then, mathematics would have been secure.

A necessary condition for the formalization was to find for each concrete theory a formal counterpart able to capture *all* its truths. In particular, all arithmetical truths should have been provable in **PA**, all truths of analysis should have been provable in (a theory equivalent to) $\mathbf{PA_2}$ and all set-theoretical truths should have been provable in **ZF**. However, like the logicistic programme also the formalistic programme was doomed to fail. In 1931 Kurt Gödel (1906–1978) proved his famous incompleteness theorems. They can both be conceived as limits put to the power of formalisms. The first theorem says that arithmetic is syntactically incomplete. More specifically, the theorem says that each theory that is (**i**) consistent, (**ii**) effective and (**iii**) contains some arithmetic (precisely, as much as Robinson's arithmetic **Q**, a weak subsystem

[3] Cp. Hilbert et al.'s "Minoritätsgutachten" of 1917.

of **PA**) is syntactically incomplete: there is, at least, one sentence of the language of the theory that is neither provable nor refutable within the theory. So, **PA** is not able to capture formally all arithmetical truths. All the more so, as to the formalized counterparts of analysis (equivalent to $\mathbf{PA_2}$) and set theory (**ZF**).

The second step of Hilbert's Program was to make mathematics secure. In particular, Hilbert required the use of infinitary parts of mathematics to be justified. It must be said that Hilbert started to talk about "ideal elements" in a later phase of the foundational research (1920–1924), in which "Hilbert's Program" takes its proper shape. In a paper presented in Leipzig in 1922 entitled "*The Logical Foundations of Mathematics* (*Die logischen Grundlagen der Mathematik*)" (Hilbert 1923), Hilbert indicates *finitary mathematics* as that part of mathematics that "has a concrete content", that is, that it operates concretely with symbols and does not use infinitary procedures and principles. It is in this context that he starts talking about "finitary logic" (the logic of finitary procedures) and of "ideal elements". Hilbert presents his paper "*On Infinite* (*Über das Unendliche*)" (Hilbert 1926) in Münster in 1925, where he explicitly speaks of "ideal elements". Here *finitary mathematics* is said to be that part of mathematics that can be rightfully considered as "secure". It does not need a justification but must itself serve as justification for *infinitary mathematics*, which is that part of mathematics that deals with actual infinity, lacks a concrete content and is moreover a possible source of contradictions. Infinitary instruments are acknowledged as "useful": they are used to prove *real* propositions. However, they must be justified, that is one has to demonstrate that their use does not lead to contradictions.

To this aim, the proof of consistency for the formalized theories **PA**, $\mathbf{PA_2}$ and **ZF** turned out to be essential. But, to serve as a justification, the proof should have used means that did not need themselves a justification. It should have used only *finitary mathematics*.

However, while the first of Gödel's theorems had shown the impossibility of completely formalizing the concrete mathematical theories, the second theorem showed that it was not possible to prove the consistency of an arithmetical theory using only means formalizable in that very same theory. More exactly, Gödel's second incompleteness theorem says that each theory, that is (**i**) consistent, (**ii**) effective and (**iii**) contains as much arithmetic as **PA** (actually, a weaker system like, e.g. **PRA**, primitive recursive arithmetic, suffices), cannot prove (the statement formalizing) its own consistency. Thus, if one assumes that *finitary mathematics* is part of **PA**, is it not possible to prove the consistency of **PA** with *finitary* means.

Thus, even the formalistic approach was denied the likelihood of success.

Intuitionism. Intuitionism was first brought forward by Luitzen E. J. Brouwer (1881–1966). Brouwer had completed his academic studies in 1907 at the University of Amsterdam with a dissertation on the foundations of mathematics (*Over de grondslagen des wiskunde*). Already at this time he had taken up a position in direct opposition to formalism and logicism, many years before he would engage in the attempt to construct intuitionistic mathematics. What was at issue was, once again, to build mathematics on an absolutely secure ground with no risk to run into antinomies. Intuitionism is in its origin a particular way of conceiving mathematics. Mathematics is the exact part of the thinking activity of an idealized knowledge-subject. Such activity grounds on our *Ur-intuition* of the flow of time. Kant's conception of time explicitly works at the background. In his doctoral dissertation of 1907 Brouwer writes: "*Mathematics can deal with no other matter than that which it has itself constructed*."[4] At a variance with Platonism, mathematical objects are *not* objects *that are there* and *stand in certain relations*, but rather mental constructions of an idealized knowledge-subject. At a variance with formalism, the role of the language is only peripheral, it is useful to communicate the results of the thinking activity of the idealized subject as well as *aide-mémoire*, but mathematics as a mental construction has not to be confused with its linguistic expression.[5]

The basic phenomenon of intuitionism, the *Ur-intuition* of time, consists, according to Brouwer, in the perception of the split-up of one moment of life into two different things which differ qualitatively. Therein consists the "first act of intuitionism" that is attributed, by Brouwer, to the mind and not to sensitiveness.

