Multiple Criteria Decision Making

Michalis Doumpos
José Rui Figueira
Salvatore Greco
Constantin Zopounidis *Editors*

New Perspectives in Multiple Criteria Decision Making

Innovative Applications and Case Studies

Springer

Multiple Criteria Decision Making

Series Editor

This book series focuses on the publication of monographs and edited volumes of wide interest for researchers and practitioners interested in the theory of multicriteria analysis and its applications in management and engineering. The book series publishes novel works related to the foundations and the methodological aspects of multicriteria analysis, its applications in different areas in management and engineering, as well as its connections with other quantitative and analytic disciplines. In recent years, multicriteria analysis has been widely used for decision making purposes by institutions and enterprises. Research is also very active in the field, with numerous publications in a wide range of publication outlets and different domains such as operations management, environmental and energy planning, finance and economics, marketing, engineering, and healthcare.

More information about this series at http://www.springer.com/series/13834

Michalis Doumpos · José Rui Figueira ·
Salvatore Greco · Constantin Zopounidis
Editors

New Perspectives in Multiple Criteria Decision Making

Innovative Applications and Case Studies

Springer

Editors
Michalis Doumpos
School of Production Engineering
and Management
Technical University of Crete
Chania, Greece

Salvatore Greco
Department of Economics and Business
University of Catania
Catania, Italy

Faculty of Business and Law
University of Portsmouth
Portsmouth, UK

José Rui Figueira
CEG-IST, Instituto Superior Técnico
Universidade de Lisboa
Lisbon, Portugal

Constantin Zopounidis
School of Production Engineering
and Management
Technical University of Crete
Chania, Greece

Audencia Business School
Institute of Finance
Nantes, France

ISSN 2366-0023 ISSN 2366-0031 (electronic)
Multiple Criteria Decision Making
ISBN 978-3-030-11481-7 ISBN 978-3-030-11482-4 (eBook)
https://doi.org/10.1007/978-3-030-11482-4

Library of Congress Control Number: 2018967753

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

An Outline of the Status and Perspective of Multicriteria Decision Analysis

For more than four decades Multiple Criteria Decision Analysis (MCDA) has consistently been one of the most active areas in Operations Research and Management Science (OR/MS). Since the pioneering work by John von Neumann and Oskar Morgenstern on utility theory, the development of decision analysis by Howard Raiffa and Ron Howard, the contributions of Abraham Charnes and William Cooper on goal programming, and those of Tjalling Koopmans and Arthur Geoffrion on the foundations of efficiency measurement and multi-objective optimization, Kenneth Arrow's contributions to social choice theory, and Bernard Roy's foundations of outranking relations, the field of MCDA made significant progress in terms of methodological development and applications.

MCDA deals with decision-making/aiding problems involving the consideration of multiple (conflicting) criteria, attributes, points of view, goals, and objectives. Such problems naturally arise in all areas of business activity, the public sector, as well as in choices made by individuals. In contrast to the traditional framework of single-objective problems, where the best option can be described by a single measure, when dealing with multiple criteria the problem becomes ill-defined because a single best solution does not exist. Therefore, various behavioral, modeling, and algorithmic issues arise, which cannot be addressed unless a systematic methodology is adopted. This procedure is not only prescriptive providing answers to a given decision problem, but also constructive, in the sense that the actors involved in the decision process progressively gain a better understanding of the problem and their preferences, that ultimately leads to nontrivial solutions to complex instances.

The field of MCDA provides an arsenal of methodologies and tools to handle the above issues, including soft approaches for problem structuring and decision modeling, techniques and models for aggregating criteria, optimization approaches, and algorithms for problems involving multiple objectives, and decision support system (DSS) implementations. Throughout its history, MCDA has followed a dynamic path of development. New types of decision models have been introduced, allowing the aggregation of different types of information (qualitative, quantitative, fuzzy, etc.), new multi-objective optimization tools have been explored for interactive decision support and combinatorial problems (e.g., metaheuristics), and advanced DSSs have been developed using improved data management/ visualization and web-based technologies. Moreover, the field has progressed in terms of behavioral issues, on aspects related to preference modeling and elicitation, the treatment of uncertainties, imprecision, and ill-determination, while also strengthening its connections with emerging data analytic technologies.

At the same time, the range of applications has been constantly widening and new areas of interest arise. Except for standard business applications (finance, logistics, marketing, human resources, etc.), many new areas now benefit from MCDA, including environmental management, energy planning, sustainable development, and various areas of the public sector and policy making.

For MCDA to maintain its success path there are several areas for future development. For instance, the extension of existing decision models to allow the modeling of more complex preference structures could provide additional flexibility to decision analysts and decision makers with more general and less restrictive tools for handling difficult decision aiding instances. More complex models require axiomatization, deep understanding of their analytical properties, and tools to make them comprehensible/accessible by decision makers. Procedures for preference modeling and elicitation using information derived from data in a robust framework could facilitate the construction of decision model and reduce the cognitive effort involved. Behavioral aspects of preference modeling are also worth the investigation, together with exploring algorithmic advances in areas such as metaheuristics, soft computing, data analytics/visualization, and computer science (e.g., web-based technologies, tools for knowledge representation and modeling, etc.).

Addressing some of these ideas and areas requires an interdisciplinary approach, combining elements from various areas in OR/MS, mathematical economics, and computer science, among others. Adopting such an interdisciplinary approach could not only lead to advances on the theory of MCDA but also promote the field in other areas.

Aims and Scope

The aim of this book is not to constitute a reference for providing an overview of standard and well-known MCDA approaches. Several other books and edited volumes have already covered this area rather comprehensively. Instead, this edited volume seeks to focus on emerging areas of research in MCDA and the perspectives in the theory and applications of the field, thus providing researchers working in this area with a collection of high-quality chapters indicating how the MCDA is currently forming and how it can be shaped in the future. It is worth noting that this covers both theoretical aspects and applied research. While the importance of the perspectives in the theory of MCDA is mostly obvious, we should emphasize that the trends and perspectives in terms of applications are also important to identify new areas that have the potential for applied MCDA research, understands the context of these domains and design new MCDA approaches that can be successfully applied in practice. With these remarks in mind, below we provide an outline of the organization and the contents of this edited volume.

Organization

The book includes 16 contributions organized in four parts covering a wide range of MCDA methodologies, recent advances, and applications.

The first part of the book includes four chapters devoted to some fundamental methodologies and MCDA concepts. In the first chapter (New Trends in Preference, Utility, and Choice: From a Mono-approach to a Multi-approach) A. Giarlotta provides a comprehensive overview of some new trends in preference modeling, utility representation, and choice rationalization. The chapter starts with the traditional "mono-approach" traditionally used in mathematical economics for describing an agent's preference structure. The recent trend towards using a "multi-approach" that relies on multiple tools is introduced and some characteristic approaches are presented. New advances in this alternative paradigm are also analyzed in relation to MCDA.

