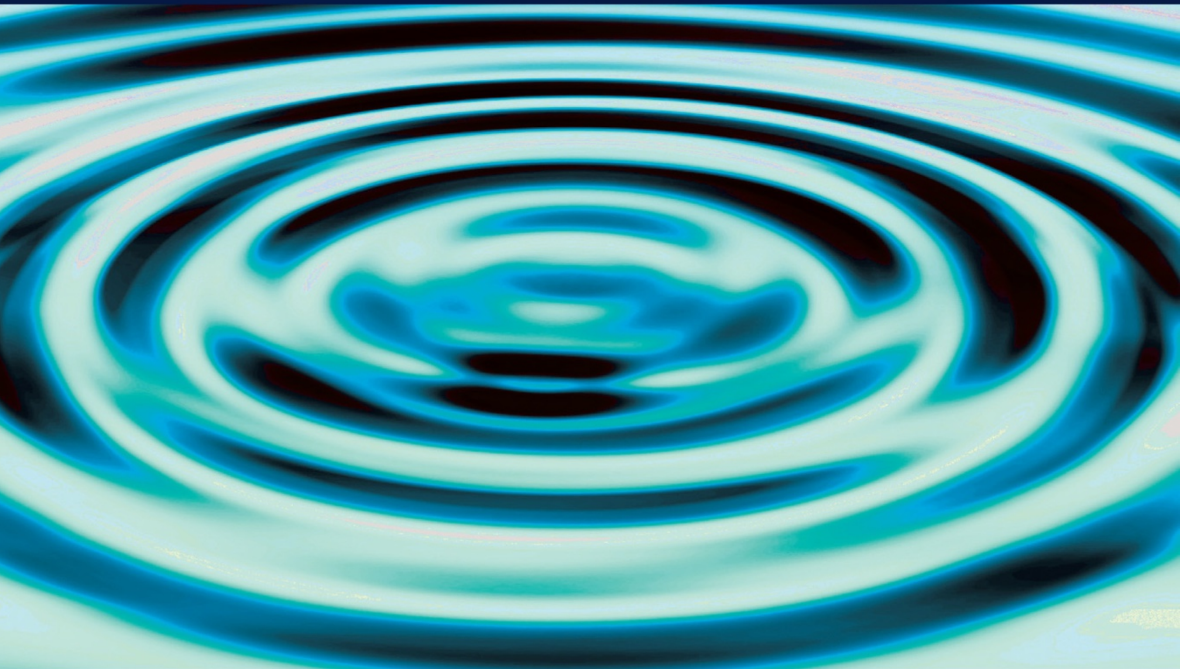
FLUID MECHANICS SERIES

Discrete Mechanics

Concepts and Applications

Jean-Paul Caltagirone



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Preface

Nature is complex; we must accept it. A concept or a theory might initially develop as a series of reproducible experiments, previously established principles, carefully conducted thought experiments and so on. Any theory must then be subjected to repeated verification, usually of an experimental nature, to validate or disprove its hypotheses, axioms and principles. Most key milestones in the history of science were intuitively understood by their authors before they were formalized in terms of equations; the equations describing a theory can be adjusted over time as the state of mathematics progresses. This was, for example, the case with Newton's laws of dynamics, which were only rephrased into the modern formulation later, by Euler.

Some phenomena continue to defy our efforts to fully understand them, such as turbulence; we are not currently capable of giving an accurate *a priori* description of the mechanisms underlying the interactions of vortices, or the processes by which they are created and dissipated. In some of the more accessible cases, simulations are our only tool for understanding turbulence at every scale and analyzing the physical behavior of turbulent flows in more depth. Simulation-based approaches rely on the equations of fluid mechanics, which are derived from Newton's second law. This perspective is expanded into a method: first, an equation is derived, which is validated in simple cases; this equation is then solved to explore the behavior of more complex phenomena. Over time, with enough simulation and exploration, the model is refined and our understanding of the phenomenon improves alongside it. The equation becomes more "intelligent" than its author.

