Mathematical and Analytical Techniques with Applications to Engineering

Rafael Martínez-Guerra
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Algebraic and Differential Methods for Nonlinear Control Theory

Elements of Commutative Algebra and Algebraic Geometry

Springer

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Preface

This book is written so the reader with little knowledge on mathematics and without experience in control theory can start from the beginning and go directly through the logical evolution of the topics. If the reader has advanced knowledge on mathematics, is fluent on abstract algebra, linear algebra, etc., will be able to read this book a little faster, even so, the reader must study the mathematics sections of the book, to make sure he is familiar with any special notation that we have presented. If the reader only knows about mathematics but has no experience in natural sciences or engineering, he must not be discouraged for the occasional use of examples and vocabulary of other fields (physical models or others). If the reader knows a little about control theory, maybe from previous courses of engineering or physics, it is possible that he will have to work slightly more at the beginning to get used to the ideas and notations in modern mathematics. When the reader is in a section that contains some mathematics that are new to him, it is required of the reader to study the examples, to do many exercises and to learn the new subject as if he has taken a course of it. At the end of the book, the reader will know not only control theory but also a little about useful mathematics. Whatever the previous experience the reader has, he will find in the book some topics that are needed to be learned for this particular area. As well, the book shows in a simple way some elementary concepts of mathematics such as basic algebraic structures (sets, groups, rings, fields, differential rings, differential fields, differential field extensions, etc., that is to say elements of commutative algebra and algebraic geometry in general) for the introduction to the theory of nonlinear control seen from a differential and algebraic point of view. Enough material has been included, rigorous proofs and illustrative exercises of each subject, as well as some examples and demonstrations are left to the reader for the good understanding of the chapters of the book. In the selection of material for this book, the authors intended to expound basic ideas and methods applicable to the study of differential equations in theory of control. The exposition is developed in such a way that the reader can omit passage that turn out to be difficult for him. The content of this book constitutes a tool that can be read by not only mathematicians, engineers and physicists but also all users of the theory of differential equations and theory of control. Control theory has become into a scientific discipline and of engineering that seems destined to have an impact in all aspects of modern society. First studied by mathematicians and engineers now it is increasingly present on areas of physics, economy, biology, sociology, psychology, etc. What is control theory and why is so important for a wide variety of specialities? Control theory is a set of concepts and techniques that are used to analyze and design various kinds of systems, independently from their physical nature and special functions. The most important aspect of control theory is the development of a quantitative model that describes the relation and interaction of a cause and effect between the variables of a system. This means that the language of control theory is mathematical and any serious attempt to learn control theory must be accompanied of mathematical precision and its comprehension. We have attempted, at writing this book, to make a pedagogic introduction to control theory that is accessible to readers with little background beyond linear and abstract elemental algebra and matrix theory while begin motivating for those mathematician readers with more sophisticated theoretical knowledge. As result, the book is more autonomous and less specialized. This book begins with the study of elementary set and map theory being a little bit abstract. Chapters 2 and 3 on groups theory and rings respectively are included because of their important relation to the linear algebra, the group of invertible linear maps (or matrices) and the ring of linear maps of a vector space being perhaps the most amazing examples of groups and rings. We deal with homomorphisms and Ideals as well as in Chap. 3 we mention some properties of integers. Also it is well known that group theory at its inception was deeply connected to algebraic equations. The success of the theory of groups in algebraic equations led to the hope that similar group-theoretic methods could be a powerful arsenal to attack the problems of differential equations. Still more, Ritt and Kolchin tackled the problem form the differential and algebraic point of view (Chap. 10 of control theory from point view of differential algebra). Chapter 4 is devoted to the matrices theory and the linear equations systems. Chapter 5 gives some definitions of permutations, determinants and inverse of a matrix. In Chap. 6 is tackled in notion of vector in real Euclidian space and in general vector space over a field, bases and dimension of a vector space as well as sums and direct sums. In Chap. 7 we treat with linear maps or linear transformations, Kernel and image of a linear map, dimensions of kernel and image, in addition, the application in linear control theory of some abstract theorems such as the concept of kernel, image and dimension of space are illustrated. Chapter 8 we shall consider the diagonalization of a matrix and the canonical form of Jordan. In Chap. 9 we attack elementary methods of solving differential equations and finally in Chap. 10 we treat the nonlinear control theory from the point of view of differential algebra.