The second chapter (Analytic Hierarchy Process and Its Extensions) by A. Ishizaka covers the analytic hierarchy process (AHP) and its extensions. AHP has traditionally been one of the most widely used methods in MCDA. The chapter first introduces the main ideas and methodological steps of AHP and then presents new advances and extensions in areas such as the analytic network process, group decision-making, variants for sorting problems, and visualization tools.

In the third chapter (Beyond Multicriteria Ranking Problems: The Case of PROMETHEE), Y. de Smet summarizes the recent developments in PROMETHEE methods, which follow the principles of outranking relations theory. PROMETHEE method have been originally introduced for multicriteria choice and ranking problems. Recently other types of problems, such as sorting and clustering, have also been addressed through variants of the PROMETHEE methods. The chapter describes some of these variants and discusses the relations between ranking, sorting, and clustering problems.

The final chapter (Preference Disaggregation for Multicriteria Decision Aiding: An Overview and Perspectives) of the first part is devoted to preference disaggregation analysis. M. Doumpos and C. Zopounidis describe the principles of this methodological stream of MCDA and its uses for constructing different types of decision models. The perspectives in this area are also discussed, in the context of robustness analysis, the use of alternative types of decision models, the optimization tools used to infer preference information from decision instances, as well as the potential of extending this area to large data.

In the first chapter (Normed Utility Functions: Some Recent Advances) of the second part, R. Mesiar, A. Kolesárová, A. Stupňanová, and R. R. Yager summarize some new results and trends in aggregation theory and introduce some new ideas that can be useful for providing multicriteria decision aiding. More specifically, the authors present two recently developed aggregation approaches, namely the *k*-additive and *k*-maxitive aggregation functions. Moreover, construction techniques are also presented.

The next chapter (Interpretation of Multicriteria Decision Making Models with Interacting Criteria) by M. Grabisch and C. Labreuche focuses on MCDA models that allow the modeling of interactions between criteria, such as the generalized-additive independence (GAI) model. The chapter further describes ways to develop an interpretation of general utility-based models through the introduction of importance indices for the decision criteria. The issue of constructing a monotone decomposition of the GAI model is also discussed.

In the last chapter (New Directions in Ordinal Evaluation: Sugeno Integrals and Beyond) of the second part of the book, M. Couceiro, D. Dubois, H. Fargier, M. Grabisch, H. Prade, and A. Rico present new directions on the use of Sugeno integrals for multicriteria evaluation problems in an ordinal setting. The chapter surveys the axiomatic characterizations of Sugeno integrals and their expression in possibilistic logic. Moreover, new developments in this area are presented such as the use of local utility functions, the notion of bipolar qualitative evaluation, as well as the use of Sugeno integrals and if-then rules for qualitative data analysis.

The first chapter (Advances and New Orientations in Goal Programming) of the third part is devoted to goal programming (GP). D. Jones and C. Romero provide an overview of the literature on different variants of GP models and proposed a conceptual distance-metric framework that unifies/describes the existing GP models. The chapter also analyzes the connections to bounded rationality and social choice functions and discusses future developments to expand the use and flexibility of GP models.

The next chapter (Robust Goal Programming with Interactive Fuzzy Coefficients), by M. Inuiguchi, is also devoted to GP, but in a fuzzy context where the goals and coefficients in the objective are fuzzy. To treat the fuzziness in such elements of an GP model, the approach of oblique fuzzy vectors is introduced. This approach extends existing methodologies for fuzzy GP by allowing the modeling of the interactions between fuzzy coefficients. Solution procedures are also discussed.

In the third chapter (Multiobjective Bilevel Programming: Concepts and Perspectives of Development) of the second part, M. J. Alves, C. Henggeler Antunes, and J. P. Costa cover the area of multi-objective bilevel programming. Multi-objective problems that have a hierarchical structure have attracted significant research interest. The chapter provides a novel view of the main concepts in this area, including the optimistic/pessimistic leader's perspectives, as well as algorithmic issues. The chapter also discusses traditional and emerging application fields as well as pitfalls in existing approaches, which may lead to new advances and improvements.

The fourth part of the book includes six chapters devoted to applications of MCDA in various emerging areas. In the first chapter (Multi-criteria Evaluation in Public Economics and Policy) of this part, G. Munda presents the contributions of MCDA techniques in public economics and policy. The chapter starts with an outline of cost–benefit analysis (CBA), which is the standard tool used in welfare economics. CBA is then systematically compared against the MCDA paradigm using ten comparison criteria, thus leading to the identification of the benefits and possibilities that MCDA tools provide in this important area.

In the next chapter (Perspectives on Multi-criteria Decision Analysis and Life-Cycle Assessment), L. C. Dias, F. Freire, and J. Geldermann discusses the combination of MCDA and life-cycle assessment (LCA) for environmental management. First the LCA framework is discussed and then the main characteristics of the MCDA perspective to environmental decision-making are outlined. Finally, an overview of the trends and perspective on the combination of the two approaches is given.

The chapter (The Monitoring of Social Innovation Projects: An Integrated Approach) of M. F. Norese, F. Barbiero, L. Corazza, and L. Sacco, presents a case study regarding the application of a MCDA approach based on the ELECTRE outranking methods for monitoring of social innovation projects by the Municipality of Turin in Italy. Except for a MCDA approach, the proposed analysis further combines other tools, such as cognitive mapping and actor network analysis to analyze the behavior of funded innovated start-up companies and to evaluate their business projects as part of an inclusive and sustainable economy.

The next chapter (Multiobjective Optimization in the Energy Sector: Selected Problems and Challenges), by C. Henggeler Antunes, illustrates the applications of multi-objective optimization approaches in the energy sector, focusing on electricity smart grids. The chapter covers issues such as unit commitment and dispatch problems, resilient systems, the usage of demand-side resources, problems associated with electric vehicles, as well as issues related to energy markets.

The area of energy systems is also the subject of the next chapter (Optimization and Multicriteria Evaluation of District Heat Production and Storage), by R. Lahdelma, G. Kayo, E. Abdollahi, and P. Salminen. The authors present a case study about the use of MCDA techniques for the evaluation of renewable energy technologies for district heating in Finland. The proposed methodology combines stochastic multicriteria acceptability analysis (SMAA) with a production planning optimization model taking into consideration various technical and economic criteria.

The book closes with the chapter (Comparison of Routing Methods in Telecommunication Networks—An Overview and a New Proposal Using a Multi-criteria Approach Dealing with Imprecise Information) by J. Clímaco, J. Craveirinha, and L. Martins, on the evaluation and comparison of routing models in telecommunication networks. The author proposes a MCDA approach based on the VIP (Variable Interdependent Parameter) software, with an additive aggregation of criteria coping with imprecise information. The formulation of the MCDA model is illustrated through an application to a problem involving the choice of a point-to-point routing method in a transport telecom network.