But, in mechanics, the laws of dynamics and their modern counterparts – the equations of motion – still remain shrouded in some mystery, with hidden surprises for the attentive observer. The equations of motion are capable of describing the behavior of many problems involving fluids, solids and waves. However, unification has not yet been achieved; for example, the formulations used for solids and fluids do not coincide, even though performing mechanics on an underlying continuum was supposed to guarantee precisely this type of consistency. The equations of continuum mechanics hold hidden structural flaws that are unraveled in this book; the origins of each issue vary, but our most important objection will be raised against the concept of a continuum itself. The results themselves offered by continuum mechanics are usually not invalid, given that they are perfectly consistent with experimental observations; however, the theory is afflicted by deeper problems that are effectively covered up by redundancy in its principles and laws.

Discrete mechanics abandons the notion of a continuum, the principle of local thermodynamic equilibrium, the state equations and constitutive equations, second-order tensors, differentiation at a point, analysis and so on. Building from the ideas proposed by Galileo, the equivalence principle and the notion of relativity, the discrete approach to mechanics begins by defining a geometric basis of space that cannot be reduced to a single point. The discrete equations of motion are then formally derived from a set of hypotheses, axioms and principles.

Jean-Paul CALTAGIRONE
November 2018

Introduction

The laws of physics were established from experimental observations by finding simple relations to describe these observations; for example, the hypothesis of linearity between the flux and the force was established as a general principle of thermodynamics. Once a relation has been chosen, a coefficient of proportionality is selected. This coefficient is typically viewed as an intrinsic physical property, even though it depends directly on the choice of equations. If the law is overly simplistic, inconsistencies will be encountered when we attempt to generalize to parameter intervals larger than those originally considered. Fortunately, the complexity of the observed physical behavior typically allows us to reconstruct enough consistency to cover a broad spectrum of the parameter space. If only the thermodynamic coefficients themselves have any true physical significance, why should we need to introduce an ideal gas law between the variables, other than for convenience?

The laws of mechanics are some of the oldest laws in physics; throughout history, Galileo, Issac Newton and Albert Einstein each left a prodigious mark on our understanding of the universe around us. The concepts of force, mass and acceleration – linked together by Newton's second law – have not changed in modern mechanics. Over the past three centuries, a great succession of physicists, mathematicians and engineers gradually perfected the laws and equations of mechanics, which today seem inexorably set in stone. Leonhard Euler, Maurice Couette, Daniel Bernoulli, Ernst Mach, George Gabriel Stokes, Henri Navier, Clifford Truesdell and many others contributed to formalizing the laws of mechanics within a modern mathematical framework. Much has been written on the epistemology of the connections between the theories of these famous individuals, and occasionally the personal relationships between them. The path taken by scientific thought over time seems natural and logical if we consider the path taken by its equations, from classical mechanics to relativistic mechanics. Sometimes, it is not entirely clear where exactly new ideas came from, especially during periods where publication could not always be taken for granted and religion sometimes interfered with the propagation of new theories. Together, this scientific work led to equations of mechanics that have been frozen in time for decades or perhaps even centuries; today, these equations allow us to simulate all kinds of motion of solids, fluids, propagating waves, etc., extremely realistically. Yet, the equations of each type of motion are very different, even though continuum mechanics was supposed to establish a unified representation of all physical phenomena within the same set of equations.

How might we have established the laws of mechanics from scratch if we had access to every observation in history? This book attempts to give an answer. We start with the following question: why did Isaac Newton formulate his fundamental laws of dynamics as an equality of forces, given that he was aware of the equivalence between inertial mass and gravitational mass? This equivalence principle is one of the cornerstones of Albert Einstein's theory of general relativity. Discrete mechanics also relies heavily on this principle, using it to eliminate the concept of mass, which is not required to describe an equality between accelerations in mechanics. The notions of rest mass and relativistic mass at the heart of relativistic mechanics clearly characterize mass as equivalent to energy. Similarly, the fundamental law of dynamics can be formulated as an equality between the acceleration experienced by a particle and the sum of the accelerations applied to this particle.

The intrinsic nature of the acceleration is not the same as that of the velocity; we can apply an absolute acceleration even if the absolute velocity of a body is not known. The classical approach to finding the absolute velocity is to consider an inertial frame of reference. We must accept that it is pointless to attempt to understand this concept of absolute velocity, as well as the trains, station masters and elevators considered by the various thought experiments of the last century. It must be abandoned; we cannot detect uniform motion. By contrast, we can now measure acceleration to extremely high accuracy, corroborating ever further that equality between masses truly represents a fundamental principle.

