History of Mathematics Education

Ole Ravn
Ole Skovsmose

Connecting Humans to Equations

A Reinterpretation of the Philosophy of Mathematics

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A Reinterpretation of the Philosophy of Mathematics

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Acknowledgements

On the following pages we have attempted to present some of the most important positions in the philosophy of mathematics. At the same time, we have tried to add new dimensions to this special domain of philosophy. It is our hope that our presentation can help to broaden the conception of mathematics as a general human concern. We see mathematics as an integral part of human social life, meaning that a philosophy of mathematics turns relevant to any kind of social theorising.

In 2011 we published the book *Matematikfilosofi* in Danish. The present book is not a translation of the Danish version; it includes reformulations, specifications and additions. During the process we realised that we were working with a new manuscript.

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Aalborg, 2018 Ole Ravn and Ole Skovsmose

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Abstracts

The book is divided into four parts, each addressing a particular dimension of the philosophy of mathematics. We have tried to capture these dimensions through four questions: Where is mathematics? How certain is mathematics? How social is mathematics? and How good is mathematics?

Part I—Where is mathematics? Mathematics is about something, but it is unclear just what kind of objects we are dealing with. We will examine three different types of answers to this question. According to Plato, mathematics is about immutable entities that constitute a world of ideas. It is real—although not palpable to our senses. We cannot sense mathematical objects, but we can grasp them by means of our rationality. This rationality, then, is a unique system of perception through which we reach beyond the capabilities of our senses and "see" the objects in the world of ideas. This is not merely an expression of Plato's personal way of thinking. Platonism in the philosophy of mathematics reoccurs time and time again. For example, great mathematicians and logicians such as Frege and Gödel voiced Platonist notions.

However, one finds other answers to the question "Where is mathematics?" Through the works of Copernicus, Kepler, Galilei and Newton, mathematics was located in a new environment as an integral part of nature. Galilei claimed that nature is written in the language of mathematics and Newton put it all in formulas. Mathematics is found in nature since the laws of nature are all mathematical. In order to understand nature, we must understand mathematics, as God seems to have created the world with a collection of mathematical formulas at his fingertips.

Historically, the question "Where is mathematics?" can also take us in a completely different direction. Kant is the great representative of a turn, which emphasises the claim that mathematics expresses conditions for human perception. The fact that nature appears to be based on a mathematical format is not due to nature, but due to the way we perceive nature. Mathematics represents characteristics of our "forms of reasoning" which we project onto our experiences. As a consequence, mathematics becomes a feature not of nature, but of our perceptual framework when we receive input from the world. Kant considered our "conceptual structures" as immutable and as a priori to any cognition.

Part II—How certain is mathematics? The apparent certainty of theorems and proofs has been considered the most fascinating aspect of mathematics. Since antiquity, certainty has been associated with theoretical knowledge. For something to count as knowledge, it must demonstrate certainty. This is a crucial idea in the classical concept of knowledge, represented by Plato and Socrates, who find that it is possible to obtain epistemic certainty. As a consequence, they reject the sceptical position: that one can—with good reason—doubt everything.

In Part II, we discuss the developments in mathematics and logic around the end of the 20th century and the beginning of the 21st. Here, the many creative constructions in the building of mathematics in the foregoing centuries were put to the test of certainty and consistency. There was a general urge towards securing the foundation of mathematics, which is sometimes referred to as the "foundational crisis." In this search for foundations, we will follow the attempts to secure mathematics made by Frege, Russell and Whitehead. In their research, the goal was to build a bridge from logic to mathematics so that mathematics could be presented as just as strict and certain as the core of propositional and predicate logic. We also consider

Hilbert's attempts to formalise mathematics in order to produce a proof that what we refer to as our mathematical knowledge is consistent and flawless – and we will discuss the setback of Hilbert's program in the emergence of Gödel's incompleteness theorems. Finally, we develop the connection between logical positivism, the young Wittgenstein's ideas on the role of science, and the developments in the philosophy of mathematics and logic.

As discussed above, the first two questions of "Where is mathematics?" and "How certain is mathematics?" bring us into the traditional orbit in the philosophy of mathematics, or what we have referred to as the classical universal-paradigm. This tradition has been preoccupied with the claim—or has just taken it for granted—that mathematics is universal. So within Part I and Part II, we let ourselves be captured by the universality paradigm of mathematics. But, as described, we are going to be engaged with further questions, bringing us to Part III and Part IV.