Acknowledgements

Sincere thanks must be expressed to all the authors who have devoted considerable time and effort to prepare excellent comprehensive works of high scientific quality and value. Without their help it would be impossible to prepare this book in line with the high standards that we have set from the very beginning of this project. José Rui Figueira also acknowledges the support from the FCT grant SFRH/BSAB/139892/2018 the under the POCH Program.

Chania, Greece	Michalis Doumpos
Lisbon, Portugal	José Rui Figueira
Catania, Italy/Portsmouth, UK	Salvatore Greco
Chania, Greece/Nantes, France	Constantin Zopounidis
November 2018	

Contents

Contributors

Elnaz Abdollahi Department of Mechanical Engineering, Aalto University School of Engineering, Aalto, Finland

Maria João Alves CeBER and Faculty of Economics, University of Coimbra, Coimbra, Portugal;
INESC Coimbra, Coimbra, Portugal

Carlos Henggeler Antunes INESC Coimbra, Department of Electrical and Computer Engineering, University of Coimbra, Coimbra, Portugal

F. Barbiero Municipality of Turin, Turin, Italy

João Clímaco Institute for Systems Engineering and Computers at Coimbra, INESC-Coimbra, University of Coimbra, Coimbra, Portugal

L. Corazza Department of Management, University of Torino, Turin, Italy

João Paulo Costa CeBER and Faculty of Economics, University of Coimbra, Coimbra, Portugal;
INESC Coimbra, Coimbra, Portugal

Miguel Couceiro Université de Lorraine, CNRS, Inria, LORIA, Nancy, France

José Craveirinha Institute for Systems Engineering and Computers at Coimbra, INESC-Coimbra, University of Coimbra, Coimbra, Portugal

Yves De Smet SMG research unit, Computer and Decision Engineering Department, Université libre de Bruxelles, Ecole polytechnique de Bruxelles, City of Brussels, Belgium

Luis C. Dias Faculty of Economics, CeBER and INESC Coimbra, University of Coimbra, Coimbra, Portugal

Michalis Doumpos Technical University of Crete, School of Production Engineering and Management, University Campus, Chania, Greece

Didier Dubois IRIT, CNRS, Université Paul Sabatier, Toulouse, France

Hélène Fargier IRIT, CNRS, Université Paul Sabatier, Toulouse, France

Fausto Freire ADAI-LAETA, Department of Mechanical Engineering, University of Coimbra, Coimbra, Portugal

Jutta Geldermann Faculty of Engineering, University of Duisburg-Essen, Duisburg, Germany

Alfio Giarlotta Department of Economics and Business, University of Catania, Catania, Italy

Michel Grabisch Université Paris I—Panthéon-Sorbonne, Paris School of Economics, Paris, France

Masahiro Inuiguchi Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka, Japan

Alessio Ishizaka Portsmouth Business School, Centre for Operational Research and Logistics, University of Portsmouth, Portsmouth, UK

Dylan Jones Department of Mathematics, Centre of Operational Research and Logistics (CORL), University of Portsmouth, Portsmouth, UK

Genku Kayo Department of Mechanical Engineering, Aalto University School of Engineering, Aalto, Finland;
School of Architecture and the Built Environment, KTH Royal Institute of Technology, Stockholm, Sweden

Anna Kolesárová Faculty of Chemical and Food Technology, Institute of Information Engineering, Automation and Mathematics, Slovak University of Technology in Bratislava, Bratislava, Slovakia

Christophe Labreuche Thales Research and Technology, Palaiseau, France

Risto Lahdelma Department of Mechanical Engineering, Aalto University School of Engineering, Aalto, Finland;
Department of Mathematics and Systems Analysis, Aalto University School of Science, Aalto, Finland

Lúcia Martins Institute for Systems Engineering and Computers at Coimbra, INESC-Coimbra, University of Coimbra, Coimbra, Portugal;
Faculty of Sciences and Technology, Department of Electrical Engineering and Computers, University of Coimbra, Coimbra, Portugal

Radko Mesiar Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Bratislava, Slovakia;
Institute for Research and Applications of Fuzzy Modelling, University of Ostrava, Ostrava, Czech Republic

Giuseppe Munda European Commission, Joint Research Centre, Unit JRC. I.1—Modelling, Indicators and Impact Evaluation, Competence Centre on Modelling, Ispra, Va, Italy

M. F. Norese Department of Management and Production Engineering, Politecnico di Torino, Turin, Italy

Henri Prade IRIT, CNRS, Université Paul Sabatier, Toulouse, France

Agnès Rico ERIC, Université Claude Bernard Lyon 1, Villeurbanne, France

Carlos Romero Group of Economics for a Sustainable Environment (ECSEN), Technical University of Madrid, Madrid, Spain

L. Sacco Unioncoop, Turin, Italy

Pekka Salminen School of Business and Economics, University of Jyväskylä, Jyväskylä, Finland

Andrea Stupňanová Faculty of Civil Engineering, Slovak University of Technology in Bratislava, Bratislava, Slovakia

Ronald R. Yager Machine Intelligence Institute, Iona College, New Rochelle, NY, USA

Constantin Zopounidis Technical University of Crete, School of Production Engineering and Management, University Campus, Chania, Greece;
Audencia Business School, Institute of Finance, Nantes, France

Part I
Basic Notions and Methods

New Trends in Preference, Utility, and Choice: From a Mono-approach to a Multi-approach

Alfio Giarlotta

Abstract We give an overview of some new trends in preference modeling, utility representation, and choice rationalization. Several recent contributions on these topics point in the same direction: the use of multiple tools—may they be binary relations, utility functions, or rationales explaining a choice behavior—in place of a single one, in order to more faithfully model economic phenomena. In this stream of research, the two traditional tenets of economic rationality, completeness and transitivity, are partially (and naturally) given up. Here we describe some recent approaches of this kind, namely: (1) utility representations having multiple orderings as a codomain, (2) multi-utility and modal utility representations, (3) a finer classifications of preference structures and forms of choice rationalizability by means of generalized Ferrers properties, (4) a descriptive characterization of all semiorders in terms of shifted types of lexicographic products, (5) bi-preference structures, and, in particular, necessary and possible preferences, (6) simultaneous and sequential multi-rationalizations of choices, and (7) multiple, iterated, and hierarchical resolutions of choice spaces. As multiple criteria decision analysis provides broader models to better fit reality, so does a multi-approach to preference, utility, and choice. The overall goal of this survey is to suggest the naturalness of this general setting, as well as its advantages over the classical mono-approach.