Discrete mechanics also abandons the idea of a continuum; the differential calculus and differentials introduced by Gottfried Wilhelm Leibniz and Isaac Newton played an essential role in formulating the laws of mechanics into their modern differential form. To reduce every variable to a point, we are forced to construct an inertial frame of reference so that we can compute derivatives in specific directions of space. If we abandon the continuum, we must therefore also abandon classical differentiation and integration. We can still scale the discrete topology down geometrically to orders of magnitude that preserve compatibility with our macroscopic view of matter. The operations of differentiation and integration are replaced by operators based on discrete differential geometry. Inertial frames of reference are replaced by local frames, in which each point of the domain only perceives an immediate neighborhood of points, edges and faces with known distances and orientations. Every point is connected by causality links defined within space-time. Nature provides plenty of examples of collective behavior: schools of sardines, flocks of starlings, etc. The presence of limits or obstacles is perceived by each object through the behavior of its immediate neighborhood over a period of time that reflects the causality between them.

The distinction between the material velocity of the medium and the wave velocity (celerity) can be preserved by viewing the latter as a parameter determined by local conditions, whereas the material velocity itself is simply a secondary variable, defined as a velocity field known only up to uniform motion. The wave velocity is bounded by the speed of light in a vacuum by the axiom proposed by Albert Einstein, but is a function of the local conditions in general. The material velocity itself is not bounded *a priori*, since none of our axioms directly imply that any such bound exists. Even though the laws of discrete mechanics can be applied to problems derived from cosmology, they will chiefly be applied to areas of classical mechanics where cause-and-effect relations are associated with finite wave velocities.

The local equilibrium hypothesis is also set aside; none of our axioms imply that an arbitrary medium is in local equilibrium, and various counterexamples can be found. Hence, state equations become useless; applying these equations generates artifacts and violates conservation equations such as the law of conservation of mass. In continuum mechanics, the state equations are typically used to close the system of equations so that it has one equation for each variable. Today, this approach seems simplistic and reveals a lack of understanding of the degree of autonomy of the equations of motion. Only some of the physical properties are worth knowing at any cost; these properties should influence the solution of course, but only via the relation that exists between the variables in the conservation equations. Including other constitutive equations among the laws of mechanics, such as rheology describing a material's behavior, is no longer justified. Like any other properties, these equations should simply be known locally and instantaneously, even if they depend on the problem variables. Thus, a drastic distinction is drawn and preserved between properties and conservation equations.

Is the tensor formulation of the equations of mechanics strictly necessary? Tensors, introduced by Woldemar Voigt in their modern form and adopted by many other renowned thinkers over the last century, were in particular employed by Albert Einstein to formulate his theory of relativity, with help from Marcel Grossmann. In the past, physicists and engineers have legitimately needed a representation of the properties of certain materials that varies as a function of the direction. But can we justify viewing tensors as an integral part of the laws of mechanics? After all, Maxwell's equations can be expressed equivalently in either vector or tensor form. Tensors are required to define the stress as the product of the gradient of the velocity or the displacement and a coefficient. Cauchy's symmetrization of the velocity tensor made matters worse. The generalization of vectors to tensors can of course be justified mathematically. But in the equations of motion, the use of tensors generates artifacts that require us to impose additional constraints to resolve the resulting indeterminacy. For example, in solid mechanics, compatibility conditions must be imposed before the displacement can be computed from the stress. Over the past two centuries, other artifacts of similar type have been introduced into the equations of continuum mechanics. Discrete mechanics adopts a different perspective of the notion of stress. Shearing in a plane can, for example, be induced by a rotational stress in the direction normal to this plane; continuum mechanics would describe the same shearing as a stress in the plane itself. Eliminating tensors from the formulation of the equations of motion is a fundamental aspect of the discrete mechanical approach. Setting aside the concepts of frame of reference and the dimensionality of space allows us to introduce the operators of differential geometry *ad hoc*.

