SECOND EDITION

# VIBRATION

## OF CONTINUOUS SYSTEMS

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# Vibration of Continuous Systems

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**Second Edition**


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# Contents

# Preface

This book presents the analytical and numerical methods of vibration analysis of continuous structural systems, including strings, bars, shafts, beams, circular rings and curved beams, membranes, plates, shells, and composite structures. The objectives of the book are (1) to make a methodical and comprehensive presentation of the vibration of various types of structural elements, (2) to present the exact analytical, approximate analytical, and approximate numerical methods of analysis, and (3) to present the basic concepts in a simple manner with illustrative examples. Favorable reactions and encouragement from professors, students and other users of the book have provided me with the impetus to prepare this second edition of the book.

The following changes have been made from the first edition:

- Some sections were rewritten for better clarity.
- Some new problems are added.
- The errors noted in the first edition have been corrected.
- Some sections have been expanded. The chapter on "Elastic Wave Propagation" has been deleted.
- A new chapter on "Vibration of Composite Structures" has been added.
- A new chapter entitled, "Approximate Numerical Methods: Finite Element Method," is added to complement the existing chapter on "Approximate Analytical Methods."

Continuous structural elements and systems are encountered in many branches of engineering, such as aerospace, architectural, chemical, civil, ocean, and mechanical engineering. The design of many structural and mechanical devices and systems requires an accurate prediction of their vibration and dynamic performance characteristics. The methods presented in the book can be used in these applications. The book is intended to serve as a textbook for a dual-level or first graduate degree-level course on vibrations or structural dynamics. More than enough material is included for a one-semester course. The chapters are made as independent and self-contained as possible so that a course can be taught by selecting appropriate chapters or through equivalent self-study. A successful vibration analysis of continuous structural elements and systems requires a knowledge of mechanics of materials, structural mechanics, ordinary and partial differential equations, matrix methods, variational calculus, and integral equations. Applications of these techniques are presented throughout. The selection, arrangement, and presentation of the material have been made based on the lecture notes for a course taught by the author. The contents of the book permit instructors to emphasize a variety of topics, such as basic mathematical approaches with simple applications, bars and beams, beams and plates, or plates and shells. The book will also be useful as a reference book for practicing engineers, designers, and vibration analysts involved in the dynamic analysis and design of continuous systems.

## Organization of the Book

The book is organized into 18 chapters and two appendices. The basic concepts and terminology used in vibration analysis are introduced in Chapter 1. The importance, origin, and a brief history of vibration of continuous systems are presented. The difference between discrete and continuous systems, types of excitations, description of harmonic functions, and basic definitions used in the theory of vibrations and representation of periodic functions in terms of Fourier series and the Fourier integral are discussed. Chapter 2 provides a brief review of the theory and techniques used in the vibration analysis of discrete systems. Free and forced vibration of single- and multidegree-of-freedom systems are outlined. The eigenvalue problem and its role in the modal analysis used in the free and forced vibration analysis of discrete systems are discussed.

Various methods of formulating vibration problems associated with continuous systems are presented in Chapters 3, 4, and 5. The equilibrium approach is presented in Chapter 3. Use of Newton's second law of motion and D'Alembert's principle is outlined, with application to different types of continuous elements. Use of the variational approach in deriving equations of motion and associated boundary conditions is described in Chapter 4. The basic concepts of calculus of variations and their application to extreme value problems are outlined. The variational methods of solid mechanics, including the principles of minimum potential energy, minimum complementary energy, stationary Reissner energy, and Hamilton's principle, are presented. The use of Hamilton's principle in the formulation of continuous systems is illustrated with torsional vibration of a shaft and transverse vibration of a thin beam. The integral equation approach for the formulation of vibration problems is presented in Chapter 5. A brief outline of integral equations and their classification, and the derivation of integral equations, are given together with examples. The solution of integral equations using iterative, Rayleigh–Ritz, Galerkin, collocation, and numerical integration methods is also discussed in this chapter.