Keywords Preference modeling · Utility representation · Choice rationalization · Completeness · Transitivity · Lexicographic order · Semiorder · $\mathbb{Z}$-product · (m, n)-Ferrers property · Bi-preference · Necessary and Possible preference · Robust ordinal regression · Multi-utility representation · Modal utility representation · Multi-rationalization · Choice resolution

A. Giarlotta (✉)
Department of Economics and Business, University of Catania, Catania, Italy
e-mail: giarlott@unict.it

M. Doumpos et al. (eds.), *New Perspectives in Multiple Criteria Decision Making*, Multiple Criteria Decision Making,
https://doi.org/10.1007/978-3-030-11482-4_1

1 Introduction

In the field of mathematical economics, the modelization of an agent's preference structure is traditionally done by means of a mono-approach, which uses a single binary relation satisfying the two basic tenets of economic rationality: (1) *completeness*, and (2) *transitivity*. (See, e.g., Chap. 1 of the classical microeconomics textbooks Mas-Colell et al. (1995) and Kreps (2013)). Under topological conditions of separability, these two properties guarantee the existence of a utility representation of preferences by a continuous real-valued function (Aleskerov et al. 2007; Bridges and Mehta 1995; Debreu 1954). Similarly, the traditional approach of revealed preference theory (Arrow 1959; Samuelson 1938) often employs complete and transitive binary relations to justify an agent's choice behavior. In some cases, the satisfaction of the two properties of completeness and transitivity has even guided the design of new economic theories: a striking instance of kind is given by the classical book "*Games and Economic Behavior*" of von Neumann and Morgenstern (1944).

By partially giving up these two properties, here we depart from traditional approaches, and examine: (a) alternative types of utility representations, (b) more refined kinds of preference structures, and (c) new forms of bounded rationality for choices. In fact, the general question that motivates this survey is the following:

(Q0) *Can we design sound theories of preference modeling, utility representation, and choice rationalization, which give up, partially or totally, the basic tenets of economic rationality?*

This paper illustrates some possible answers to question (Q0).

Specifically, first we deal with preference representations in a lexicographically ordered codomain (Chipman 1971; Fishburn 1974), thus extending the classical real-valued representation. This approach provides a description of preferences that fail to have a real-valued representation (Beardon et al. 2002a, b). Successively, we describe some novel types of preference structures, which are formed by nested and intertwined pairs of binary relations (Giarlotta and Watson 2018b). In this bi-preference approach, the two properties of transitivity and completeness are coherently spread over the two components. This feature makes these structures well suited to applications in operations research and economics. In particular, special types of bi-preferences, called necessary and possible (Giarlotta and Greco 2013), have already been successfully employed as a modeling tool in multiple criteria analysis (Greco et al. 2008). Under suitable conditions, bi-preferences can be represented by a doubly indexed family of utility functions: this is the so-called modal utility representation (Giarlotta and Greco 2013), which adapts to bi-preferences the recently introduced multi-utility representation of a preorder (Evren and Ok 2011; Ok 2002).

In parallel to a multi-approach to preference and utility, we also develop a theory of choice multi-rationalization. Samuelson's theory of revealed preferences (Arrow 1959; Houthakker 1950; Samuelson 1938) postulates that choices are observed, and preferences can be derived from them. The class of rationalizable choices is especially significative in this respect, since it codifies all types of choice behavior that can be explained by means of the maximization of a single binary relation. However, the theory of revealed preferences yields a sharp rational/irrational dichotomy, since any non-rationalizable choice behavior is bluntly classified as "irrational". With the goal of smoothening this dichotomy, several new theories of *bounded rationality* (Simon 1955, 1982) have naturally emerged over the last few years (Cherepanov et al. 2013; Kalai et al. 2002; Manzini and Mariotti 2007; Masatlioglu and Nakajima 2013; Rubinstein and Salant 2006). Here we describe a general setting for the multi-rationalizability of a choice (Cantone et al. 2018c), which may employ more than one binary preference to explain the behavior of an economic agent, thus broadening the classical notion of mono-rationalizability. We also sketch the main features of a recently introduced methodology in choice theory, called "resolution". This methodology, which is an adaptation of an analogous technique in general topology (Fedorcuk 1968; Watson 1992), studies the inner structure of a complex choice process (Cantone et al. 2018a) on the basis of a notion of delegations of tasks. This yields a decomposition (and explanation) of a complex selection process into independent and simpler decisional units, typically distributed in a hierarchical way.

Multiple criteria decision analysis (Greco 2005; Greco et al. 2010a) provides powerful analytical tools to handle complex real life problems, offering more flexible modelizations than mono-criterion techniques. Similarly, *mutatis mutandis*, a multi-approach to the theories of preference, utility representation, and choice rationalization yields a more realistic representation of economic phenomena rather than the classical mono-approach. The purpose of this work is to give an overview of a multi-approach to these theories, also suggesting its naturalness, feasibility, and potential.

Organization of the Paper

The remainder of this survey is organized into three main sections, a conclusive section, and an appendix.

Section 2 (The Mono-approach). We start in Sect. 2.1 with a historical discussion about the two properties of transitivity and completeness. Successively, we provide an overview of basic notions and classical results in preference modeling (Sect. 2.2), utility representations (Sect. 2.3), and choice rationalization (Sect. 2.4). These theories use a single tool for the description of an agent's behavior/attitude. In summarizing their main achievements, we shall also detect some shortcomings, and indicate possible ways of coping with the arising issues.

Section 3 (The Transition). Here we sketch a few recent approaches to the theories described in Sects. 2.2–2.4. These techniques, which suggest the use of multiple tools to represent economic behavior, address some shortcomings of classical theories and pave the way for more general approaches to these topics. Specifically, we describe: utility representations using lexicographic orderings as a codomain (Sect. 3.1), universal characterizations of semiorders based on shifted lexicographic products (Sect. 3.2), Ferrers properties describing a discrete evolution of transitivity (Sect. 3.3), choice correspondences rationalizable by well-structured revealed preferences (Sect. 3.4), and a process detecting the inner structure of a choice in terms of delegations of tasks (Sect. 3.5). The goal of this section is to provide the reader with a natural justification and a smooth transition toward a multi-approach.

Section 4 (The Multi-approach). Here we finally describe some very recent developments in the theories described in Sects. 2.2–2.4, which employ multiple tools rather than a single one. Specifically, in Sect. 4.1 we introduce bi-preference structures, and describe their advantages over mono-preferences. In Sect. 4.2, we deal with particular types of bi-preferences, called necessary and possible, which have been already used in multiple criteria decision analysis. In Sect. 4.3, we recall the notion of a multi-utility representation, and show how bi-preferences are representable by a suitably indexed type of multi-utility representation, called modal. Within the theory of choice rationalization, we provide in Sect. 4.4 an overview of the recent bounded rationality approaches, which use multiple binary rationales to explain a choice behavior. Finally, in Sect. 4.5 we describe a natural extension of the notion of choice resolution to a multiple and iterated setting.

Section 5 concludes this contribution.
The **Appendix** contains two figures, which graphically describe some results.
Neither original results nor proofs appear in this survey.

2 The Mono-approach

To keep the presentation as much self-contained as possible, this section recalls the classical setting of the theories of preference modeling, utility representation, and choice rationalization.