The inappropriate usage of certain laws of physics can obscure inconsistencies in the system of equations; in fluid mechanics, the equations of motion are necessarily accompanied by the equations of conservation of mass. The Navier–Stokes equations do not conserve mass when used autonomously. There is a simple reason for this: the velocity-dependent contribution to the accumulation of the pressure is missing. The Navier–Stokes equations are incomplete, whereas the Navier–Lamé equations for solids are self-sufficient. Applying conservation of mass compensates for this deficiency, but it should not need to be invoked here in principle.

It is common practice in mathematics and mechanics to couple together the equations of the boundary conditions and the initial conditions. According to Jacques Hadamard (1902), a problem is said to be well posed if it satisfies certain force or displacement conditions formulated in terms of partial derivatives at the boundary of the domain. In particular, Hadamard remarks that the displacement field can only be expressed up to rigid motion. This perspective is incompatible with discrete mechanics; it is difficult to apply boundary conditions to the displacement or the velocity when these quantities are only defined up to a constant. It will, therefore, be essential to introduce any conditions on the boundary or within the domain directly into the discrete formulation itself, phrased in terms of the only persistent quantities, the stresses; these stresses are applied using discrete operators that filter out uniform translational and rotational motion. The boundary conditions are an integral element of the mathematical formulation and are no longer viewed as external conditions, as is the case in continuum mechanics. Similarly, and for the same reasons, the initial conditions cannot be imposed directly upon the velocity variable. The initial state of a system is entirely defined by the state of its stresses, corresponding to a state of mechanical equilibrium. This equilibrium state is defined as the state that satisfies the equations of motion exactly; the evolution of a physical system takes the form of a succession of equilibrium states.

The primary objective when establishing a physical model or an equation is to predict the future from a fixed current state. The predictive strength of a model is limited by various factors, for example relating to the quality of our knowledge of this current state, the inherently chaotic and turbulent nature of the evolution of nonlinear dynamic systems and so on. Because of progress in mathematics over the past century, we now have a solid understanding of these questions, but our current perspective of the physical model is still problematic. How can we predict the state in a fixed neighborhood of space-time given a perfectly determined state? In continuum mechanics, this cannot be done without introducing constitutive equations to connect the variables; for example, the state equations are used to close the system of equations and connect the states of the system at different times, even though this diverges from the intended role of these equations. This approach implicitly adopts the hypothesis of local equilibrium, which violates conservation of mass over time. To establish a deterministic prediction that is consistent with conservation principles, we must guarantee continuity in time as well as in space, together with causality in time to establish a continuous history of the evolution of the system. Discrete mechanics introduces the principle of accumulation of stresses: the variations in the velocity or the displacement modify the pressure and shear stresses, which preserves the continuous history of the system. The variable in the equations of the motion, the velocity, is an instantaneous quantity that is only used to accumulate the stresses. Thus, the discrete equations of motion are autonomous and do not depend on any state equations or conservation equations, e.g. a separate law describing the conservation of mass.

The current notions of a continuum, constitutive equations, boundary conditions and thermodynamics led to a physical model that attempts to mimic reality as accurately as possible with a system of partial differential equations. But once this system has been established, what do we do with it? Since the system is formulated at a point in accordance with the hypothesis of a continuum, before we can apply it to a space, such as a three-dimensional space, we must first discretize the partial derivatives by finite differences or some other suitable variational formulation. Regardless of the formulation, this numerical discretization step is necessary to transform the problem into a linear problem that can be solved using a topology with a fixed or variable mesh; this step introduces new sources of error. The behavior of the numerical model cannot always be improved by applying higher order numerical schemes, and the same is true of the final simulation. By contrast, discrete mechanics does not require an additional discretization of the space, and the discrete equations of motion are ready to be used directly. The initial topology of the formulation of the model is the same as that of the discretization, and the discrete operators are the same as the operators of both the equations of motion and the boundary conditions. The properties of continuous operators also coincide with those of the discrete topology; this is not true, in general, for classical numerical methods on unstructured topologies.

The principle of objectivity or material frame-indifference introduced by Clifford Truesdell states that the reaction of a material under a stress is independent of the direction of observation; the response of the material is invariant under change of reference. The Cauchy stress tensor is objective, but the rate-of-rotation tensor is not; this is one of the obstacles presented by formulations based on the curl of the velocity. The curl is directly related to rigid-body translational and rotational motion. For rigid rotational motion, the rotation tensor can be expressed in terms of its dual, the curl vector. This point, as well as the objectivity of the discrete equations of motion, will be discussed later, but our formulation of the problem will ultimately diverge from continuum mechanics, where the answers given in the literature vary and the discussion remains open. The fact that the equations of motion do not require any constitutive equations or global frame of reference will greatly simplify our answer to this question.