The common solution procedure based on eigenvalue and modal analyses for the vibration analysis of continuous systems is outlined in Chapter 6. The orthogonality of eigenfunctions and the role of the expansion theorem in modal analysis are discussed. The forced vibration response of viscously damped systems are also considered in this chapter. Chapter 7 covers the solution of problems of vibration of continuous systems using integral transform methods. Both Laplace and Fourier transform techniques are outlined together with illustrative applications.

The transverse vibration of strings is presented in Chapter 8. This problem finds application in guy wires, electric transmission lines, ropes and belts used in machinery, and the manufacture of thread. The governing equation is derived using equilibrium and variational approaches. The traveling-wave solution and separation of variables solution are outlined. The free and forced vibration of strings are considered in this chapter. The longitudinal vibration of bars is the topic of Chapter 9. Equations of motion based on simple theory are derived using the equilibrium approach as well as Hamilton's principle. The natural frequencies of vibration are determined for bars with different end conditions. Free vibration response due to initial excitation and forced vibration of bars are both presented, as is response using modal analysis. Free and forced vibration of bars using Rayleigh and Bishop theories are also outlined in Chapter 9. The torsional vibration of shafts plays an important role in mechanical transmission of power in prime movers and other high-speed machinery. The torsional vibration of uniform and nonuniform rods with both circular and noncircular cross-sections is described in Chapter 10. The equations of motion and free and forced vibration of shafts with circular cross-section are discussed using the elementary theory. The Saint-Venant and Timoshenko–Gere theories are considered in deriving the equations of motion of shafts with noncircular cross-sections. Methods of determining the torsional rigidity of noncircular shafts are presented using the Prandtl stress function and Prandtl membrane analogy.

Chapter 11 deals with the transverse vibration of beams. Starting with the equation of motion based on Euler–Bernoulli or thin beam theory, natural frequencies and mode shapes of beams with different boundary conditions are determined. The free vibration response due to initial conditions, forced vibration under fixed and moving loads, response under axial loading, rotating beams, continuous beams, and beams on an elastic foundation are presented using the Euler–Bernoulli theory. The effects of rotary inertia (Rayleigh theory) and rotary inertia and shear deformation (Timoshenko theory) on the transverse vibration of beams are also considered. The coupled bending-torsional vibration of beams is discussed. Finally, the use of transform methods for finding the free and forced vibration problems is illustrated toward the end of Chapter 11. In-plane flexural and coupled twist-bending vibration of circular rings and curved beams is considered in Chapter 12. The equations of motion and free vibration solutions are presented first using a simple theory. Then the effects of rotary inertia and shear deformation are considered. The vibration of rings finds application in the study of the vibration of ring-stiffened shells used in aerospace applications, gears, and stators of electrical machines.

The transverse vibration of membranes is the topic of Chapter 13. Membranes find application in drums and microphone condensers. The equation of motion of membranes is derived using both the equilibrium and variational approaches. The free and forced vibration of rectangular and circular membranes are both discussed in this chapter. Chapter 14 covers the transverse vibration of plates. The equation of motion and the free and forced vibration of both rectangular and circular plates are presented. The vibration of plates subjected to in-plane forces, plates on elastic foundation, and plates with variable thickness is also discussed. Finally, the effect of rotary inertia and shear deformation on the vibration of plates according to Mindlin's theory is outlined. The vibration of shells is the topic of Chapter 15. First, the theory of surfaces is presented using shell coordinates. Then the strain-displacement relations according to Love's approximations, stress–strain, and force and moment resultants are given. Then the equations of motion are derived from Hamilton's principle. The equations of motion of circular cylindrical shells and their natural frequencies are considered using Donnel–Mushtari–Vlasov and Love's theories. Finally, the effect of rotary inertia and shear deformation on the vibration of shells is considered.