> Mathematics arises when the subject of two-ness, which results from the passage of time, is abstracted from all special occurrences. The remaining empty form [the relation of n to n+1] of the common content of all these two-nesses becomes the original intuition of mathematics and repeated unlimitedly creates new mathematical subjects.[6]

According to Brouwer, a mathematical proof is a mental construction. It is essential, for the intuitionistic view, to conceive mathematical objects as results of finite mental construction processes. This conception goes hand in hand with a strongly epistemic view about truth: the truth of a proposition depends, in an essential way, on the knowledge of the subject.

Another characteristic feature of intuitionism is the refusal of the actual infinite. Only finite reasoning can be justified. Measurable infinities are admitted in the sense that the step from n to n+1 can be repeated unlimitedly and is understood as principle of formation of the sequence of natural numbers. As Dummett puts it: "[T]he thesis that there is no completed infinity means, simply, that to grasp an infinite structure is to grasp the process which generates it, that to refer to such a structure is to refer to that process, and that to recognize the structure as being infinite is to recognize that the process will not terminate."[7] This view contrasts markedly with classical mathematics, which deals with the infinite process as a whole: "It is, however, integral to classical mathematics to treat infinite structures as if they could be completed and then surveyed in their totality."[8] To top it off, "an infinite process is spoken of as if it were merely a particularly long finite one".[9]

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[4]Brouwer 1975, 51.

[5]Cp. Heyting's Introduction to Brouwer 1975, xiv.

[6]Cit. in Kline 1972, 1199–2000.

[7]Dummett 1977, 56.

[8]loc.cit., 56.

[9]loc.cit., 57.

Intuitionistic Logic. Arendt Heyting (1881–1996) makes intuitionistic logic systematic in his work *Die Formalen Regeln der intuitionistischen Logik* (1930). While classical logic makes use of non-epistemic concepts of truth and falsehood (the truth of a statement is independent of its being known), intuitionistic logic grounds on a basically epistemic conception of them. Truth and falsehood are not properties that a statement has independently of its being known. That a statement is true means that the knowledge-subject has direct evidence for it or can exhibit a proof, i.e. a suitable mental construction, for it. Similarly, that a statement is false means the knowledge-subject knows that no matter how his knowledge will develop, he will never be able to exhibit a proof for it. Thus, while intuitionism reads the sentence "that p, is true" as "the epistemic subject is in possession of a proof for the proposition, that p", the assertion "that p, is false", in symbols: "$\neg p$", is saying something much stronger than "the epistemic subject is not in possession of a proof for p". "$\neg p$" means "the epistemic subject is in possession of a proof of the impossibility of p", i.e. he knows he will never have evidence for p, no matter how his cognitive process develops further. On the basis of these reflections, intuitionism does not include, among others, the law of the excluded middle in the tautologies of logic. There are statements, such as Goldbach's Conjecture (every even, positive integer greater than two is the sum of two primes), that are at the present undecided. Intuitionism emphasizes the temporal aspect of knowledge. The fact that someday, it will be perhaps possible to determine whether Goldbach's Conjecture is true or false does not change the current indecision. Potential truth values cannot replace actual ones. Only if a proper mental construction lays before the epistemic subject, he or she knows that p, or, respectively, that $\neg p$. The excluded middle ($p \vee \neg p$) as valid logical principle is dropped.

After this argumentation, it becomes clear that intuitionistic logic cannot admit the formal counterpart of indirect reasoning:

$$\frac{\begin{array}{c}[\neg \mathrm{p}]\\ \vdots \\ \bot\end{array}}{\mathrm{p}}$$

Another point of disagreement between classical and intuitionistic (predicate) logic is constituted by the interpretation of the quantifiers. Classical logic deals with quantified statements *as if* the multiplicity of objects the quantifier ranges over would be finite. To say it roughly: when the classical logician writes '$\forall x \alpha(x)$' he or she takes for granted that it is in principle possible to verify, for each single element of the domain in question, that it has the property of being α. But this is possible only if the quantifier ranges over a collection of only finitely many entities.