Some authors consider thermodynamics a science in its own right; in particular, its extension to the thermodynamics of irreversible processes by Ilya Prigogine (1977) led to an agitated yet fruitful confrontation with the ideas of rational mechanics formulated by Clifford Ambrose Truesdell and Walter Noll. Truesdell's thoughts on the history of science and, in particular, the field of mechanics offer insight that still remains relevant in modern times. Some aspects of this mechanical–thermodynamic interaction raise obstacles that are incompatible with deterministic views of the process of deriving equations. In particular, the Clausius–Duhem inequality for the entropy leads to a condition on the viscosity, compression and shear coefficients. The symmetry conditions required for a description of an isotropic medium lead to the same result: indeterminacy in the volume viscosity. Stokes' choice to hypothesize that the volume viscosity is zero has proven inadequate to say the least. However, the ramifications of this decision have been fully mitigated by applying the law of conservation of mass in connection with the equations of motion. Discrete mechanics is able to eliminate the indeterminacy in the volume viscosity by showing that the compression and shear stresses are two distinct concepts, each with its own coefficient that is unrelated to the other. The derivation of the conservation laws in a discrete medium does not involve any equations describing the behavior of the material, any constitutive equations or any thermodynamics. The thermophysical properties must simply be known.

Surface or shock discontinuities can pose major difficulties when modeling physical phenomena; their nature can vary greatly: fissures in solid materials, faults in porous materials, phase changes, interfaces between immiscible fluids, shock waves in compressible flows, etc. To formulate these problems mathematically, equations are established on each subdomain of the medium, with jump conditions imposed on certain scalar or vector variables at the interfaces. Implementing these jumps is always tricky and often depends on the numerical methodology. In discrete mechanics, the jumps are fully integrated into the formulation, within the conservation equations. The discontinuity is formulated as the gradient of a phase function in the equations of motion or in the equations describing the conservation of flux. For example, this allows us to find an exact solution for the Laplace problem of a capillary pressure jump in a droplet. The discrete equations of motions are capable of describing compressible flows and, in particular, the propagation of shock waves.

Helmholtz-Hodge decomposition is not an axiom as such but rather a mathematical result, the fundamental theorem of vector calculus, established by Hermann Ludwing Ferdinand von Helmholtz, and extended by the work of William Vallance Douglas Hodge on differential geometry. The theorem states that any vector may be decomposed into a component with zero divergence (the solenoidal part) and another component with zero curl (the irrotational part). The theorem introduces notions of scalar and vector potentials that have clear physical interpretations in some fields of physics, including electromagnetism. Strangely, even though it has been used in mechanics to project onto a divergence-free field, the theorem itself has never been viewed as an intangible principle. Although some connectedness assumptions are required, Helmholtz-Hodge decomposition can be used as an axiom to derive the conservation equations; this is the approach taken by the theory of discrete mechanics developed in these pages. The Navier-Lamé equations are to some extent analogous – they give a decomposition into the sum of a gradient and a curl – but much like the Navier-Stokes equations, they only hold within the framework of continuum mechanics. In discrete mechanics, the acceleration decomposes naturally into the two components cited above, and the equations of motion can be presented as a Helmholtz-Hodge extractor for the scalar and vector potentials.

The field of discrete mechanics offers a new paradigm founded on the concept of a discrete medium in which space consists of oriented edges at every scale of observation. The concepts of continuum, derivatives, and global frames of reference are abandoned. The equations of discrete mechanics may be derived axiomatically from the clearly identified principles described below, leading to a consistent formulation that gives a unified description of multiple distinct domains of physics.