Part III—How social is mathematics? Historically, it has been considered obvious that we have to discover the world of mathematics. The mathematician might be travelling deep into this fascinating continent of mathematics, and report on some of its features and creatures. Thus, it seems obvious that the Pythagorean Theorem was discovered, and that it is a discovery that there are infinitely many prime numbers. By contrast, it has not yet been discovered whether any even number can be written as the sum of two primes. In the 20th century, however, it became doubted whether mathematics is indeed about discovery and instead it was suggested that mathematics is a human invention. According to this interpretation, mathematics today could have looked very different. Notions such as "group" and "vector space" refer to structures that are defined as part of the mathematical research process. It seems eccentric to speak of a vector space as being "discovered."

In Part III, this perspective offers us a way to discuss "How social is mathematics?" and how we should interpret the role of human beings in the construction of mathematics. What role does the individual mathematician play in this process, and what is the role of the mathematical research community as a whole?

We start out by investigating the philosophy of mathematics called intuitionism. This refers to the position that mathematics is entirely a human construction, although a very particular type of construction. Thus, intuitionism presents the construction process as strictly individual and mental. We challenge this account of human construction by reference to the works of Lakatos and his account of the socio-historical construction process of mathematical concepts. Lakatos presents this construction as taking place in the mathematical community. Finally, we look into the notion of mathematics as measure as presented by the later Wittgenstein. On this view, mathematics is to be considered as an expression of social conventions. One can think of grammar rules in our language as an example of such conventions, and if one interprets mathematics as language, it seems consequential to consider mathematical structures as manifestations of linguistic conventional usage. In this way, we reach an interpretation of mathematics as a profoundly social construction.

Part IV—How good is mathematics? The fact that mathematics is able to *describe* the world seems obvious. The descriptive power of mathematics seems overwhelming; just consider the role of mathematics in astronomy and in natural science in general. Mathematics provides the essential tool when one has to provide a "world view."

One has experienced the strength of mathematical descriptions in terms of their capacity to provide simplified models of reality. This brings us to the idea that one *acts* through the use of mathematics. Whether we choose to use one or another mathematical model as a representation of, say, the economic situation, has an impact on what we see and what decisions we might make. However, mathematics can also operate without us being in full control of the human modelling of reality. Today's society is submerged in mathematical structures of an extremely complex nature, which interact with each other and form the society in which we live. Mathematical action may therefore also be of this more uncontrollable nature.

No actions are made in an ethical vacuum. Actions can have all kinds of qualities. They can be benevolent, risky, dangerous, needed, exciting or boring, to name only a few potentials. This also applies to acts that include mathematics. Mathematics-based action does not derive a special status due to mathematics being involved. Instead, they open a range of ethical issues that we address through the question "How good is mathematics?"

In the concluding Chapter 12, we reconsider our whole project presented in the book, as well as the possibility that more dimensions could be added to a philosophy of mathematics than the four that we have addressed. Thus, we see our work not as a concluded project, but as one that calls for many more steps. As we argue, mathematics is a social construction, as are the philosophy of mathematics and the traditions of which it consists. It will not be easily changed by a couple of new dimensions—a non-universal paradigm of mathematics is still quite a novel conception.

Chapter 1

This chapter addresses ontological questions as formulated in ancient Greece. Mathematics is about something, but it is unclear what kind of objects mathematics is dealing with. The chapter examines different suggestions. According to Plato, mathematics is about immutable entities that constitute a world of ideas. This world is real—although not palpable to our senses. We cannot sense mathematical objects, but we can grasp them by means of our rationality. This rationality, then, is a unique system of perception through which we reach beyond the capabilities of our senses and "see" the objects in the world of ideas, including the real mathematical objects.

The location of mathematical objects in a world of ideas is not merely an expression of Plato's personal way of thinking. Platonism in the philosophy of mathematics reoccurs time and time again. For example, great mathematicians and logicians such as Frege and Gödel voiced Platonist notions. Thus the chapter addresses Platonism after Plato, Platonism before Plato, as well as Plato's Platonism. Furthermore, the chapter examines the idea of axiomatisation, and how this structures Euclid's *Elements*. This work got a paradigmatic significance, not only with respect to the formulation of mathematical knowledge, but with respect to knowledge in general.