Mexico City, Mexico

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To the memory of my father
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Juan Javier Montesinos-García

Contents

Notations and Abbreviations

$\mathbb{N}$	The Set of natural numbers
$\mathbb{Z}$	The Set of integers numbers
$\mathbb{Q}$	The Set of rational numbers
$\mathbb{R}$	The Set of real numbers
$\mathbb{C}$	The Set of complex numbers
$A, B, \ldots$	Capital letters represent arbitrary sets
$x, y, \ldots$	Lowercase letters represent elements of a set
$A \subset B$	A is subset of B
$x \in A$	x is element of A
$A \cup B$	The union of two sets
$A \cap B$	The intersection of two sets
A^c	Complement set of A
$A \backslash B$	Difference of sets A and B
$\emptyset$	Empty set
$\iff$	Necessary and sufficient condition
$\forall$	For all
$\sim$	Equivalence relation
$a \equiv b \bmod n$	a is congruent with b module n
A/R	Quotient set
$*$	Binary operation for groups
$\det(A)$	The determinant of a square matrix $A \in \mathbb{R}^{n \times n}$
$\|A\|$	The determinant of a square matrix $A \in \mathbb{R}^{n \times n}$
$\mathrm{tr}(A)$	The trace of a matrix A
$A^\top$	The transpose of a matrix A
$\{\cdots\}$	Set
$(a_{ij})_{i,j}$	$m \times n$ matrix with entries a_{ij}, $1 \leq i \leq m, 1 \leq j \leq n$
$\mathrm{rank}(A)$	Rank of a matrix A
A^{-1}	Inverse of A
$\dot{y} = \frac{dy}{dt}$	First derivative of y with respect to t
E_λ	Eigenspace of E corresponding to eigenvalue λ

$\dim(V)$	Dimension of V
$\mathscr{L}(V, W)$	Space of linear transformations of V in W
Ker $\boldsymbol{\varphi}$	Kernel of $\boldsymbol{\varphi}$
diff trd°	Differential transcendence degree
$\square$	Designation of the end of a proof
$<(>)$	Less (greater) than
$\leq(\geq)$	Less (greater) than or equal to
$\forall$	For all
$P(n)$	A statement
$\cong$	Isomorphism
G/N	Quotient group
$K\langle u\rangle$	Differential field generated by the field K, u and its time derivatives
K/k	Differential field extension $k \subset K$

Chapter 1
Mathematical Background

Abstract This chapter focuses on the basic concepts and algebra of sets as well as a brief introduction to the theory of functions and the well known principle of mathematical induction. All this background will be needed as a tool to understand the theory of linear algebra and differential equations as set forth in the following chapters.

1.1 Introduction to Set Theory

Definition 1.1 A **set** A is any collection of objects called **elements** of A that is well defined (same nature) [1–4]:

$$A = \{a_1, a_2, a_3, \ldots, a_{n-1}, a_n\}$$

- $a_i \in A$ denotes a_i is an element of set A.
- $x \notin A$ denotes x is not an element of set A.

Hereinafter, capital letters, $S, T, \ldots$ denote sets and each of the elements of a set is denoted by a lowercase letters, $x, y, \ldots,$ etcetera.

Example 1.1 Let $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers (positive integers), $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$ the set of integers, $\mathbb{Q} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\right\}$ the set of rational numbers and $\mathbb{R}$ the set of real numbers. It is clear that $-5 \in \mathbb{Z}$, $\frac{3}{5} \in \mathbb{Q}$, $\sqrt{23} \in \mathbb{R}$, $\pi \notin \mathbb{N}$, $\sqrt{23} \notin \mathbb{Z}$.

According to the number of elements of each set, the set can be **finite** or **infinite**. For example, if we consider the set of grains of sand in the world, although the amount is very large and possibly we could never finish counting, this set is finite. The numbers set are infinite. The set with no elements is called **empty set** [5–8]. This set is unique and it is represented by Ø. On the other hand, a finite set consisting of one element is called **singleton**.