List of Symbols

$\cdot$	scalar product (inner product)
$\times$	vector product (cross product)
$\otimes$	tensor product
$:$	contracted tensor product
∇	nabla operator, gradient
$\nabla\cdot$	divergence
$\nabla\times$	curl
$\nabla_p\times$	primal curl
$\nabla_d\times$	dual curl
$\nabla^2(*)$,	$\nabla\cdot\nabla(*)$, Laplacian
tr	trace of a tensor
$\frac{d}{dt}$	material derivative, total derivative
$\frac{\partial}{\partial t}$	partial derivative with respect to time
α	isothermal expansion coefficient
α_l	attenuation factor of longitudinal waves
α_t	attenuation factor of transverse waves
β	thermal expansion coefficient
δ	length of an edge in the primal topology
ε	local porosity of a porous medium
χ_T	isothermal compressibility coefficient
χ_S	isentropic compressibility coefficient
δ_{ij}	Kronecker delta
ε_{ij}	components of the strain tensor
ϕ	scalar potential of the acceleration

γ	heat capacity ratio, surface tension
κ	mean curvature of an interface
κ_l	longitudinal curvature of the primal topology at a point
κ_t	transverse curvature on a face
λ	compression viscosity
$\boldsymbol{\varphi}$	heat flux density
μ	shear viscosity on a face
μ_{dm}	discrete shear eddy viscosity
μ_{sm}	eddy viscosity
μ_t	turbulent viscosity
ν	kinematic viscosity
ρ	density
ρ_m	density on a face
ρ_v	density on an edge
σ	surface tension per unit mass, Poisson coefficient
τ	time constant
ψ	stream function
$\boldsymbol{\gamma}$	acceleration
$\boldsymbol{\varepsilon}$	strain rate tensor
$\boldsymbol{\omega}$	shear-rotation stress potential
$\boldsymbol{\omega^o}$	equilibrium shear-rotation stress potential
$\boldsymbol{\sigma}$	stress tensor
$\boldsymbol{\tau}$	viscous stress tensor
$\boldsymbol{\xi}$	phase indicator
$\boldsymbol{\psi}$	vector potential of the acceleration
$\boldsymbol{\psi^o}$	equilibrium vector potential of the acceleration
Σ	local discontinuity on the edge Γ
Φ	dissipation function
$\mathbf{\Phi}$	heat flux
Γ	edge in the primal topology, curvilinear contour
Ω	volume of a domain
$\mathbf{\Omega}$	rate-of-rotation tensor, spin tensor
$\mathbf{\Psi}$	vector potential
(x, y, z)	Cartesian coordinates
(r, θ)	polar coordinates
(r, θ, z)	cylindrical coordinates
(r, θ, φ)	spherical coordinates
$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$	unit vectors
$\mathcal{A}$	area of a surface
$\mathcal{D}$	domain, control volume
$\mathcal{L}$	linear operator

$\mathcal{L}$ Lamb vector
$\mathcal{M}$ molar mass
$\mathcal{N}$ non-linear operator
$\mathcal{P}$ power
$\mathcal{V}$ volume
$\mathcal{S}$ surface of a face in the primal topology

a thermal diffusivity
c_p specific heat at constant pressure
c_v specific heat at constant volume
c_l longitudinal celerity, longitudinal wave velocity
c_t transverse celerity, transverse wave velocity
d distance between the points or vertices $[a, b]$
e specific internal energy
f scalar function
k thermal conductivity, turbulent kinetic energy
h specific enthalpy
m mass
p pressure, scalar potential
p^o equilibrium scalar potential
p^* dynamic pressure
p_B Bernoulli pressure
q heat production per unit volume
q_m mass flow rate
q_v volume flow rate
r ideal gas constant
s specific entropy, curvilinear coordinates
t time
v specific volume

D flow rate
D_h hydraulic diameter
E Young's modulus, total energy
J Jacobian of the transformation
R molar gas constant
L reference distance, latent heat
S entropy
T temperature
T^o equilibrium temperature
T_0 reference temperature
V_0 reference velocity

$\mathbf{f}$	body force
$\mathbf{g}$	acceleration due to gravity, mass force
$\mathbf{q}$	momentum
$\mathbf{t}$	unit tangent vector
$\mathbf{v}'$	fluctuation in the velocity
$\mathbf{v}$	perturbation in the velocity
$\mathbf{D}$	strain rate tensor
$\mathbf{F}$	force
$\mathbf{I}$	identity matrix or tensor
$\mathbf{K}$	permeability tensor
$\mathbf{M}$	mobility tensor
$\mathbf{N}$	outward normal to a free surface
$\mathbf{T}$	stress
$\mathbf{V}$	component of the velocity along the edge Γ
$\|\mathbf{W}\|$	modulus of the velocity
$\mathbf{W}$	velocity
$\overline{\mathbf{W}}$	averaged velocity