2.1 The Two Classical Tenets of Rationality

A preference structure on a set X of alternatives is usually modeled by a binary relation R on X. Traditionally, R is assumed to "behave well", in the sense that it satisfies suitable ordering properties. The two classical properties that are assumed to hold for R are:

(Completeness) for any distinct $x, y \in X$, either xRy or yRx (or both)[1];
(Transitivity) for any $x, y, z \in X$, if xRy and yRz, then xRz.

The reasons for which R is often supposed to be both complete and transitive are several, some being related to their economic significance, some others to their mathematical tractability. However, both properties have been questioned by eminent scholars over time.

[1]Notice that, since x and y are distinct, this formulation of completeness does not imply reflexivity.

In their monumental work *Theory of Games and Economic Behavior* (von Neumann and Morgenstern 1944), von Neumann and Morgenstern already acknowledged, albeit rather elusively, that preferences may naturally be incomplete (pp. 19–20):

> We have conceded that one may doubt whether a person can always decide which of two alternatives … he prefers. If the general comparability assumption is not made, a mathematical theory … is still possible. It leads to what may be described as a many dimensional vector concept of utility. This is a more complicated and less satisfactory set-up, but we do not propose to treat it systematically at this time.

In fact, von Neumann and Morgenstern limited their analysis to complete (and transitive) preferences, due to the mathematical amenability of this simplified setting, and never published details about the mentioned "many dimensional vector concept of utility".

In his seminal paper on incomplete preferences, Aumann (1962) suggested (p. 449) an interpretation of von Neumann and Morgenstern's statement:

> What they probably had in mind was some kind of mapping from the space of lotteries to a canonical partially ordered euclidian space, rather than the real-valued mappings we use here; but it is not clear to me how this approach can be worked out.

Aumann's criticism of the completeness property was quite direct (p. 446):

> Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint.

Since Aumann's work, many other authors started abandoning the axiom of completeness as a basic feature of rational behavior. On the topic, Bewley (1986) and Ok (2002) attentively elaborate on the links between the notion of rationality and the incompleteness of preferences.

In their systematic analysis of the multi-utility representation of preferences, Evren and Ok (2011) mention several behavioral phenomena which naturally yield incompleteness, e.g., status-quo bias (Apesteguía and Ballester 2009; Masatlioglu and Ok 2005), intransitive choice (Manzini and Mariotti 2007), choice deferral (Kopylov 2009), and indecisiveness in revealed preferences (Eliaz and Ok 2006). Similarly, incompleteness has been a main focus in various decision models used in operations research and management science (Danan 2010; Greco et al. 2008; Masin and Bukchin 2008), financial economics (Rigotti and Shannon 2005), political economics (Levy 2004; Roemer 1999), and game theory (Bade 2005). Further, several recent studies on (in)decisions under risk and uncertainty use incomplete preorders to model preferences (Dubra et al. 2004; Ghirardato et al. 2003, 2004; Gilboa et al. 2010; Maccheroni 2004; Nau 2006; Ok et al. 2012). Last but not least, following the seminal work of Bernard Roy (1985, 1990a, b), there is a large number of multiple criteria decision methodologies which explicitly take into account incompleteness of preferences as a natural feature of the decision maker's attitude (Greco et al. 2010a).

The axiom of transitivity was possibly harder to abandon, even if probably questioned before completeness. In his well-known paper, Tversky (1969) was still advocating the importance of transitivity in the modelization of preferences, since its violation could cause unpleasant phenomena of "money pump" (Davidson et al. 1955).[2] This attitude was however contrasted by other authors, who had already been designing economic models in which transitivity was partially or totally abandoned. The probabilistic choice model proposed by Luce (1959) can be regarded as a pioneering example of intransitive preferences in economic theory. The obstinate insistence of some economists to employ transitive models even brought Sen (1971) to declare that revealed preference theory is "obsessed with transitivity". In their recent paper, Bleichrodt and Wakker (2015) argue that the year 1982 was a sort of "breaking point" in the economic literature, since transitivity was given up in three seminal papers related to *regret theory*: the axiomatic approach of Fishburn (1982), a decision analysis oriented paper by Bell (1982), and the fundamental contribution of Loomes and Sudgen (1982). From an experimental point of view, there are many papers in mathematical psychology explaining intransitivity of preferences by random models, insofar as the subject's preferences vary over time from one type of ordering to another: see, e.g., Regenwetter et al. (2010, 2011) for some models of this kind, and Davis-Stober et al. (2018) for a recent method to test these models.

In the same stream of research that opposes the blunt assumption of fully transitive preferences, we ought to mention the extraordinary amount of literature on semiorders, interval orders, and similar preference structures, which describe forms of rational behavior characterized by weaker forms of transitivity. Anticipated by the intuitions of Fechner (1860), Poincaré (1908), Georgescu-Roegen (1936), Armstrong (1939), and Halphen (1955), research on intransitive preference structures had its definitive consecration by the seminal papers of Luce (1956) and Fishburn (1970), who formally introduced the notions of semiorder and interval order, respectively. Their approaches are based on the idea of weakening the axiom of transitivity, rather than abandoning it all together. Indeed, Luce's famous coffee/sugar example suggests that the transitivity of the associated indifference should be somehow weakened and regulated, whereas the transitivity of the strict preference may be retained as a natural assumption of rational behavior.

The recently introduced weak (m, n)-Ferrers properties go exactly in the direction of considering binary structures with a transitive strict preference but a possibly intransitive indifference (Giarlotta and Watson 2014a). Originally designed to provide a combinatorial extension of the Ferrers condition and semitransitivity—which coincide, respectively, with weak $(2, 2)$-Ferrers and weak $(3, 1)$-Ferrers—these properties display a finite taxonomy of enhanced forms of the transitivity of the strict preference. In fact, roughly speaking, weak (m, n)-Ferrers properties classify transitive strict preferences by means of the types of forbidden mixed cycles of preference/indifference (see Sect. 4.2 in Cantone et al. (2016)). It follows that such an approach may be relevant for economic applications insofar as weak (m, n)-Ferrers properties prompt a possible recognition of money-pump effects due to the presence of mixed cycles of a certain length and type.

[2]See Sect. 3.3 of this survey for a discussion on this point in relation to the so-called (m, n)-Ferrers properties.

Strict (m, n)-Ferrers properties (Giarlotta and Watson 2014a, 2018a; Öztürk 2008) go even further in weakening the assumption of transitivity, since they do not even postulate the transitivity of the strict preference. These properties yield an infinite taxonomy of intransitive preference structures, which are connected to other types of money-pump phenomena.

In this paper, we shall also mention some new approaches to preference modeling in which both basic tenets of economic rationality are only partially retained, being "spread over" two binary relations (see Sects. 4.1 and 4.2 on bi-preferences and NaP-preferences, respectively).