Current Foundations

Set Theory

Set theory is rather well known to be a quite efficient framework for foundational issues: mathematical as well as metamathematical objects can easily be represented within it, it gives clear answers to old philosophical questions on basic assumptions, allowed inferences and gapless proofs. Set-theoretic accounts, however, have to face one main challenge: the independence phenomenon, or syntactic incompleteness of set theory.

Let us recall that **ZFC** originated, in the first place, as an answer to the problems arisen within the context and set of problems of Cantor's *naïve* set theory. As is well known, Zermelo suggested a first axiomatization in 1908,[10] which was further developed to **ZFC**. **ZFC** as foundational framework helps to give an answer to the following fundamental questions:

(i) *Which are the basic assumptions of mathematics?*
(ii) *What is an allowed inference?*
(iii) *What counts as a proof without gaps?*

Indeed, one takes (**i**) the basic assumptions to be the axioms of **ZFC**, (**ii**) an allowed inference to be such that it only employs inference rules whose premises and conclusions are written in the language of first order logic, and (**iii**) a proof without gaps to be a derivation of a sentence from the **ZFC**-axioms through the use of allowed inferences.

Set theory can represent most *mathematical* objects from any mathematical area. There are standard representations of simple mathematical objects such as the following of natural numbers:

$$0 = \varnothing, 1 = \{\varnothing\}, 2 = \{\varnothing, \{\varnothing\}\}, \dots, \mathrm{n} + 1 = \mathrm{n} \cup \{\mathrm{n}\}, \dots$$

Functions are represented as sets of ordered pairs, groups $(G, \cdot)$ as the ordered pairs of a set G and an operation $\cdot$, and so on.

Set theory can represent *metamathematical* objects as well, for instance, through Gödel's coding.

It is worth mentioning that we do not need the full power of **ZFC** to formalize a number of metamathematical concepts. In particular, for that part of mathematics needed to set up **ZFC**, only some recursion theory is actually needed.

It is the representing property that makes it possible to take set theory as foundational framework for mathematics, metamathematics and yet, as some argue,

[10] Zermelo 1908.

for all mathematical reasoning. *De facto* "foundational framework" and "set theory" are most often used as tantamount.[11]

Nonetheless, the *independence phenomenon* constitutes no minor problem for any set-theoretic account. For, if set theory cannot *decide* every sentence of its own language, how can it work as a suitable framework for all mathematical reasoning? We recall that a theory **T** is syntactically complete when it decides every sentence **A** of its language. Sentences that are neither provable nor refutable in a theory are called "independent sentences". Set theory happens to be more sensible to the independence phenomenon than other mathematical areas, and it is often the use of a set-theoretic framework that makes the independence problem come to the fore in different mathematical fields, such as in operator algebra.[12]

A number of set theorists appear to be quite indifferent to the philosophical implications of the *independence phenomenon*. Others consider independence to be avoidable and search for stronger methods to decide whether an independent sentence is true or false. Two main questions arise in this context:

(i) Are *all* set-theoretic sentences either true or false?
(ii) If **ZFC** is extended to a stronger theory which axioms should be added?

Question **i.** is usually referred to as "realism/pluralism debate". "Realism" denotes that particular position in the philosophy of mathematics that takes *every* set-theoretic sentence to be either true or false, "pluralism", at variance, denotes the view that there are set-theoretic sentences that are true in one mathematical reality and false in another.[13] Ontological talk is often used to formulate question **i.**. Under the heading "universism" one usually understands the view that set theory has a unique model (as argued, for instance, by W. Hugh Woodin), under "multiversism" the view that there is a multiverse of various different universes of sets (as argued, for instance, by Joel D. Hamkins.)[14]

Answers to question **ii.** are often articulated in proposing criteria for new axioms. Already Kurt Gödel had suggested that new axioms could be justified either *intrinsically* or *extrinsically*.[15] We can render the idea of an *intrinsically* justified axiom by saying that it flows from the very nature of set or that it roots in the concept of set. The *extrinsic* justification plays, nowadays, a major role in Penelope Maddy's approach. Roughly, an axiom is justified if it meets the practical goals of set theorists. A further criterion welcomed by all parties (even those who are critical towards both intrinsicality and extrinsicality) is *maximality*. Shortly, *there exist as many sets as there can possibly exist*. A problem here is constituted by the fact that there are different ways to formally capture this property and that, depending

[11] See, for instance, Caicedo et al. (eds.) 2017.