Remark 1.1 Ø and {Ø} are totally different. The first one is a set without elements, whereas the another set is a singleton with the unique element that is Ø.

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Definition 1.2 Let A, B two set. It is said that A is a **subset** of B, $A \subset B$ (or $B \supset A$), if and only if every element of A is element of B, i.e., $a \in A \Rightarrow a \in B$, $\forall a \in A$. (If $A \neq B$ we call A a proper subset of B).

According to this new concept, note that if $a \in A$ then we can to write an equivalent notation for this fact, i.e. $\{a\} \subset A$. Other examples of subsets are given by the following chain of inclusions: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, the larger field, $\mathbb{C}$ is called an extension field of $\mathbb{R}$, similarly $\mathbb{R}$ is an extension field of $\mathbb{Q}$, this concept will have relevance in Chap. 10.

Remark 1.2 Let A an arbitrary set, then the empty set satisfies $\emptyset \subset A$ for all set A. To prove this fact, let us suppose that the statement is false. It is false only if $\emptyset$ has an element that does not belong to A. Since $\emptyset$ has not elements, then is not true and therefore $\emptyset \subset A$ for every A.

Remark 1.3 Let A, B, C any set, according to the Definition 1.2,

- $A \subset A$. This property states that the inclusion is **reflexive** (every set is a subset of itself).
- The inclusion is **antisymmetric**. This property is written as follows: $A \subset B$ and $B \subset A$ then $A = B$.
- If $A \subset B$ and $B \subset C$ then $A \subset C$, i.e., the inclusion is **transitive**.

The condition of equality between two sets is given from the following axiom.

Property 1.1 (Axiom of extension): *Two sets are equal if and only if they have the same elements.*

The axiom of extension states that if A and B are set such that $A \subset B$ and $B \subset A$ then A and B have the same elements and therefore $A = B$. This is the more general form to prove the equality between two set and this is the necessary and sufficient condition. Note that equality is **symmetric**, i.e., if $A = B$ then $B = A$. In particular, when $A = A$ it said that equality is **reflexive**. Finally, the relationship of equality is **transitive**. This property is analogous to the property of transitivity for subsets, i.e., $A = B$ and $B = C$ then $A = C$ [10–12].

1.1.1 Set Operations and Other Properties

Sometimes, when we have two set, A and B that are not necessarily equals, it is necessary to define a third set whose elements are given by the above sets. So that, it is necessary to define an operation in the sets with that feature. Before the definition is established, an important axiom that determines these new operations:

Property 1.2 (Axiom of unions): *For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.*

Although is complicated to define a set namely **universe**, that contains every set, in the next definition we will suppose that there exists a set (non empty) that contains a finite number of sets.

Definition 1.3 Let A, B two subsets of a set S. The **union** of two sets A and B is the set

$$A \cup B = \{s \in S \mid s \in A \text{ or } s \in B\}$$

where, we say $s \in A \cup B$ if and only if $s \in A$, $s \in B$ or s is in both sets.

The other important operation is the intersection between two sets, defined as follows.

Definition 1.4 Let A, B two subsets of a set S. The **intersection** of two sets A and B is the set

$$A \cap B = \{s \in S \mid s \in A \text{ and } s \in B\}$$

where, we say $s \in A \cap B$ if and only if $s \in A$ and $s \in B$.

Remark 1.4 These operations can be generalized for more sets. Let $\{A_\alpha\}_{\alpha \in I}$ a collection of subsets of S, then:

- $\bigcup\limits_{\alpha \in I} A_\alpha = \{s \in S \mid s \in A_\alpha \text{ for some } \alpha \in I,\ I \text{ is an index set}\}$
- $\bigcap\limits_{\alpha \in I} A_\alpha = \{s \in S \mid s \in A_\alpha \text{ for all } \alpha \in I,\ I \text{ is an index set}\}$

In any application of set theory, all sets that intervene can be considered as subsets of a fixed set. This set is called **universal set** (or simply universe) and it is denoted by $\mathscr{U}$.

Definition 1.5 Let the universe $\mathscr{U}$ and an arbitrary subset A of $\mathscr{U}$, the complement of set A respect to $\mathscr{U}$, to the set of all elements that do **not** belong to A but that belong to $\mathscr{U}$ [11, 13].