Chapter 16 presents vibration of fiber-reinforced composite structures and structural members. The composite material mechanics of laminates including constitutive relations, stress analysis under in-plane and transverse loads as well as free vibration analysis of rectangular plates and beams are presented in this chapter. Chapter 17 is devoted to the approximate analytical methods useful for vibration analysis. The computational details of the Rayleigh, Rayleigh-Ritz, assumed modes, weighted residual, Galerkin, collocation, subdomain collocation, and least squares methods are presented along with numerical examples. Finally, the numerical methods, based on the finite element method, for the vibration analysis of continuous structural elements and systems are outlined in Chapter 18. The displacement approach is used in deriving the element stiffness and mass matrices of bar, beam, and linear triangle (constant strain triangle or CST) elements. Numerical examples are presented to illustrate the application of the finite element method for the solution of simple vibration problems.

Appendix A presents the basic equations of elasticity. Laplace and Fourier transform pairs associated with some simple and commonly used functions are summarized in Appendix B.

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# About the Author

Dr. Singirisu S. Rao is a Professor in the Department of Mechanical and Aerospace Engineering at the University of Miami, Coral Gables, Florida. He was the Chairman of the Mechanical and Aerospace Engineering Department during 1998–2011 at the University of Miami. Prior to that, he was a Professor in the School of Mechanical Engineering at Purdue University, West Lafayette, Indiana; Professor of Mechanical Engineering at San Diego State University, San Diego, California; and the Indian Institute of Technology, Kanpur, India. He was a visiting research scientist for two years at NASA Langley Research Center, Hampton, Virginia.

Professor Rao is the author of eight textbooks entitled: *The Finite Element Method in Engineering*; *Engineering Optimization*; *Mechanical Vibrations*; *Reliability-Based Design*; *Vibration of Continuous Systems*; *Reliability Engineering*; *Applied Numerical Methods for Engineers and Scientists*; *Optimization Methods: Theory and Applications*. He co-edited a three-volume *Encyclopedia of Vibration*. He has edited four volumes of *Proceedings of the ASME Design Automation and Vibration Conferences*. He has published over 200 journal papers in the areas of multiobjective optimization, structural dynamics and vibration, structural control, uncertainty modeling, analysis, design and optimization using probability, fuzzy, interval, evidence and gray system theories. Some 34 Ph.D. students have received their degrees under the supervision of Professor Rao. In addition, 12 post-doctoral researchers and scholars have conducted research under his guidance.

Professor Rao has received numerous awards for his academic and research achievements. He was awarded the *Vepa Krishnamurti Gold Medal for University First Rank* in all the five years of the BE (Bachelor of Engineering) program among students of all branches of engineering in all the Engineering Colleges of Andhra University. He was awarded the *Lazarus Prize for University First Rank* among students of Mechanical Engineering in all the Engineering Colleges of Andhra University. He received the *First Prize* in the James F. Lincoln Design Contest open to all M.S. and Ph.D. students in the USA and Canada for a paper he wrote on automated optimization of aircraft wing structures from his Ph.D. dissertation. He received the *Eliahu I. Jury Award for Excellence in Research* from the College of Engineering, the University of Miami, in 2002; he was awarded the *Distinguished Probabilistic Methods Educator Award* from the Society of Automotive Engineers (SAE) International for *Demonstrated Excellence in Research Contributions in the Application of Probabilistic Methods to Diversified Fields, Including Aircraft Structures, Building Structures, Machine Tools, Air Conditioning and Refrigeration Systems, and Mechanisms* in 1999; he received the American Society of Mechanical Engineers (ASME) *Design Automation Award* for *Pioneering Contributions to Design Automation, particularly in Multiobjective Optimization, and Uncertainty Modeling, Analysis and Design Using Probability, Fuzzy, Interval, and Evidence*

*Theories* in 2012; and he was awarded the *ASME Worcester Reed Warner Medal* in 2013 for *Outstanding Contributions to the Permanent Literature of Engineering, particularly for his Many Highly Popular Books and Numerous Trendsetting Research Papers*. In 2018, Dr. Rao received the *Albert Nelson Marquis Lifetime Achievement Award* for demonstrated unwavering excellence in the field of Mechanical Engineering.