[12] Hereto cp. Farah 2011.

[13] See, among others, the symposia on Set-theoretic Pluralism (https://sites.google.com/site/pluralset).

[14] See Hamkins 2012.

[15] Gödel 1947.

on this formal rendering, different existence axioms are introduced, to wit, large cardinal axioms or forcing axioms.[16]

The philosophy of set theory is today an active research field, in which new ideas are regularly brought into discussion, available concepts are constantly refined and farsighted perspectives are developed. The first chapter of the present volume gives more detailed insights into debated topics in current philosophy of set theory.

Homotopy Type Theory/Univalent Foundations

Part II of the present volume pursues two complementary goals, namely, **(i)** to explain the basic ideas of HoTT and **(ii)** to present special properties of it.

Classic set theory, as we saw, can represent all mathematical objects *via* sets, i.e. by *one kind* of objects. Type theory, on the contrary, works with a hierarchy of different *kinds* of objects, called "*Types*". The idea of a type-sensitive foundation has become more and more important along with the progressive development of *data-types*-talk in computer science. Moreover, actual mathematical practice seems to be aware of type differences even when single mathematicians plead for untyped frameworks: nobody would try to differentiate a vector space!

In type theory each term has a type.

Let "*nat*" be the type of natural numbers. Thus, "5 is a natural number" will be written as follows:

$$\mathbf{5} : \boldsymbol{nat}.$$

Operations are type-sensitive too. The addition on natural numbers, e.g., takes two objects of the kind *nat* as input and gives one object of the kind *nat* as output:

$$+ : \boldsymbol{nat} \times \boldsymbol{nat} \rightarrow \boldsymbol{nat}.$$

As already said, the type-hierarchy of mathematical objects and operations makes it possible to ascribe to each term its own type. Under "rewrite systems" is usually meant the reduction of complex terms to their *normal forms*. For instance, "5 + 2" and "7" are names of the very same object. The latter has a well-determined place in the natural number series. We write:

$$\mathbf{5} + \mathbf{2} \twoheadrightarrow \mathbf{7},$$

to indicate the *reduction* of the complex term "5 + 2" to its normal form "7".

[16]Cp. Incurvati 2017, 162 for Large Cardinal Axioms, and 178ff. for Forcing Axioms.

Type theory was introduced by Bertrand Russell as a way out from the difficulties come into being within the context and set of problems of *naive* set theory. At a variance with Russell's, current type theory has the additional problem of reducing terms to their normal form (**rewrite problem**). This makes it necessary to look for a formal counterpart for the operation represented by "$\twoheadrightarrow$".

This operation is to be found formally implemented, for the first time, in combinatory logic,[17] a logic which was introduced by Schönfinkel with the aim to drop quantifiers from the language of predicate logic. Combinatory logic is usually seen as the forerunner of the λ-calculus. In this regard it is worth mentioning that A. Church elaborated, around 1930, a framework for a paradox-free reconstruction of mathematics based on the notion of a type-free function-in-intension. The original theory was proved inconsistent around 1933 by S. Kleene and J.B. Rosser. Subsequently, its sound fragment, the so-called untyped λ-calculus, was isolated and employed in the context of the theory of computability. In this theory the abstraction operator "λ" plays a fundamental role. Such approach is based on the procedural conception of function, i.e. its conception as rule, prescription, algorithm in a general sense.

Indeed, two natural ways of thinking of functions are often appealed to. The first way is the relational way: it takes a function to be a type of relation. The second way takes a function to be a rule, or algorithm in a general sense, as we said above. Under the extensional reading, the concept of function and the concept of set are interdefinable. We reduce functions to sets by means of the notion of "graph of a function" and, conversely, we reduce sets to functions by means of the notion of "characteristic function of a set". *This interdefinability no longer obtains when we consider functions as rules*, since the latter notion always implies the idea of a procedure that is more or less effective. Note that an implicit type-distinction is present in the set-theoretical account of functions: functions, on the one hand, and arguments and values, on the other hand, live at different levels of the cumulative hierarchy. Thus, from a set-theoretical perspective the application of a function to itself makes no sense: from $f : A \to B$ it follows, in standard set theory with the axiom of foundation, that $f \notin A$. By contrast, in the λ-calculus it does make sense to apply an operation to itself: intuitively, operations can be thought of as finite lists of instructions which, therefore, do not differ in principle from the arguments on which they act: both are, loosely speaking, finite pieces of data. The active role (behaving as an operation) and the passive role (behaving as an argument) of certain elements F, G of this untyped universe is only position-determined. When we write $F(G)$, F is a rule that is applied to the argument G; but this does not prevent us from considering also the application $G(F)$ of G to F, where now G is the rule.