In analogous manner the union an intersection operations have some particular properties. We establish the following assertion. Note that each assertion should be demonstrated.

1. **Associativity**
    - $A \cup (B \cup C) = (A \cup B) \cup C$
    - $A \cap (B \cap C) = (A \cap B) \cap C$
2. **Commutativity**
    - $A \cup B = B \cup A$
    - $A \cap B = B \cap A$
3. **Distributivity**
    - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
    - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. **Idempotency**
   - $A \cup A = A$ for all set A.
   - $A \cap A = A$ for all set A.
5. **De Morgan's laws**
   - $(A \cup B)^c = A^c \cap B^c$
   - $(A \cap B)^c = A^c \cup B^c$

To prove each one of the claims, it is necessary to establish the equality between both sets. That is to say, we prove the two inclusions:

$$A = B \Leftrightarrow A \subset B \text{ and } B \subset A$$

For example, for the case of associativity of union, we have the following proof. Let $x \in A \cup (B \cup C)$, then

$$\begin{aligned} x \in A \cup (B \cup C) &\Leftrightarrow x \in A \text{ or } x \in (B \cup C) \\ &\Leftrightarrow x \in A \text{ or } x \in B \text{ or } x \in C \\ &\Leftrightarrow x \in (A \cup B) \text{ or } x \in C \\ &\Leftrightarrow x \in (A \cup B) \cup C \end{aligned}$$

Thus $A \cup (B \cup C) = (A \cup B) \cup C$.

At the end of this section, we will establish a proposition with more properties. Now, we establish the following definition.

Definition 1.6 Let A, B subsets of a set S. A and B are said to be **disjoint** if they have not elements in common, that is to say: $A \cap B = \emptyset$.

Note, that this definition can be extended to any family of sets: A family of sets is **pairwise disjoint** or **mutually disjoint** if every two different sets in the family are disjoint ($A_i, A_j \in \mathscr{A}, i, j \in I, A_i \cap A_j = \emptyset$).

Another important operation is the difference between two sets, defined as follows.

Definition 1.7 Let A, B subsets of a set S. The **difference** of the sets A and B is the set of all elements of set A that are not belong to the set B, i.e.,

$$A \setminus B = A - B = \{s \in S \mid s \in A\ ,\ s \notin B\} = A \cap B^c$$

where B^c denotes the complement of B.

Example 1.2 Let $K = \{1, 2, 3, 4, 5\}$, $L = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Note that $K \subset L \subset M$. According to the difference of sets, we have that

$$
\begin{aligned}
L - K &= \{6, 7, 8\} \\
M - L &= \{9, 10, 11, 12\} \\
M - K &= \{6, 7, 8, 9, 10, 11, 12\}
\end{aligned}
$$

Note as these sets are related by the union between them, and by their number of elements denominated cardinality of the set, i.e.:

$$M - K = (M - L) \cup (L - K)$$

and $7 = card(M - K) = card(M - L) + card(L - K) = 4 + 3$. This idea will be generalized and stated in Chap. 10 as a theorem for a tower of fields and their transcendence degree.

Remark 1.5 According to the above definition, the complement of a set A can be written as follows:

$$A^c = \{x \mid x \in \mathscr{U}\ and\ x \notin A\} = \mathscr{U} \setminus A$$

that is, the difference of the universe with the set A.

The properties of difference of set are the following:

1. For every set A, $A \setminus A = \emptyset$.
2. For every set A, $A \setminus \emptyset = A$.
3. For every set A, $\emptyset \setminus A = \emptyset$.
4. Let A and B are nonempty sets. If $A - B = B - A$ then $A = B$

The latter property can be demonstrated as follows. By definition, we have

$$
\begin{aligned}
A - B &= \{x \mid x \in A \text{ and } x \notin B\} \\
B - A &= \{x \mid x \in B \text{ and } x \notin A\}
\end{aligned}
$$

If $A - B = B - A$ then every element of $A \setminus B$ is an element of $B \setminus A$, but in the set $A \setminus B$ there are only elements of A. In the same manner, all element of $B \setminus A$ is an element of $A \setminus B$ but in the set $B \setminus A$ there are only elements of B, therefore, $A = B$.