Bi	Biot number
Da	Darcy number
M	Mach number
Ma	Marangoni number
Ra	Rayleigh number
Re	Reynolds number
We	Weber number

1

Fundamental Principles of Discrete Mechanics

This chapter is dedicated to the foundations of discrete mechanics. The notion of space is defined directly as a set of topological elements: edges and surfaces. These geometric elements exist at every scale and cannot be reduced to a point like in a continuum; as a result, we must abandon the concept of local differentiation, as well as inertial and non-inertial frames of reference. Some of the classical principles of physics can be kept, such as the weak equivalence principle and the principle of relativity, and some new physical principles are encountered for the first time, such as Hodge–Helmholtz decomposition. We also require new axioms and hypotheses: the accumulation of stresses and the duality of mechanical actions of all kinds.

1.1. Definitions of discrete mechanics

1.1.1. *Notion of discrete space–time*

A method of positioning ourselves within space and time is essential if we wish to represent the universe around us, whether the universe of our daily lives, or the wider universe governed by the laws of general relativity. Positioning systems (GPS and Galileo) have become indispensable tools for many human activities such as transportation, well-drilling and so on. But in fact, to move toward a nearby target, we do not need to know our position exactly with respect to some absolute reference; we simply need to know the path to our target. The various theories of mechanics (Newtonian, quantum, continuum, relativistic, etc.) do not contradict each other – quite the opposite – but the connections between them have not yet been definitively established. Each theory of mechanics only describes a part of reality. The concepts, analysis tools and hypotheses of each theory vary. Ultimately, a unified theory of mechanics might not be strictly necessary.

Discrete Mechanics: Concepts and Applications,
First Edition. Jean-Paul Caltagirone.

The theory of discrete mechanics presented here assumes that there exists a time, the present, that describes the state of a physical system instantaneously. Although this image of the present exists as such, an observer located within space can only perceive its environment at later moments in time, since waves (light, sound, tidal waves) travel at finite velocity. The present can, therefore, only be perceived by an exterior observer in the form of a mathematical model that provide an instantaneous description of every phenomenon in the physical system.

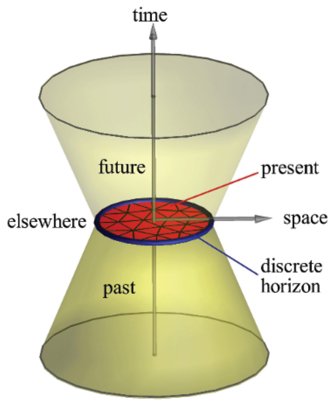

Figure 1.1. *Light cone in space–time. For a color version of this figure, see www.iste.co.uk/caltagirone/mechanics.zip*

Figure 1.1 shows the configuration of space–time in discrete mechanics; this model is borrowed from cosmology, where the present only makes sense for events unfolding at the origin. Any event that can influence or be influenced by an event unfolding at the origin is contained in the two cones whose summits are joined at the origin: the lower cone, which represents the past, and the upper cone, which represents the future. This light cone defines a so-called causal structure. For example, since the distance between the Earth and the Sun is large, we only receive light from the Sun 8 min after it was emitted. Any light signal emitted from the Earth would take just as long to reach the Sun. Events that occur during this period of time cannot be perceived by an observer; these events are said to be located elsewhere. In cosmology, the displacement $\mathcal{A}\,\mathcal{B}$ of a point in space–time is represented by the four-vector $(c\,dt, d\mathbf{x})$, and the present is restricted to the origin. In both discrete and continuum mechanics, the present unites all elements within a single causal structure; in Figure 1.1, the boundary of this structure is a circle, which we shall call the discrete horizon. Every event in this space is linked by cause and effect, the radius of the circle r_h is independent of time and every event within the circle is known instantaneously. Even if a specific observer located at some point of this space cannot directly perceive every event unfolding in the present instantaneously, the instantaneous field of all problem variables exists and can be represented by a mathematical model. Time is assumed to unfold linearly.