Martin-Löf's type theory provides an alternative foundational framework inspired by constructive mathematics. In particular, the requirements we recalled above as to intuitionistic logic are taken to hold; to wit, the exhibition of a witness for any existential assumption, the existence of a proof for one of the

[17] Cp. Schönfinkel 1924 and Church 1932.

horns of a disjunction, if the disjunction is taken to be true, or the exhibition of the proof that a sentence **A** leads to contradiction if this sentence is taken to be false. The implementation of such requirements within the theory is referred to as *internalization* of the **B**rouwer–**H**eyting–**K**olmogorov interpretation (**BHK**-interpretation) of the logical operators. In addition, constructive type theory presents the particular feature of admitting dependent types.

HoTT is a form of type theory based on constructive type theory[18] endowed with some elements from topology that can be used to construct models for logical systems.[19] One important reading of HoTT sees types as spaces and terms as points. Dependent types correspond to fibrations in topology, and the identity between terms is rendered by the concept of *path* between points. At a variance with set theory there is here no need for a logic next to or on top of it: constructive types already act as a formal calculus for deduction. HoTT (in the mentioned interpretation) along with the corresponding models has properties that naturally bring to the fore candidates for new axioms, as for instance, the univalence axiom.[20] Voevodsky's univalent foundations programme that emerged from these investigations is but one good example for a foundational framework, which, moreover, turns out to be well suited as a background theory for the implementation of modern mathematics in formal proof systems such as Coq.[21]

HoTT is a very promising approach. Even the Institute of Advanced Study in Princeton has dedicated a *Special Year on Univalent Foundations of Mathematics* (2012/2013), during which topologists, computer scientists, category theorists and mathematical logicians have collaborated to jointly promote HoTT.

The Contributions

Let us now give a short overview of all contributions.

[18] Cp. Martin-Löf 1975.

[19] Hofmann and Streicher 1998 constructed a model of Martin-Löf's type theory in the category of groupoids. Moerdijk 2011 conjectured that there exists a general link between Quillen models (models that contain, in particular, classical homotopy of spaces and simplicial sets) and type theory. Awodey, Warren and Voevodsky proved fundamental connections between homotopy theory and type theory.

[20] This approach interprets types as spaces or homotopy types. This makes it possible to directly work with spaces, without having to define a set-theoretic topology. The univalence axiom additionally holds in several possible models. Hereto see Voevodsky 2009.

[21] In the univalent perspective one uses a modified type theory, which is extended by certain axioms, such as the univalence axiom. This approach can very well be implemented as theoretical background of formal proof system, such as Agda and Coq.

Part I: Current Challenges for the Set-Theoretic Foundations

Mirna Džamonja and **Deborah Kant** attempt in their chapter "Interview with a Set Theorist" to give insights into current set-theoretic practice with a focus on independence and forcing. After giving some technical remarks, Džamonja describes the introduction and adoption of the forcing method in set theory and they present important forcing results. In a next section, they discuss the meaning of the word "axiom", differing between ZFC-axioms, large cardinal axioms and forcing axioms, and mentioning the questions of existence and truth in set theory. Then, some experience-based statements on set theory are revealed in a conversation on surprising events. Džamonja and Kant argue that such statements are part of set-theoretic knowledge. They give three hypotheses and encourage further research on set-theoretic practice.

Laura Fontanella provides in her chapter "How to Choose New Axioms for Set Theory" a mathematically highly informed overview on specific arguments for and against new axiom candidates for set theory. First, she discusses common justification strategies and focuses on Gödel's and Maddy's ideas. Second, she considers in detail the axiom of constructibility, large cardinal axioms (small ones and large ones), determinacy hypotheses, Ultimate-L and forcing axioms.

Claudio Ternullo considers in his chapter "Maddy on the Multiverse" Maddy's objections to multiverse approaches in set theory and suggests counter-arguments to her concerns. He elaborates in particular on the role of set theory as a generous arena, and as fulfilling the goal of conceptual elucidation, and on the issues of the metaphysics and possible axiomatization of the multiverse. In particular, Ternullo identifies two forms of multiversism – instrumental and ontological multiversism – and while he agrees with Maddy that all metaphysical considerations relating to the status and prospects of the multiverse should be disregarded, he argues that a multiverse theory can fulfil set theory's foundational goals, in particular generous arena and conceptual elucidation, and that, ultimately, it might also be axiomatized.