**This book is accompanied by a book companion site: www.wiley.com/go/rao/vibration**

# 1

# Introduction: Basic Concepts and Terminology

## 1.1 CONCEPT OF VIBRATION

Any repetitive motion is called *vibration* or *oscillation*. The motion of a guitar string, motion felt by passengers in an automobile traveling over a bumpy road, swaying of tall buildings due to wind or earthquake, and motion of an airplane in turbulence are typical examples of vibration. The theory of vibration deals with the study of oscillatory motion of bodies and the associated forces. The oscillatory motion shown in Fig. 1.1(*a*) is called *harmonic motion* and is denoted as

$$x(t) = X \cos \omega t \tag{1.1}$$

where $X$ is called the *amplitude of motion*, $\omega$ is the *frequency of motion*, and $t$ is the time. The motion shown in Fig. 1.1(*b*) is called *periodic motion*, and that shown in Fig. 1.1(*c*) is called *nonperiodic* or *transient motion*. The motion indicated in Fig. 1.1(*d*) is *random* or *long-duration nonperiodic vibration*.

The phenomenon of vibration involves an alternating interchange of potential energy to kinetic energy and kinetic energy to potential energy. Hence, any vibrating system must have a component that stores potential energy and a component that stores kinetic energy. The components storing potential and kinetic energies are called a *spring* or *elastic element* and a *mass* or *inertia element*, respectively. The elastic element stores potential energy and gives it up to the inertia element as kinetic energy, and vice versa, in each cycle of motion. The repetitive motion associated with vibration can be explained through the motion of a mass on a smooth surface, as shown in Fig. 1.2. The mass is connected to a linear spring and is assumed to be in equilibrium or rest at position 1. Let the mass $m$ be given an initial displacement to position 2 and released with zero velocity. At position 2, the spring is in a maximum elongated condition, and hence the potential or strain energy of the spring is a maximum and the kinetic energy of the mass will be zero since the initial velocity is assumed to be zero. Because of the tendency of the spring to return to its unstretched condition, there will be a force that causes the mass $m$ to move to the left. The velocity of the mass will gradually increase as it moves from position 2 to position 1. At position 1, the potential energy of the spring is zero because the deformation of the spring is zero. However, the kinetic energy and hence the velocity of the mass will be maximum at position 1 because of the conservation of energy (assuming no dissipation of energy due to damping or friction). Since the velocity is maximum at position 1, the mass will continue to move to the left, but against the resisting force due to compression of the spring. As the mass moves from position 1 to the left, its velocity will gradually decrease until it reaches a value of zero at position 3. At position 3, the velocity and hence the kinetic energy of the mass will be zero and the