Figure 1.2 shows two spaces. The first has a finite horizon as its boundary, and the second is a sphere without a boundary but which is nonetheless finite; in both cases, all events unfolding on these surfaces are connected by the propagation of various types of waves through space. On the space with a boundary, events will necessarily be influenced by boundary conditions, which will be defined later. On the sphere, interactions will cumulate as the system evolves over time.

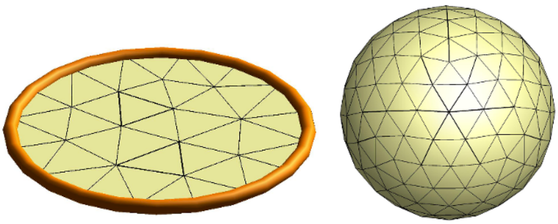

Figure 1.2. *Space with a discrete horizon as a boundary (left) and a sphere without a boundary but which is nonetheless finite (right)*

Thus, the discrete horizon defines a space on which separate phenomena can be described by a mathematical model at the same moment in time. For example, atmospheric models give an image of the present time and can be used to predict the weather over the next few days. Neglecting absorption, sound waves require over 300 h to travel the $40,000$ km of the circumference of the Earth, and light waves require slightly over 0.1 s. Even if an event at one point is not perceived instantaneously from another point, we can still construct an instantaneous image of the atmospheric currents. To make accurate predictions, we need a good representation of the present, which can be achieved by collecting a large amount of precise data. However, the chaotic and to some extent random nature of the turbulent evolution of atmospheric flows limits the prediction range of the model to just a few days.

The approach adopted by discrete mechanics draws heavily from the classical view of mechanics, where every interaction is defined directly. The interactions conventionally described as “actions at a distance”, such as variations in gravity due to the Moon and Sun, are predictable and can be taken into account in the mathematical model. However, we cannot represent the cause-and-effect relations of more rapid events, such as the collapse of two black holes producing the gravitational waves predicted by general relativity.

Nevertheless, Newtonian mechanics is an alternative that is compatible with reality. Newtonian mechanics is widely thought to only be valid at velocities far below the speed of light, but in fact Newton's theory has remarkable properties when extended to the propagation of waves. The velocity is not the only relevant distinction between the relativistic and Newtonian theories of mechanics; relativistic kinematics and dynamics are other examples. Ultimately, the objective of the discrete perspective presented here is to investigate whether Newtonian mechanics is capable of describing all types of fluid and solid behavior, as well as the propagation of all types of waves.

1.1.2. *Notion of a discrete medium*

The perspective presented and explored here abandons the hypothesis of a continuum, which defined all of the problem variables, physical properties, etc., at every point. In continuum mechanics, Newton's law of dynamics, also known as Newton's second law, is formulated at a point. To express the spatial variation of the vector quantities on which the theory is based, we are forced to define the concepts of frame of reference and differentiation. Newton himself contributed to the development of infinitesimal calculus, even though he originally represented vectors as bipoints [NEW 90]. Later work expanded this continuous approach, which has various advantages, but also disadvantages that can generate artifacts.

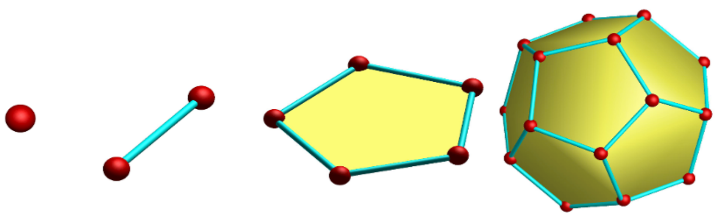

Figure 1.3. *Construction of a discrete medium from points, edges, surfaces and volumes*

The field of discrete mechanics is built upon connected objects such as those shown in Figure 1.3. First, we consider points that are not absolutely positioned within space; we shall work within a local frame of reference that positions objects relatively to one other. Two points and a straight line define an edge, or bipoint. This introduces two important ideas: the distance d between the points and a direction, also defined in relative terms. From multiple edges, we can construct a surface, which is necessarily planar; to represent non-planar surfaces, we can reduce them to triangles, which are planar by definition. Finally, by assembling multiple planar surfaces, we can construct volumes. The concept of dimension (one, two or three) is abandoned. For example, a (2D) plane constructed from three points simply defines