Chapter 2

This chapter gives the Renaissance and rationalist philosophers of the 16th and 17th century have the word. The Renaissance is generally characterised by the belief that human reason can provide insight in the organisation of the world. The world was still considered God's creation, but it became increasingly common to view it as machine, which functioning the deity has not interfered with since its creation.

The chapter starts out with an account of the so-called scientific revolution and the break with Aristotelian physics that it represents. The move from a geocentric to a heliocentric worldview becomes analysed in detail. The use of mathematics for describing nature is a central element in this move, and the chapter examines the tying together of explanations of nature and mathematics that took place during this tumultuous time. Infinitesimals challenged mathematical ontology. If mathematics is the “language of nature,” infinitesimals must somehow relate to entities in reality. But as such, they are rather unmanageable, for how can a world that has actual extension be built by units so small that they have no extent? How many infinitesimals have to be added up in order to turn into a natural object?

Chapter 3

This chapter presents how, among others, Hume and Kant locate mathematics in the human apparatus of cognition. According to Hume, there exists two kind of knowledge, namely knowledge concerning “facts” derived from sense perception, and knowledge of “quantities or numbers.” While knowledge of facts is strictly empirical, knowledge concerning quantities of numbers is analytical. And analytical knowledge can be characterised as only concerning conceptual relationships.

To Kant, mathematics tells something about the way in which we experience the world, and not about the world as such. Mathematics applies to nature, but this is not due to any resemblance with nature. Mathematics fits nature because it represents how we, human beings, necessarily must experience nature. There is no ontological mathematics-nature unity, but there is a unity between mathematics and categories for human understanding. Thus Kant provides a radical relocation of mathematics. It was not any longer to be found in some eternal world of ideas, nor in nature, but in configurations provided by the human mind. Finally, the chapter addresses mathematics entities like Cantor’s Set, Sierpinski’s Triangle, and the Peano’s curve, which all constitute challenges for the Kant’s interpretation of mathematics.

Kant eliminated the possibility of scepticism with respect to mathematics and also with respect to the most fundamental statements in physics, like Newton’s laws.

Chapter 4

This chapter investigates the logicist programme that became influential in the first part of the 20th century. Logicism pursues the idea of reducing mathematics to the secure foundation that has been established in logic. It suggests how mathematical concepts can be defined in terms of logical concepts, and how mathematical theorems can be derived from logical proposition.

The chapter considers Frege’s critique of important philosophic conceptions of mathematics, before it presents his elaboration of the logicist programme as provided in his *Begriffsschrift*. Then follows a presentation of Russell and Whitehead’s continued detailed elaboration in *Principia Mathematica*. This elaboration was initiated by Russell’s discovery of a paradox that apparently destroyed the solidity of the logical foundation of mathematics in the format suggested by Frege. Russell communicated his discovery of the paradox to Frege, who got deeply chocked. While Frege gave up solving the paradox and paralysed in his logical endeavours, Russell moved on and tried to locate a solution that could save the logicist programme. In his own time, Frege was an unknown German mathematician with a taste for logic. His work remained unknown to the wider public until Russell discovered and developed his ideas. Later, it became evident that Frege was the most important person in the development of modern logic.

Chapter 5

This chapter discusses a programme for restoring certainty in mathematics. It addresses the establishment of meta-mathematics through which one could try to demonstrate that mathematics is without contradictions. Meta-mathematics should establish a critical mathematical investigation of mathematics itself. This strategy for a mathematical self-critique was formulated by Hilbert, who did not, as Frege and Russell had suggested, try to establish a secure foundation for mathematics in logic. Hilbert advocates a different way out of the foundational crisis caused by the appearance of logical paradoxes in the heart of mathematics. The occurrence of such paradoxes indicates that one has allowed oneself too much: drawn conclusions that are not valid, used axioms that give rise to inconsistencies, or the like. Hilbert wants to analyse the mathematical axiomatics and proof methods in order to ensure that they, when properly chosen, cannot lead to contradictions.