Penelope Maddy clarifies her view on multiversism in her "Reply to Ternullo on the Multiverse". She points out that extrinsic reasons for new axioms cannot simply be rejected by evaluating them as practical, since multiverse views also use extrinsic considerations as theoretical. Categorizing important forms of multiversism, as Ternullo did, she considers theory multiversism as an explicit form of instrumental multiversism, and heuristic multiversism. The main problem for theory multiversism is in her view that there is only one axiomatic multiverse theory (given by John Steel), but this theory is not intended to replace ZFC. Maddy rejects the claim that a multiverse theory is a better foundational theory than ZFC. However, she embraces the heuristic value of multiverse thinking for set theory.

Philip Welch presents in his chapter "Proving Theorems from Reflection" some open problems of analysis which are independent of ZFC. For their solution, he suggests a global reflection principle. First, he describes important relations between large cardinal axioms, determinacy principles and regularity properties of projective sets of reals. Second, he explains how reflection principles can justify large cardinal axioms. Welch considers a family $\mathcal{C}$ of the mereological parts of V and formulates the global reflection principle which states that the structure (V, $\in$, $\mathcal{C}$) be reflected to some $(V_\alpha, \in, V_{\alpha+1})$. This principle justifies the existence of certain large cardinals, which implies that all projective sets of reals have the regularity properties. This answers the problems presented at the beginning.

Part II: What are Homotopy Type Theory and the Univalent Foundations?

Thorsten Altenkirch motivates and explains in his chapter "Naive Type Theory" the use of homotopy type theory. He starts his presentation from a clear position by arguing that HoTT is a better foundation of mathematics than set theory. In the first half of the chapter, he revises important basic notions of type theory such as judgements, propositions, the Curry-Howard equivalence, functions, induction and recursion. Throughout the chapter, his explanations are supplemented by many exercises which invite the reader to use the introduced definitions. The second half is dedicated to HoTT. While referring to set-theoretic concepts for clarification, he literally denotes some classical principles as forms of lying and elaborates on several type-theoretic versions of AC. Altenkirch concludes with the topic of higher inductive types and gives an extended example of the definition of the integers.

Benedikt Ahrens and **Paige Randall North** focus in their chapter "Univalent Foundations and the Equivalence Principle" on the notion of equivalence of mathematical objects, which is one fundamental notion of the univalent foundations. Their aim is to prove the equivalence principle for different domains D, and defining equivalence through reference to D-properties and D-structures such that it coheres with mathematical practice. They review the univalence principle which holds in Voevodsky's simplicial set model and prove the equivalence principle for propositions, sets and monoids. Ahrens and North close their presentation with categories and show that the equivalence principle does not hold for arbitrary categories but for univalent categories.

Ulrik Buchholtz presents the construction of higher structures in his chapter "Higher Structures in Homotopy Type Theory". He distinguishes between two aims of HoTT: On the one hand its use as a foundation of mathematics in the univalent foundations, and on the other hand, the study of structures that are needed in mathematics. He introduces many technicalities to show in which specific aspects the constructions are complicated. Elementary HoTT is defined as MLTT with univalence, pushouts and propositional resizing, and can already be most needed. There remain, among others, the problems of constructing $(\infty,1)$-categories and developing a meta theory for HoTT/UF. Here, Buchholtz offers detailed and optimistic considerations on the challenges and possible means that could solve these problems.

In his chapter "Univalent Foundations and the UniMath Library: The Architecture of Mathematics" **Anthony Bordg** first explains the univalent foundations of mathematics, the Univalence Axiom and the so-called homotopy levels. Then, he gives insights into the UniMath library of formalized mathematics based on the univalent foundations, underlying in the process the compatibility of this library with classical reasoning supported by the Law of Excluded Middle and the Axiom of Choice. Second, he analyses some challenges for large-scale libraries of formalized mathematics. Bordg devotes his third part to an investigation of parallels between ideas from the architect Christopher Alexander and some organic features of mathematics. He argues that these features need to be preserved, hence they should become desirable properties of libraries of formalized mathematics if one wants these libraries to be scalable and a sustainable way of doing mathematics.