2.2 *Preference Modeling*

Here we summarize the basic terminology in preference theory. Two good sources of information on this topic—as well as on utility representations, which is the topic of the next section—are the textbooks by Bridges and Mehta (1995) and Aleskerov et al. (2007).

Henceforth, X is a nonempty (possibly infinite) set of alternatives (courses of action, etc.), and $\Delta(X) = \{(x, x) : x \in X\}$ is the *diagonal* of X.

Definition 2.1 A reflexive binary relation on X is referred to as a *weak preference* on X, and is henceforth denoted by $\succsim$; the pair $(X, \succsim)$ is generically called an *ordered set*. The following relations are derived from a weak preference $\succsim$ on X: its *strict preference* $\succ$ (the asymmetric part of $\succsim$), its *indifference* $\sim$ (the symmetric part of $\succsim$), and its *incomparability* $\perp$ (the symmetric part of the complement of $\succsim$). These relations are formally defined as follows for each $x, y \in X$:

$$\begin{aligned} x \succ y &\overset{\text{def}}{\iff} (x \succsim y) \wedge \neg(y \succsim x) \\ x \sim y &\overset{\text{def}}{\iff} (x \succsim y) \wedge (y \succsim x) \\ x \perp y &\overset{\text{def}}{\iff} \neg(x \succsim y) \wedge \neg(y \succsim x). \end{aligned}$$

Given an ordered set $(X, \succsim)$, the set of *maximal elements* of $A \subseteq X$ is defined by

$$\max(A, \succsim) := \{x \in A : (\nexists y \in A)\ y \succ x\}.$$

The *composition* of two weak preferences $\succsim_1$ and $\succsim_2$ on X is the binary relation $\succsim_1 \circ \succsim_2$ on X defined as follows for all $x, y \in X$:

$$x(\succsim_1 \circ \succsim_2)\, y \overset{\text{def}}{\iff} (\exists z \in X)\ x \succsim_1 z \succsim_2 y.$$

Notice that a weak preference $\succsim$ is (i) complete if and only if its incomparability $\perp$ is empty, and (ii) transitive if and only the inclusion $\succsim \circ \succsim \subseteq \succsim$ holds. Whenever $\succsim$ is complete, the set of maximal elements of $A \subseteq X$ can be also written as $\max(A, \succsim)$

$:= \{x \in A : (\forall y \in A)\ x \succsim y\}$. Finally, observe that, even when X is finite, the set $\max(A, \succsim)$ may be empty, due to the possible presence of strict cycles (see Definition 2.2).

Definition 2.2 A weak preference $\succsim$ on X is called (x, y, z, w are arbitrary elements of X):

- *complete* (or *total* or *connected*) if $x \succsim y$ or $y \succsim x$ always holds ($x \neq y$);
- *antisymmetric* if $x \succsim y$ and $y \succsim x$ implies $x = y$ (equivalently, $\sim$ is the diagonal of X);
- *acyclic* if there are no $x_1, x_2, \ldots, x_n \in X$, with $n \geq 3$, such that $x_1 \succ x_2 \succ \cdots \succ x_n \succ x_1$;
- *quasi-transitive* if $\succ$ is transitive, i.e., ($x \succ y$ and $y \succ z$) implies $x \succ z$[3];
- *Ferrers* if ($x \succsim y$ and $z \succsim w$) implies ($x \succsim w$ or $z \succsim y$);
- *semitransitive* if ($x \succsim y$ and $y \succsim z$) implies ($x \succsim w$ or $w \succsim z$);
- an *interval order* if it is Ferrers;
- a *semiorder* if it is Ferrers and semitransitive;
- a (*partial*) *preorder* if it is transitive;
- a *partial order* if it is an antisymmetric preorder;
- a *total preorder* if it is a complete preorder;
- a *linear order* if it is an antisymmetric total preorder.

Accordingly, the pair $(X, \succsim)$ is called, e.g., a *semiordered set*, a *preordered set*, a *partially ordered set* (also called a *poset*), a *linearly ordered set* (also called a *linear ordering* or a *chain*), etc.

Notice that (i) any total preorder is trivially a semiorder, (ii) any semiorder is trivially an interval order, (iii) an interval order is both complete and quasi-transitive, and (iv) any quasi-transitive weak preference is acyclic. Moreover, the indifference derived from a preorder is an equivalence relation, but the same does not hold for the indifference associated to a semiorder (hence, a fortiori, for that of an interval order). Observe also that if X is finite, then an acyclic relation on X always has maximal elements for each nonempty subset of X.

Next, we recall some notions due to Fishburn (1970), which play an important role in the theory of preferences, especially for defining notions of (semi)continuity as well as for preferences that are interval orders and semiorders (but also for bi-preference structures, see Sects. 4.1 and 4.2): the "traces" of a weak preference.

Definition 2.3 Let $\succsim$ be a weak preference on X. For each $x \in X$, let

(weak lower section of x) $x^{\downarrow,\succsim} := \{w \in X : x \succsim w\}$,
(weak upper section of x) $x^{\uparrow,\succsim} := \{w \in X : w \succsim x\}$,
(strict lower section of x) $x^{\downarrow,\succ} := \{w \in X : x \succ w\}$,
(strict upper section of x) $x^{\uparrow,\succ} := \{w \in X : w \succ x\}$.

[3] In case $\succsim$ is complete, then the following statements are equivalent: (i) $\succsim$ is quasi-transitive; (ii) for each $x, y, z \in X$, $x \succ y \succsim z$ implies $x \succsim z$; (iii) for each $x, y, z \in X$, $x \succsim y \succ z$ implies $x \succsim z$.

Define three binary relations[4] on X as follows for each $x, y \in X$:

$$
\begin{array}{llll}
\textit{(left trace of } \precsim\textit{)} & x \succsim^* y & \stackrel{\text{def}}{\Longleftrightarrow} & y^{\downarrow,\succsim} \subseteq x^{\downarrow,\succsim}, \\
\textit{(right trace of } \succsim\textit{)} & x \succsim^{**} y & \stackrel{\text{def}}{\Longleftrightarrow} & x^{\uparrow,\succsim} \subseteq y^{\uparrow,\succsim}, \\
\textit{(global trace of } \succsim\textit{)} & x \succsim_0 y & \stackrel{\text{def}}{\Longleftrightarrow} & x \succsim^* y \ \wedge \ x \succsim^{**} y.
\end{array}
$$

The next lemma collects some enlightening results about traces: see, e.g., Fishburn (1985), Monjardet (1978), Pirlot and Vincke (1997).

Lemma 2.4 *Let* $\succsim$ *be a weak preference on* X.