This programme appears extremely ambitious, for how can one ensure that one cannot, sometime in the future, end up in deducing contradictions in some mathematical theories? Mathematics is continually developing, and one can imagine that in the future, one may come to prove theorems that contradict what is proved today. One thing is to guard against already known contradictions, but how can one guards against future and new contradictions? The programme is not only ambitious; it turns out to be unrealistic as well. This unrealism is shown by Gödel, who demonstrates the impossibility of proving the consistency of a mathematical system that includes a certain degree of complexity. With the insight provided by Gödel's two incompleteness theorems, the meta-mathematical programme becomes revealed as an illusion.

Chapter 6

This chapter presents the position of logical positivism, which assumes that mathematics and logic play a particular role in science. This line of thought also holds the key to understanding the formalist interpretation of mathematics that can be considered a further development of the meta-mathematical programme.

A particular input to logical positivism is provided by Wittgenstein's *Tractatus*. But what are the messages in the *Tractatus*? In his own preface, Wittgenstein summarises them in two points: first, that philosophical problems are caused by misunderstandings of the logic of language; and secondly, what is possible to say can be said clearly. In this way, a limit for human knowledge becomes drawn. This limit coincides with the limits of language. We do not possess epistemological tools that reach any further. Outside language exists, epistemologically speaking, dark-nightly nothing. When talking about language, Wittgenstein has in mind the formal language provided by mathematics and logic. This leads to the slogan that mathematics is the language of science. Wittgenstein also makes the observation that mathematics is composed of tautologies, which points towards the formalist programme, according to which mathematics can be identified with pure formal structures. Within formalism, all ontological issues with respect to mathematics becomes eliminated. Mathematics is not about anything. Mathematics is a game with symbols without content.

Chapter 7

This chapter explores Brouwer's conception of mathematics. Brouwer is the principal proponent of the direction in the philosophy of mathematics referred to as intuitionism. Brouwer is also concerned about the paradoxes that troubled Frege, Russell, Whitehead and Hilbert. However, in order to eliminate the paradoxes, he suggests a radically different remedy. A basic idea of intuitionism is that the very language we use to communicate mathematics gives rise to mathematical problems and paradoxes. This also applies to formalised languages. Brouwer finds that mathematical formalisms are only imprecise and limited representations of mathematics. When we, for instance, denote the set of natural numbers as *N*, we provide a linguistic construction within the recorded mathematics, which does not have anything to do with intuitive mathematical constructions. Brouwer's thesis is that the linguistic representations of mathematics have departed from the intuitive mathematical constructions. Linguistic expressions bring us to believe that we express ourselves about mathematical entities, while we actually become seduced by language. According to Brouwer, linguistic formulations are the source of paradoxes and inconsistencies.

According to Brouwer, mathematics develops through a particular mathematical intuition shared by all human beings. This way, Brouwer supports the thesis that new mathematics is created. Brouwer's constructivism opens up the possibility of interpreting mathematics as a process. The chapter summarises Heyting's presentation of mathematical processes through an invented dialogue between an intuitionist, a formalist, a classic and other characters in mathematics.

Chapter 8

This chapter addresses Lakatos' version of constructivism. Lakatos shows how mathematical notions, theorems and proofs become constructed through a dialogical process. He illustrates how the logic of mathematical discovery operates as an interactive process within the mathematical community. In opposition to logical positivism, Popper, who inspired Lakatos profoundly, moved from being interested in verification to being interested in falsification. In parallel with this, he also turned his attention from "the context of justification" to "the contexts of discovery." This way scepticism and fallibilism come into focus. Lakatos reinterprets Popper's ideas, and this way he formulates fallibilism as a position in the philosophy of mathematics.

Lakatos introduces a neo-empiricist perspective on mathematics, which he refers to as quasi-empiricism in the sense that mathematics to some extent resembles natural sciences. Mathematics is about concept development, and it is driven forward by particular observations concerning conceptual connections. With reference to Euler's polyhedron theorem, Lakatos provides a detailed specification of how this development takes place, which he condenses in terms of the method of proofs and refutations. This method includes many features, as for instance the emergence of proof-generated concepts. Lakatos finds that mathematical statements have their origin in experience, but his naturalism differs from classic versions, as for instance suggested by Mill. Instead of searching for an empirical basis for mathematical notions and theorems in terms of sensory experiences, Lakatos considers the basis of mathematical reasoning to be quasi-empirical objects.

Chapter 9

This chapter addresses Wittgenstein's interpretation of mathematics, although not in the form expressed in the *Tractatus*, but as it later become elaborated. Wittgenstein emphasises the profound social component in mathematical constructions. He interprets mathematics as basically a rule-following system, and he sees mathematical rules as being similar to grammatical rules. Rule-following can be interpreted as a social practice, or as a social construction.