1.2 Equivalence Relations

Definition 1.8 Let A a nonempty set. A **relation** in A is a subset R of $A \times A$.

Remark 1.6 We write $a \sim b$ if $(a, b) \in R$.

Definition 1.9 A relation R in A it is said to be:

1. **Reflexive**, if $a \sim a \ \forall\, a \in A$
2. **Symmetric**, if $a \sim b \Rightarrow b \sim a \ \forall\, a, b \in \ A$
3. **Transitive**, if $a \sim b$ and $b \sim c \Rightarrow a \sim c \ \forall\, a, b, c \in \ A$

Definition 1.10 A relation R in A it is said to be **equivalence relation** if R is reflexive, symmetric and transitive [14, 15].

Example 1.3 Let $A = \mathbb{Z}$. Let $a, b \in A$. It said to be a and b have the same parity, if a and b are both even or are both odd. We say that a and b are related, $a \sim b$, if a and b have the same parity. Indeed, this is a equivalence relation:

- $a \sim a \Leftrightarrow (-1)^a = (-1)^a$, $a - a = 0$ is even.
- $a \sim b \Leftrightarrow (-1)^a = (-1)^b \Leftrightarrow (-1)^b = (-1)^a \Leftrightarrow b \sim a$.
- $a \sim b \Leftrightarrow (-1)^a = (-1)^b$ and $b \sim c \Leftrightarrow (-1)^b = (-1)^c$. By virtue of the properties of equality, we have $(-1)^a = (-1)^b = (-1)^c$. Therefore $a \sim c$.

Definition 1.11 Let $m, n \in \mathbb{Z}$. It is said that m divides n if there exists $q \in \mathbb{Z}$ such that $n = mq$. In other words, m is a factor of n or n it is a **multiple** of m.

Remark 1.7 The notation for divisibility of numbers is the following:

- If m divides n, we write $m \mid n$.
- In other case, i.e., if m not divides n, we write $m \nmid n$.

Definition 1.12 Let $n \in \mathbb{N}$, $a, b \in \mathbb{Z}$. It said to be a is **congruent** with b **module** n if $n \mid (b - a)$. We write $a \equiv b \mod n$. The congruence module n is a equivalence relation.

Definition 1.13 Inclusion is not an equivalence relation. Note that $\mathbb{N} \subset \mathbb{Z}$ but $\mathbb{Z} \not\subset \mathbb{N}$ ($\subset$ is not symmetric).

Definition 1.14 Let $\sim$ an equivalence relation in A. The **equivalence class** of $a \in A$ is denoted by the following set:

$$\mathscr{C} = [a] = \{x \in A \mid x \sim a\}$$

The elements of $[a]$ are the elements of A that are related with a. If $\mathscr{C}$ is an equivalence class, then any element of $\mathscr{C}$ is a **representative** of $\mathscr{C}$.

Example 1.4 In the case of the parity we have:

- $[0] = \{\dots, -4, -2, 0, 2, 4, \dots\} = \{2m \mid m \in \mathbb{Z}\}$
- $[1] = \{\dots, -5, -3, -1, 1, 3, 5, \dots\} = \{2m + 1 \mid m \in \mathbb{Z}\}$

Example 1.5 According to $a \equiv b \mod n$, if $n = 5$, we have:

- $[0] = \{\dots, -10, -5, 0, 5, 10, \dots\} = \{5m \mid m \in \mathbb{Z}\}$
- $[1] = \{\dots, -9, -4, 1, 6, 11, \dots\} = \{5m + 1 \mid m \in \mathbb{Z}\}$

- $[2] = \{\dots, -8, -3, 2, 7, 12, \dots\} = \{5m + 2 \mid m \in \mathbb{Z}\}$
- $[3] = \{\dots, -7, -2, 3, 8, 13, \dots\} = \{5m + 3 \mid m \in \mathbb{Z}\}$
- $[4] = \{\dots, -6, -1, 4, 9, 14, \dots\} = \{5m + 4 \mid m \in \mathbb{Z}\}$

Theorem 1.1 *If $\sim$ is an equivalence relation in A, then*

$$A = \bigcup_{a \in A} [a]$$

In addition, if $[a] \neq [b] \Rightarrow [a] \cap [b] = \emptyset$.