- $\succsim^*$, $\succsim^{**}$, $\succsim_0$ *are preorders contained in* $\succsim$.
- $\succsim^* \circ \succsim \subseteq \succsim$ *and* $\succsim \circ \succsim^{**} \subseteq \succsim$.
- $\succsim_0 \circ \succsim \subseteq \succsim$ *and* $\succsim \circ \succsim_0 \subseteq \succsim$.
- $\succsim$ *is an interval order* $\Longleftrightarrow$ $\succsim^*$ *is a total preorder* $\Longleftrightarrow$ $\succsim^{**}$ *is a total preorder.*
- $\succsim$ *is a semiorder* $\Longleftrightarrow$ $\succsim_0$ *is a total preorder.*
- $\succsim$ *is a preorder* $\Longleftrightarrow$ $\succsim = \succsim_0$.
- $\succsim$ *is a total preorder* $\Longleftrightarrow$ $\succsim = \succsim_0$ *is complete.*

Many classical results on preferences are related to the possibility of (continuously) representing them by a utility function, a topic that is analyzed in the next section. There are also other issues arising from the traditional mono-approach to preference modeling, mostly due to the limited expressive power of a single binary relation. In this respect, a general question is:

(Q1) *Can we use binary relations to represent preferences in a more flexible and realistic way?*

We shall address question (Q1) in Sects. 4.1 and 4.2, where we suggest how a bi-preference approach may enhance the modeling power of a binary representation of agents' preference structures by taking into account two different kinds of "attitudes".

2.3 Utility Representations

In this section we deal with the classical setting of real-valued utility representations of binary preferences. Two are the basic issues, the first purely order-theoretic and the second topological:

[4]We follow the approach described in Bouyssou and Pirlot (2004), defining all traces in terms of weak sections, instead of defining strict traces first and then deriving weak traces. The difference is immaterial whenever dealing with complete and quasi-transitive preferences, in particular for interval orders and semiorders. Notice also that the notion of global trace has been recently revised from a different perspective, and renamed *transitive core* (Nishimura 2018).

(Q2) *Can we can represent a total preference relation by a real-valued utility function?*
(Q3) *Can we make this utility function continuous?*

To start, we give the basic elements to properly formulate and then address question (Q2).[5]

Definition 2.5 A binary relation $\succsim$ on X is *representable in* $\mathbb{R}$ if there is a function $u\colon X \to \mathbb{R}$ such that, for all $x, y \in X$, we have

$$x \succsim y \iff u(x) \geq u(y).$$

In this case, the function u is a *utility representation* of $(X, \succsim)$ in $\mathbb{R}$. (We also say that $(X, \succsim)$ is *order-embeddable* or *embeddable* in $\mathbb{R}$.) The chain $(\mathbb{R}, \geq)$ is the *base* of the representation.

An obvious necessary condition for the representability of a weak preference $\succsim$ in $\mathbb{R}$ is that $\succsim$ must be a total preorder, i.e., it satisfies the two classical properties of transitivity and completeness. This condition is also sufficient for the cases in which the ground set X is finite or countably infinite (see, e.g., Chap. 1 of Bridges and Mehta (1995)). In the general case, however, we need an additional property of "separability" to ensure representability.

The first characterization of representability in $\mathbb{R}$ is most likely the following (Cantor 1895; Milgram 1939):

Theorem 2.6 (Cantor 1895; Milgram 1939) *A linear ordering $(X, \succsim)$ is order-embeddable in $\mathbb{R}$ if and only if it includes a countable subset that is weakly order-dense in X.*[6]

Similar characterizations were given by Birkhoff (1948). Nevertheless, due to an imperfect communication in the scientific community, until the early 1950s economists considered all preference relations as representable in $\mathbb{R}$. In other words, the concepts of "preference" and"utility" were (wrongly) considered equivalent. For a salient instance of this kind, let us cite Hicks (1956, p. 19):

> If a set of items is strongly ordered, it is such that each item has a place of its own in the order; it could, in principle, be given a number.

If the above statement were to hold, then *every* total preorder would be representable in $\mathbb{R}$, and the concepts of preference and utility would coincide, which is false.

[5]The literature also examines weaker forms of representability of a single binary relation, e.g., the existence of (continuous, semicontinuous) *Richter-Peleg* utility functions (Alcantud et al. 2016; Peleg 1970; Richter 1966). We shall deal with this topic in Sect. 4.3, where we also discuss some shortcomings of this notion, and introduce multi-utility representations.

[6]A set $Y \subseteq X$ is *weakly order-dense* in X if, for each $x_1, x_2 \in X$ such that $x_1 \succ x_2$, there is $y \in Y$ with the property that $x_1 \succsim y \succsim x_2$. Such a set is often called *Debreu order-dense*, and the existence of a countable Debreu order-dense is referred to as *Debreu-separability* (Bridges and Mehta 1995).

In his celebrated paper on the *Open Gap Lemma*, Debreu (1954) finally exhibited an example of a natural preference that is non-representable in $\mathbb{R}$: the lexicographic plane $\mathbb{R}^2_{\text{lex}} = (\mathbb{R}^2, \precsim_{\text{lex}})$. Several characterizations of representability followed, for instance (Fleischer 1961):

Theorem 2.7 (Fleischer 1961) *A chain* $(X, \precsim)$ *is representable in* $\mathbb{R}$ *if and only if it has at most countably many jumps and the topological space* $(X, \tau_{\precsim})$ *is separable.*[7]

For an extensive overview of the topic, the reader is referred to Bridges and Mehta (1995), Mehta (1998).

In 2002, Beardon et al. (2002a, b) systematically analyzed the structure of total and transitive preferences that fail to be representable in $\mathbb{R}$, and obtain a striking subordering classification of them. Their characterization (Beardon et al. 2002a) can be suggestively rephrased as follows:

Theorem 2.8 (Beardon et al. 2002a) *A chain is non-representable in* $\mathbb{R}$ *if and only if it is* (i) *long or* (ii) *large or* (iii) *wild.*[8]

(Here by "long" we mean that it contains a copy of the first uncountable ordinal[9] ω_1 or its reverse ordering $\omega_1{}^*$; by "large" we mean that it contains a copy of a non-representable subordering of the lexicographic plane $\mathbb{R}^2_{\text{lex}}$; and by "wild" we mean that it contains a copy of an *Aronszajn line*, which is defined as an uncountable chain such that neither ω_1 nor $\omega_1{}^*$ nor an uncountable subordering of $\mathbb{R}$ embeds into it.) Some more recent results in this direction, which use lexicographic orders as modeling tools, are mentioned in Sect. 3.1.

Next, we deal with question (Q3), that is, the existence of a *continuous* real-valued representation. To describe the topological setting, we recall the notions of (i) the continuity of a preorder, and (ii) the order topology induced by a preorder. (For all undefined topological notions, the reader may consult the classical textbook by Munkres (2000).)

Definition 2.9 Given a topological space (X, τ), a preorder $\precsim$ on X is *continuous*[10] if $\precsim$ is a closed subset of the topological product $X \times X$.