As an alternative to the theory of language and mathematics as presented in *Tractatus*, the later Wittgenstein does not propound another theory of the same universal scope. In his opinion, no universal theory about language can be formulated. His central term in this clarification is "language game." There are many different kind of games: football, handball, tennis, badminton, draughts, chess, bingo … But there is no essence underlying the fact that we call them games. Wittgenstein's point is that it is a hopeless to try to provide a universal definition of "game." And further: what can be said about "game" also applies to "mathematics.". It is hopeless to try to answer the question: "What is mathematics?"

Chapter 10

This chapter considers what it means if one move beyond the thesis of isolation, which assumes that mathematics operates and develops according to its own intrinsic priorities. In general, it is important to considered to what extent mathematics becomes formed through worldviews, metaphysical assumptions, ideologies, political priorities, economic conditions and technologies. The chapter addresses such social structurings of mathematics, and concentrates on metaphysics, technology and the market.

First, it is exemplified how general worldviews can shape mathematics. It becomes illustrated how mathematical disciplines become structured by contemporary ideas and social trends. This is exemplified with reference the change in priorities with respect to research in differential equations, which in turn reflects the rise and fall of the mechanical worldview. Second, the importance of technological tools for the development of mathematics becomes addressed. In particular, it becomes illustrated how the computer is shaping mathematics: not only by changing features of the mathematical research practice, but also by changing conceptions of what counts as a mathematical proof. Finally, the chapter considers the commodification of knowledge that has taken place and the impact this has on the formation of mathematics. In brief, one finds market values in the shaping of mathematical research priorities. These observations all challenge the thesis of isolation.

Chapter 11

This chapter considers reasons for abandoning the thesis of neutrality, which assumes that mathematics does not incorporate any value judgements. The chapter considers to what extent mathematics forms part of technological and political actions by establishing new forms of actions as well as by providing justifications for actions. Like any form of action, also mathematics-based actions are not neutral. They are expressions of particular interests as well as of political or economic priorities.

The different formats of mathematics-based actions become illustrated. One format is the fabrication of fictions, which refers to the possibility that mathematics can be used in identifying new technological alternatives. Fabrication of facts refers to the possibility that implementation of algorithmic procedures create new structures, for instance with respect to production and control. Fabrications of risks refer to the possibility that when mathematics comes to make part of automatics procedures, for instance in form of automatics piloting, new forms of accidents turn possible. Finally, it becomes pointed out that mathematics provides illusions of objectivity. All such mathematics-based fabrications become part of our reality. The fabrications of fictions, facts, risks and illusions do not take place in an ethical vacuum. They merge with a number of other forms of actions, making ethical neutrality in mathematics impossible. The chapter is concluded by an example referring to Google's search engine, which is a powerful mathematical way of structuring knowledge.

Chapter 12

This chapter addresses different conceptions of the philosophy of mathematics. Classic positions become characterised as two-dimensional by concentrating on ontological and epistemological issues. As an alternative, a four-dimensional philosophy of mathematics become presented by expanding the philosophy to include a social and a ethical dimension as well.

The book has elaborated upon a four-dimensional philosophy of mathematics, but it does not make any claim about the adequate number of dimensions. Its main point has been to move beyond any two-dimensional philosophy, and in this move to establish human beings as having an all-important role in mathematics. By having opened a space for a humanised conception of mathematics—as opposed to the traditional anti-human conceptions—even more dimensions may emerge, as for instance an aesthetic and a political dimension. This leads to the more general question: What could it mean to move beyond the borders set by the Western tradition in the philosophy of mathematics? In fact, one comes to acknowledge the possibility that a philosophy of mathematics may stretch beyond the borders set by philosophy itself.

Preface

Mathematics is universal. Such a claim has been consistently repeated throughout history. Furthermore, mathematical theorems seem ahistorical; when something has been proved, it is lifted away from historical temporality and established in a realm of eternal truths. In this way, mathematics has been presented as different from all other sciences by producing knowledge that lasts and holds true for everyone everywhere.

The universality of mathematics has been outlined in many different ways by various philosophies of mathematics and it has been assumed by many to be an indisputable fact.