Andrei Rodin's article "Models of HoTT and the constructive view of theories" can be read in two ways: As the development of new formal tools to do philosophy of science and as a case study of the usefulness of the new developments around the Univalent Foundation program. On the philosophical side of things, he develops the constructive view of (scientific) theories in contrast to axiomatic approaches and those based on set theory. A crucial distinction here is between rules and axioms in their role as first principles. These general thoughts find their way to precise formal counterparts and similar to the part before both axiomatic deduction system and model theoretic tools based on set theory are contrasted with proof-theoretic alternatives benefitting from the tools of Martin-Löf Type Theory and Homotopy Type Theory. While still being a first study, it offers a very detailed account of the technical apparatus.

Part III: Comparing Set theory, Category Theory, and Type Theory

Neil Barton and **Sy Friedman** deal with the problem of different foundational perspectives – categorial and set-theoretic – in their chapter "Set theory and Structures" by presenting a modification of set theory to better respect structural discourse. In the first part, they respond to various concerns about both category-theoretic and set-theoretic foundations such as the problem of a determinate subject matter for category theory and too little isomorphism invariance in set theory. In a second part, they present a class theory with structures (NBGS), in which material set theory is combined with structures in such a way that a higher degree of isomorphism invariance is provable. Barton and Friedman advocate methodological pluralism for the foundations of mathematics and encourage further research on combining foundational theories.

Mirna Džamonja faces in her chapter "A New Foundational Crisis in Mathematics, Is It Really Happening?" the question of a potential rivalry between different foundational theories in mathematics. Her main aim is to describe the current situation of the foundations of mathematics and to resolve some worries. In order to illustrate some differences, she describes type theory, elaborates on identity and univalence and explains the topology behind the univalent foundations. She concludes by claiming that there does not have to be a unique foundation of mathematics. Džamonja argues for pluralism and that the univalent foundations and set theory can well complement each other.

Ansten Klev elucidates in his chapter "A Comparison of Type Theory with Set Theory" fundamental conceptual differences between Martin-Löf's type theory and set theory. He argues that type theory is better suited to clarify mathematical notions. In a first part, he describes types to be kinds or sorts of objects. In contrast, he explains sets as pluralities which can contain objects of arbitrary different types. In a second part, he considers in detail the syntax of type theory and presents types, propositions, terms, judgements and contexts (for hypothetical judgements). In his third part on functions, the advantage of having functions as primitive notions in type theory is highlighted. The final part is dedicated to identity, in which Klev elaborates on the difference between propositional and judgemental identity in type theory.

Penelope Maddy addresses in her chapter "What Do We Want a Foundation to Do?: Comparing Set-Theoretic, Category-Theoretic, and Univalent Approaches" the question of foundational theories by isolating rigorously the foundational jobs that are done by a suggested theory. First, she argues that set theory does the jobs risk assessment, generous arena, shared standard, and metamathematical corral. Second, she analyses that the foundational job that category theory was argued to be better suited for is essential guidance. Third, she introduces a new foundational job, proof checking, that is done by the univalent foundations. With her new terminology, Maddy provides a theoretic framework which is very well suited for a thorough comparison and discussion of foundational theories in mathematics.

Part IV: Philosophical Thoughts on the Foundations of Mathematics

In "Formal and Natural Proof: A Phenomenological Approach" **Merlin Carl** elaborates on the relation between formal proofs in the sense of a derivation and proofs in the sense we can find them in mathematical discourse. He motivates a positive connection *via* Gödel's Completeness Theorem and success of the generation of deduction with formal mathematics.

A crucial problem is the nature of formal and natural proofs. A classical dispute is the one between Rav and Azzouni, while the second explains the strong agreement of mathematicians and the statements they prove with a link to underlying derivation, the first claims that a derivation presupposes a sufficient understanding of the natural proof, which makes it absolute.

This is a challenge for the connection that the author answers in two ways. *Via* the success of formal mathematics and especially *via* the Naproche system, which tries to bridge the gap with linguistic methods including a historic case study, where a formalization attempt would have helped to find a flaw in a proof attempt. And *via* phenomenological methods, namely by enlarging Husserl's distinction between distinctiveness and clarity of judgements, to the mathematical proofs.

Michèle Friend provides with "Varieties of Pluralism and Objectivity in Mathematics" a revised version of an article published in *Mathematical Pluralism*, Special Issue of the *Journal of the Indian Council of Philosophical Research*, Mihir Chakraborty and Michèle Friend (Guest Editors) Springer. JICPR Vol. 34.2 pp. 425 – 442. DOI 10.1007/s40961-061-0085-3. ISSN: 0970-7794. pp. 425 – 442.