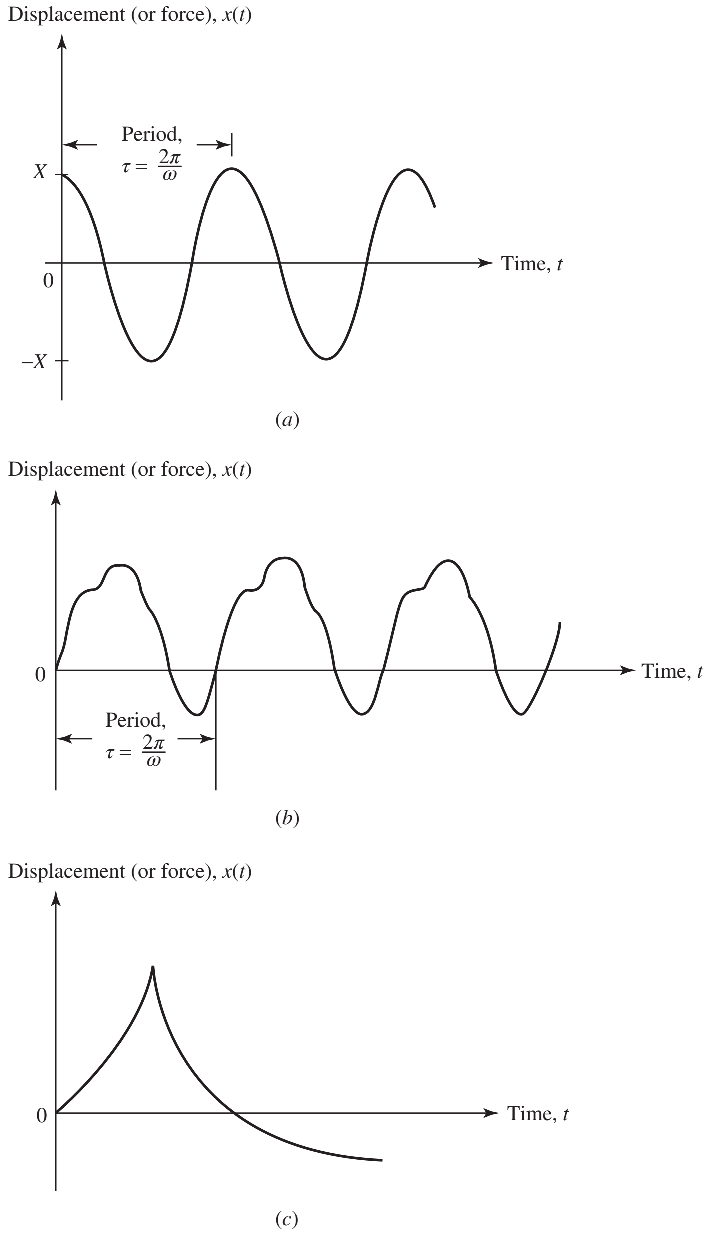
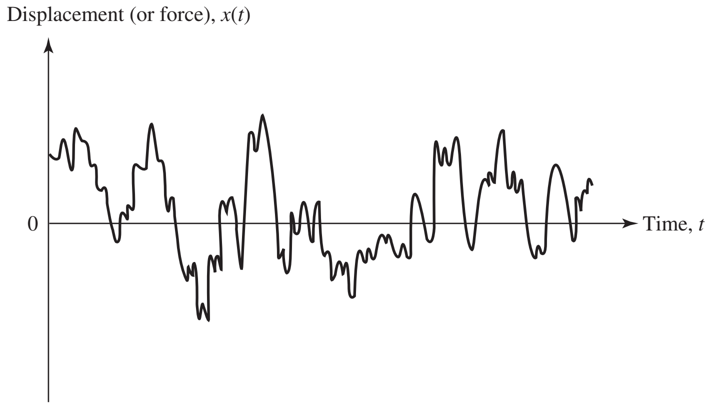



**Figure 1.1** Types of displacements (or forces): (*a*) periodic, simple harmonic; (*b*) periodic, nonharmonic; (*c*) nonperiodic, transient; (*d*) nonperiodic, random.




(*d*)