At the same time, we are familiar with the fact that mathematics is an integral part of almost all forms of technology and involved in many of our activities in everyday life. Mathematics finds its way into production and into decision-making processes. It is integrated into strategies for optimisation and automation, in medicine as well as in warfare. Mathematical algorithms are at the core of electronic processes such that computers and all forms of electronic packages contain condensed mathematical algorithms. But mathematics is also involved in more subtle manners in creating ways of seeing, interpreting and discussing a number of phenomena in our surroundings like the GDP or the global warming issues—it is even integrated in our leisure through the way we cook or play games.

But *is* mathematics actually universal? Or is the situation rather that many have been seduced by this particular interpretation of mathematics?

What happens if we start thinking about what an alternative to the universality of mathematics could look like? Leaving universality behind—in the sense that mathematics is the same no matter where you are or who you are—we will introduce the idea that humans and social settings create and shape mathematics from scratch.

This alternative line of human-centred reasoning has not been a popular choice in the philosophy of mathematics. We will refer to those philosophies that remain in the universality-paradigm as the "classical" philosophies of mathematics. For these classical philosophies, the essential questions to address are about ontology and epistemology, while human or social actions are rarely mentioned as having anything to do with mathematics. These philosophies address questions of how we can conceive of mathematics as part of the universe and about how and why mathematics represents certain knowledge. In this sense, classical philosophies of mathematics are two-dimensional, dealing mostly with ontological and epistemological issues.

In our exposition here, we will argue that a philosophy of mathematics holds at least four dimensions, as outlined by the following four questions: Where is mathematics? How certain is mathematics? How social is mathematics? How good is mathematics?

Each of the four parts of the book will be focused on one of these dimensions. There will, of course, be overlapping issues, but the point is that if we leave one of these dimensions behind when thinking about mathematics, we will lose the depth and breadth needed to comprehend what mathematics is about.

As we argue above, the four questions are not the usual questions in the philosophy of mathematics—well, at least the last two questions are not. Classical philosophy of mathematics has always been preoccupied with discussing the certainty of our mathematical knowledge and the source from which it springs—that is, finding the place where mathematics can be located or otherwise reasoned to belong. By posing these first two questions, we follow the tradition in the philosophy of mathematics.

In the Western history of the philosophy of mathematics, reflections about mathematical connections to technology, everyday use of mathematics, the actual construction of mathematics, and especially to relations to the human lifeworld in general, have been largely absent. In presenting the two final parts of this book, we introduce the role of human beings and the social into a philosophy of mathematics. Understanding the relation between mathematics and human beings, or even understanding the processes in a community of human beings producing or using mathematics, must, from our perspective, be a key element in any philosophy of mathematics. As a consequence, we also need to address the two questions "How social is mathematics?" and "How good is mathematics?"

In this way, we want to present a four-dimensional philosophy of mathematics. There could, however, be other dimensions as well, a point to which we are going to return in Chapter 12. One could think of the questions we address here as concerning ontology, epistemology, language and ethics. By not merely addressing ontology and epistemology, but also addressing language and ethics, we seek to make a contribution to the humanisation and socialisation of the philosophy of mathematics.

Hence, we aim to challenge and change the scope of those fundamental questions addressed in the philosophy of mathematics, but we will do this in a historical, step-by-step approach. We will start out from the classical outline of a philosophy of mathematics and locate what have been considered as the most important positions in the philosophy of mathematics. By doing this, we also describe how we ourselves have been introduced to the philosophy of mathematics: at universities dominated by a classical Western interpretation of this philosophy. We have experienced the classical universality-paradigm at close range and we do not want to leave behind this seduction in any postulatory way; instead, we shall use it actively to dissolve the paradigm of universality from within.

First, we acknowledge the value of what has been thought about mathematics throughout centuries and the importance of knowing this history when we reflect on mathematics today. Also, we have the intention of developing a book that can become a challenging resource in educational curricula that deal with conceptions of mathematics. In this sense, it is important for us to present many of the classical ideas about mathematics as well as their challengers. And finally, it is our intention to observe how important lines of thinking from the middle of the 20th century and onwards point to radically different trajectories of thinking about mathematics than the classical universality-paradigm. In this way, we set out not just to develop two new dimensions in any philosophy of mathematics but also to confront the classical paradigm about mathematics even on the ontological and epistemological issues.

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