She offers an alternative to the realist and traditional accounts of objectivity in mathematics by elaborating the pluralistic position. To be more precise, there are several ways in which a philosophy of mathematics can be pluralistic, namely, epistemology, foundations, methodology, ontology and truth. Friend gives an overview on these different approaches, delivers thereby an interesting typography and analyses how each variation can explain objectivity in mathematics.

Graham Priest's chapter "From the Foundations of Mathematics to Mathematical Pluralism" tells the story of different foundational endeavours pinned at historically important actors, namely, Frege, Russell, Zermelo, Brouwer and Hilbert. These more historical aspects yield to the recent history of category theory. In § 9 and 10 the author introduces paraconsistent accounts of mathematics, motivated by impossible manifolds, which can be formalized by paraconsistent tools. This roundtrip to the study of such different mathematical structures motivates the final vote towards a more pluralistic philosophy of mathematics.

Roy Wagner's chapter "Does Mathematics Need Foundations?" is an important step back from our endeavour of the comparison of different foundational theories. It starts with a revisit of the classical foundational enterprises: Russell, Brouwer and Hilbert but also critical accounts like Poincaré's and Wittenstein or the maverick tradition. This offers insights into a variety of schools of thoughts in a broad narrative of the problems and possibilities of foundational thinking. The chapter ends with a very interesting case study from anti-foundationalist mathematics, namely, Kerala mathematics in the fourteenth to sixteenth century.

The chapter offers an account which preserves a lot of positive aspects often connected to the foundations of mathematics, like shared standards and risk assessment, and shows that this is not only a metaphysical possibility but historical fact.

Part V: Foundations in Mathematical Practice

Nathan Bowler's chapter "Foundations for the Working Mathematician, and for Their Computer" discusses the aspects of the candidates discussed in this volume which are reflected in the mathematical practice. The chapter focuses mainly on

ZFC but considers some advantages of categorical or type-theoretic foundations. A key distinction is in how far we differentiate between human-friendly and machine-friendly aspects of the foundations.

Bernhard Fisseni, Martin Schmitt, Bernhard Schröder and Deniz Sarikaya's Article "How to frame a mathematician: Modelling the cognitive background of proofs" uses the tool of frame semantics, developed in linguistics and artificial intelligence, as a basis for describing and understanding proofs. The key idea of frames is that concepts offer some roles (also: features, attributes): When talking about "selling" something, we know, for example, that the transfer involves a seller, a buyer, a sold item, money, etc., and, if not given explicitly, we assign default entities from the discourse universe to (some of) those roles. The general argument is that we proceed similarly when comprehending proofs. As an illustration, the authors develop a frame representation for induction proofs. They use a formalism based on feature structures, a general data structure prominent in formal linguistics.

Lawrence C. Paulson gives in his chapter "Formalising Mathematics in Simple Type Theory" an intriguing insight into the formalization of mathematics. Starting from the conviction that there will always be different formalisms that are better suited than others for specific purposes, he carefully delineates the possibilities of simple type theory. He shares his personal perspective, which is based on long experience with different automated proof systems and presents the original code of a theorem on stereographic projections in HOL Light and its translation into Isabelle/HOL. Paulson shows here how translations from one proof system to another are accomplished, and he emphasizes the importance of such translational work for the future use of proof systems in mathematics.

In "Dynamics in Foundations: What Does It Mean in the Practice of Mathematics?" **Giovanni Sambin** gives insight in the programme of *dynamic constructivism.* The first part gives a philosophical motivation and explains the interplay between mathematical practice and the philosophy of mathematics. We learn (without too many technical details) about a possible shared arena for classical and constructive mathematics, the minimalist foundation (developed jointly by the Author and M. E. Maietti). The second part offers a proof of concept for dynamic constructivism, we learn about point free topology and other theories developed by the author in this new paradigm. This is especially important given the motivation in the first section asking from a foundation to foster new and creative pieces of mathematics. The last part reflects on the benefits of the adaption to this position and draws a parallel to fruitful developments in the sciences *via* the vocabulary from philosophy of science, namely by elaborating how the adaption of this position yields a helpful new perspective - a new paradigm.

The Editors

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Contents

Part I
Current Challenges for the Set-Theoretic Foundations