---

[7]A *jump* in an ordered space $(X, \precsim)$ is a pair $(a, b) \in X^2$ such that $a \succ b$ and there is no point $c \in X$ such that $a \succ c \succ b$. The topology $\tau_{\precsim}$ is the *order topology* induced by $\precsim$. The topological space $(X, \tau_{\precsim})$ is *separable* if it contains a countable set D that intersects each nonempty open set. See Munkres (2000) for topological notions.

[8]This is not the terminology originally used by the authors.

[9]An *ordinal* is a well-ordered set $(X, <)$ such that each $x \in X$ is equal to its initial segment $\{y \in X : y < x\}$. The finite ordinals are the natural numbers. The first infinite ordinal is the set ω_0 of all natural numbers, endowed with the usual order. The first uncountable ordinal is the set ω_1 of all countable ordinals, endowed with the natural order. The famous *continuum hypothesis*, formulated by George Cantor in 1878, says that the cardinality of $\mathbb{R}$ is equal to ω_1 (as a cardinal). In 1963, Paul Cohen proved that the continuum hypothesis is independent from the axioms of ZFC (Zermelo-Fraenkel axiomatic set theory, plus the Axiom of Choice), in sense that there are models in which it is true, and models in which it is false (because $|\mathbb{R}| > \omega_1$ holds). See the classical textbook by Kunen (1980) for ZFC axiomatic set theory.

[10]Here we use the notion of continuity employed in some standard textbooks in microeconomic theory, such as Mas-Colell et al. (1995, p. 46). Other authors sometimes employ a weaker notion

It can be shown that a *complete* preorder $\succsim$ on (X, τ) is continuous if and only if (i) all weak upper sections $x^{\uparrow,\succsim}$ and lower sections $x^{\downarrow,\succsim}$ are closed subsets of (X, τ) if and only if (ii) all strict upper sections $x^{\uparrow,\succ}$ and lower sections $x^{\downarrow,\succ}$ are open subsets of (X, τ). Conditions (i) and (ii) are sometimes called, respectively, *closed semicontinuity* and *open semicontinuity*, whereas their joint satisfaction is called *bi-semicontinuity*: see Sect. 4.1. Notice that bi-semicontinuity does not imply continuity for incomplete preorders.[11]

Definition 2.10 Given a preordered set $(X, \succsim)$, the *order topology* $\tau_{\succsim}$ on X induced by $\succsim$ is the topology having as a subbasis the family of all strict upper and lower sections (equivalently, the topology having as a basis the family of all open intervals).

An immediate consequence of Definitions 2.9 and 2.10 is that for any totally preordered set $(X, \succsim)$, the order topology $\tau_{\succsim}$ is the coarsest topology on X such that $\succsim$ is continuous.

There are many results dealing with continuous real-valued utility representations of a total preorder. The most classical theorems in this field are due to Eilenberg (1941) and Debreu (1954, 1964):

Theorem 2.11 (Eilenberg 1941) *In a connected separable topological space, any continuous total preorder is continuously representable in* $\mathbb{R}$.

Theorem 2.12 (Debreu 1954, 1964) *In a second countable topological space, any continuous total preorder is continuously representable in* $\mathbb{R}$.

A miscellany of representation results followed (in the 1970s): let us recall, among others, the approaches due to Jaffray (1975a), Neuefeind (1972), Peleg (1970), Richter (1980), and Sondermann (1980). A common denominator of many approaches to the topic is the *Open Gap Lemma*, which was (incorrectly) proved by Debreu (1954), and then corrected by the same author ten years later (Debreu 1964). For our purpose, the most relevant consequence of this result is the following:

Corollary 2.13 *If a total preorder on a topological space is representable in* $\mathbb{R}$, *then it is continuously representable in* $\mathbb{R}$.

The above result brings back the problem of the continuous representability of a total preorder to that of its mere representability, on which Theorem 2.8 by Beardon et al. (2002a) certainly sheds some light. However, Theorem 2.8 mostly provides

---

of continuity: see, e.g., Sect. 1.6 of Bridges and Mehta (1995). However, from the point of view of applications, the distinction between the various notions of continuity is often immaterial. See also Evren and Ok (2011, p. 555), and Gerasímou (2013, pp. 2–3).

[11] Herden and Pallack (2002) provide a very simple counterexample to the equivalence between continuity and bi-semicontinuity for incomplete preferences: in fact, they show that the relation of equality is a bi-semicontinuous non-continuous preorder in any topological space that is T_1 but not Hausdorff. On the topic, see also Gerasímou (2013), who characterizes continuity in terms of closed semicontinuity and a property of "local expansion" of transitivity (Theorem 1 in Gerasímou (2013)).

negative information, since several total preorders typically fail to be representable. Thus, it appears natural to seek more refined classifications of non-representable preferences. More precisely, the (new) questions are:

(**Q2**$'$) *Can we detect weaker forms of representability for non-representable preferences?*

(**Q3**$'$) *Can we make these weaker forms of representability continuous?*

A possible approach to questions (Q2$'$) and (Q3$'$) is to establish a "degree of representability" of total preferences by using more descriptive codomains rather than the set of real numbers. In this respect, codomains (different from $\mathbb{R}$) ensuring that the content of Corollary 2.13 is preserved—in the sense that the representability of a total preorder implies its continuous representability—look quite appealing. This brought Herden and Mehta (2004) to formulate the notion of a *Debreu chain*, which is a linear ordering such that the representability in it also ensures the existence of a continuous representation. (Thus, by Corollary 2.13 the linear ordering of the reals is the prototype of a Debreu chain; however, it is not the only one.)

In the same direction of research, some other authors extended the notion of a Debreu chain to that of a *pointwise Debreu* and *locally Debreu* chain (Caserta et al. 2008), also considering lexicographic products satisfying these properties (Giarlotta and Watson 2009). We shall deal with these recent approaches that aim at enlarging the representability of preference relations in Sect. 3.1, where we consider representations with lexicographic codomains. Further, in Sect. 3.2 we will present a universal description of semiorders by means of embeddings into modified forms of lexicographic products.

Nevertheless, the issues mentioned in the last two paragraphs are not the only ones. In fact, further problems on representability arise for the lack of representations of preferences that fail to fully possess the classical tenets of economic rationality. More precisely, the issue—which is obviously related to the question (Q1) formulated in Sect. 2.2—is the following:

(Q4) *How can we represent more refined preference structures by means of utility functions?*

We shall present possible ways to address question (Q4) in Sect. 4.3, where we deal with multiple and modal utility representations of both a single preference and a pair of preferences.

2.4 Choice Rationalization

Here we recall some elementary definitions on choices. We also summarize the basics of the theory of revealed preferences, pioneered by Samuelson (1938) and successively developed by several eminent scholars: see, among many others, Arrow (1959, 1963), Chernoff (1954), Hansson (1968), Herzberger (1973), Houthakker (1950), Plott (1973), Richter (1966), Sen (1971, 1986, 1993). For further details,