**Figure 1.1** (*continued*)

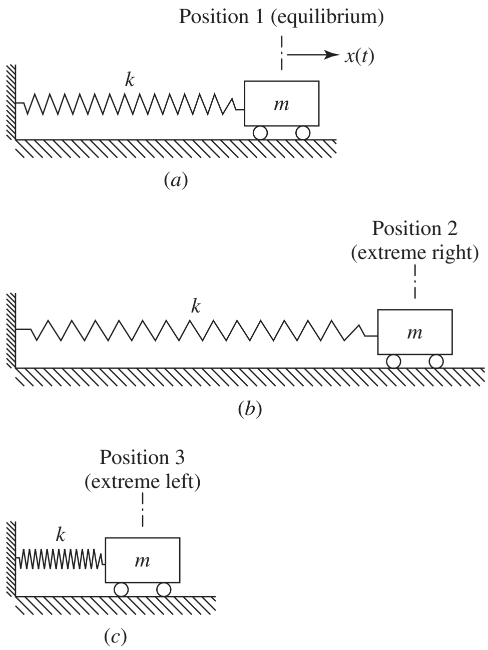



**Figure 1.2** Vibratory motion of a spring–mass system: (*a*) system in equilibrium (spring undeformed); (*b*) system in extreme right position (spring stretched); (*c*) system in extreme left position (spring compressed).

deflection (compression) and hence the potential energy of the spring will be maximum. Again, because of the tendency of the spring to return to its uncompressed condition, there will be a force that causes the mass $m$ to move to the right from position 3. The velocity of the mass will increase gradually as it moves from position 3 to position 1.

At position 1, all the potential energy of the spring has been converted to the kinetic energy of the mass, and hence the velocity of the mass will be maximum. Thus, the mass continues to move to the right against increasing spring resistance until it reaches position 2 with zero velocity. This completes one cycle of motion of the mass, and the process repeats; thus, the mass will have oscillatory motion.

The initial excitation to a vibrating system can be in the form of initial displacement and/or initial velocity of the mass element(s). This amounts to imparting potential and/or kinetic energy to the system. The initial excitation sets the system into oscillatory motion, which can be called *free vibration*. During free vibration, there will be an exchange between the potential and the kinetic energies. If the system is conservative, the sum of the potential energy and the kinetic energy will be a constant at any instant. Thus, the system continues to vibrate forever, at least in theory. In practice, there will be some damping or friction due to the surrounding medium (e.g. air), which will cause a loss of some energy during motion. This causes the total energy of the system to diminish continuously until it reaches a value of zero, at which point the motion stops. If the system is given only an initial excitation, the resulting oscillatory motion eventually will come to rest for all practical systems, and hence the initial excitation is called *transient excitation* and the resulting motion is called *transient motion*. If the vibration of the system is to be maintained in a steady state, an external source must continuously replace the energy dissipated due to damping.

## 1.2 IMPORTANCE OF VIBRATION

Any body that has mass and elasticity is capable of oscillatory motion. In fact, most human activities, including hearing, seeing, talking, walking, and breathing, also involve oscillatory motion. Hearing involves vibration of the eardrum, seeing is associated with the vibratory motion of light waves, talking requires oscillations of the larynx (tongue), walking involves oscillatory motion of legs and hands, and breathing is based on the periodic motion of the lungs. In engineering, an understanding of the vibratory behavior of mechanical and structural systems is important for the safe design, construction, and operation of a variety of machines and structures.

The failure of most mechanical and structural elements and systems can be associated with vibration. For example, the blade and disk failures in steam and gas turbines and structural failures in aircraft are usually associated with vibration and the resulting fatigue. Vibration in machines leads to rapid wear of parts, such as gears and bearings, to loosening of fasteners, such as nuts and bolts, to poor surface finish during metal cutting, and excessive noise. Excessive vibration in machines causes not only the failure of components and systems but also annoyance to humans. For example, imbalance in diesel engines can cause ground waves powerful enough to create a nuisance in urban areas. Supersonic aircraft create sonic booms that shatter doors and windows. Several spectacular failures of bridges, buildings, and dams are associated with wind-induced vibration, as well as oscillatory ground motion during earthquakes.

In some engineering applications, vibrations serve a useful purpose. For example, in vibratory conveyors, sieves, hoppers, compactors, dentist drills, electric toothbrushes, washing machines, clocks, electric massaging units, pile drivers, vibratory testing of materials, vibratory finishing processes, and materials processing operations, such as casting and forging, vibration is used to improve the efficiency and quality of the process.