Proof First, we prove the double contention of subsets.

1. Let $a \in A$. Then $a \in [a]$ and therefore $a \in \bigcup\limits_{a \in A} [a]$. This implies that $A \subset \bigcup\limits_{a \in A} [a]$.
2. Given that $[a] \subset A \; \forall a \in A$, therefore $\bigcup\limits_{a \in A} [a] \subset A$.

Hence

$$A = \bigcup_{a \in A} [a]$$

Finally we will use the contrapositive form to prove the last statement. Suppose that $[a] \cap [b] \neq \emptyset$. Let $c \in [a] \cap [b]$, then we have $c \sim a$ and $c \sim b$, therefore $a \sim b$. Now, if $x \in [a]$, then $x \sim a$ and $x \sim b$ (since $a \sim b$) and hence $x \in [b]$. This implies $[a] \subset [b]$ and in a similar manner, $[b] \subset [a]$. Therefore $[a] = [b]$. □

Definition 1.15 Let A a nonempty set. A **partition** of A, is a collection $\{A_\alpha\}_{\alpha \in I}$ of subsets of A such that:

$$A = \bigcup_{\alpha \in I} A_\alpha$$

In addition, if $\alpha \neq \beta$ then $A_\alpha \cap A_\beta = \emptyset$. A_α are so called **elements of the partition**.

Remark 1.8 Let A a nonempty set. An equivalence relation in A gives rise to a partition of A. The elements of this partition, are the equivalence classes. Reciprocally, a partition of A generates an equivalence relation. In conclusion, let $a, b \in A$. a and b are said to be related if and only if they are in the same element of the partition.

Definition 1.16 Let A a nonempty set and R (or $\sim$) an equivalence relation in A. A is said to be a **quotient set** denoted by:

$$A/R = A/\sim = \{[a] \mid a \in A\}$$

Example 1.6 Let $n \in \mathbb{N}$. Consider the congruence module n in $\mathbb{Z}$ (that is an equivalence relation). The quotient set of $\mathbb{Z}$ is given by

$$\mathbb{Z}/n\mathbb{Z} = \{[0], \dots, [n-1]\}$$

Sometimes, $\mathbb{Z}/n\mathbb{Z}$ is denoted by $\mathbb{Z}/n$ or $\mathbb{Z}_n$. As particular cases, we have:

1. $\mathbb{Z}/2\mathbb{Z} = \{[0], [1]\}$
2. $\mathbb{Z}/5\mathbb{Z} = \{[0], [1], [2], [3], [4]\}$

1.3 Functions or Maps

Definition 1.17 Let A, B nonempty sets. A **function** or **map** is a relation given by a set A and a set B with the following property: each element $a \in A$ is related to exactly one and only one element $b \in B$. In other words, a function or map is a rule that assigns to each element a of A [16, 17], one and only one element b of B. If $f : A \to B$ is a function, the set A is called **domain** of the function and the set B is the **codomain** or range of f. A function f is commonly denoted by:

$$\begin{aligned} f : A &\to B \\ a &\longmapsto b \end{aligned}$$

The notation $f(a) = b$ is equivalent to say $a \longmapsto b$. The **image of** $a \in A$ is $f(a) \in B$. The **image of the function** f is:

$$\operatorname{Im} f = \{f(a) \mid a \in A\} \subset B$$

Example 1.7 1. Let f defined by:

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ a &\longmapsto a^2 \end{aligned}$$

It is not hard to see that some elements are given by:

$$f(2) = 4,\ f(-2) = 4,\ f(5) = 25,\ f(-5) = 25,\ f(0) = 0.$$

2. Let g defined by:

$$\begin{aligned} g : \mathbb{Z} &\to \mathbb{Z} \\ n &\longmapsto 2n + 1 \end{aligned}$$

Some elements using the correspondence rule, are the following:

$$g(2) = 5, g(-2) = -3, g(5) = 11, g(-5) = -9, g(0) = 1.$$

3. Let A and B nonempty sets. The functions

$$\begin{aligned} \pi_1 : A \times B &\to A \\ (a, b) &\longmapsto a \end{aligned}$$