

The background of the cover is a dark, almost black, space filled with vibrant, wavy, horizontal lines. These lines are primarily purple and magenta, with a bright, glowing red and white center that creates a sense of depth and movement, resembling a lens flare or a reflection on a polished surface.

SECOND EDITION

VIBRATION

OF CONTINUOUS SYSTEMS

SINGIRESU S. RAO

WILEY

Vibration of Continuous Systems

Vibration of Continuous Systems

Second Edition

Singiresu S. Rao

University of Miami

WILEY

This edition first published 2019
© 2019 John Wiley & Sons, Inc.

Edition History

John Wiley & Sons Ltd (1e, 2007)

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, except as permitted by law. Advice on how to obtain permission to reuse material from this title is available at <http://www.wiley.com/go/permissions>.

The right of Singiresu S Rao to be identified as the author of this work has been asserted in accordance with law.

Registered Offices

John Wiley & Sons, Inc.. 111 River Street. Hoboken, NJ 07030, USA

Editorial Office

The Atrium, Southern Gate, Chichester, West Sussex, P019 8SQ, UK

For details of our global editorial offices, customer services, and more information about Wiley products visit us at www.wiley.com.

Wiley also publishes its books in a variety of electronic formats and by print-on-demand. Some content that appears in standard print versions of this book may not be available in other formats.

Limit of Liability/Disclaimer of Warranty

While the publisher and authors have used their best efforts in preparing this work, they make no representations or warranties with respect to the accuracy or completeness of the contents of this work and specifically disclaim all warranties, including without limitation any implied warranties of merchantability or fitness for a particular purpose. No warranty may be created or extended by sales representatives, written sales materials or promotional statements for this work. The fact that an organization, website, or product is referred to in this work as a citation and/or potential source of further information does not mean that the publisher and authors endorse the information or services the organization, website, or product may provide or recommendations it may make. This work is sold with the understanding that the publisher is not engaged in rendering professional services. The advice and strategies contained herein may not be suitable for your situation. You should consult with a specialist where appropriate. Further, *readers* should be aware that websites listed in this work may have changed or disappeared between when this work was written and when it is read. Neither the publisher nor authors shall be liable for any loss of profit or any other commercial damages, including but not limited to special, incidental, consequential, or other damages.

Library of Congress cataloging-in-Publication Data

Names: Rao, Singiresu S., author.

Title: Vibration of continuous systems / Singiresu S Rao, University of Miami.

Description: Second edition. | Hoboken, NJ, USA : John Wiley & Sons Ltd,

[2019] | Includes bibliographical references and index. |

Identifiers: LCCN 2018041496 (print) | LCCN 2018041859 (ebook) |

ISBN 9781119424253 (Adobe PDF) | ISBN 9781119424277 (ePub) |

ISBN 9781119424147 (hardcover)

Subjects: LCSH: Vibration—Textbooks. | Structural dynamics—Textbooks.

Classification: LCC TA355 (ebook) | LCC TA355 .R378 2019 (print) |

DDC 624.1/71—dc23

LC record available at <https://lccn.loc.gov/2018041496>

Cover image: © Veleri/iStock/Getty Images Plus

Cover design: Wiley

Set in 10.25/12pt and TimesLTStd by SPi Global, Chennai, India

10 9 8 7 6 5 4 3 2 1



Contents

Preface xv

Acknowledgments xix

About the Author xxi

1 Introduction: Basic Concepts and Terminology 1

- 1.1 Concept of Vibration 1
- 1.2 Importance of Vibration 4
- 1.3 Origins and Developments in Mechanics and Vibration 5
- 1.4 History of Vibration of Continuous Systems 7
- 1.5 Discrete and Continuous Systems 12
- 1.6 Vibration Problems 15
- 1.7 Vibration Analysis 16
- 1.8 Excitations 17
- 1.9 Harmonic Functions 17
 - 1.9.1 Representation of Harmonic Motion 19
 - 1.9.2 Definitions and Terminology 21
- 1.10 Periodic Functions and Fourier Series 24
- 1.11 Nonperiodic Functions and Fourier Integrals 25
- 1.12 Literature on Vibration of Continuous Systems 28

References 29

Problems 31

2 Vibration of Discrete Systems: Brief Review 33

- 2.1 Vibration of a Single-Degree-of-Freedom System 33
 - 2.1.1 Free Vibration 33
 - 2.1.2 Forced Vibration under Harmonic Force 36
 - 2.1.3 Forced Vibration under General Force 41
- 2.2 Vibration of Multidegree-of-Freedom Systems 43
 - 2.2.1 Eigenvalue Problem 45

2.2.2 Orthogonality of Modal Vectors 46

2.2.3 Free Vibration Analysis of an Undamped System Using Modal Analysis 47

2.2.4 Forced Vibration Analysis of an Undamped System Using Modal Analysis 52

2.2.5 Forced Vibration Analysis of a System with Proportional Damping 53

2.2.6 Forced Vibration Analysis of a System with General Viscous Damping 54

2.3 Recent Contributions 60

References 61

Problems 62

3 Derivation of Equations: Equilibrium Approach 69

- 3.1 Introduction 69
- 3.2 Newton's Second Law of Motion 69
- 3.3 D'Alembert's Principle 70
- 3.4 Equation of Motion of a Bar in Axial Vibration 70
- 3.5 Equation of Motion of a Beam in Transverse Vibration 72
- 3.6 Equation of Motion of a Plate in Transverse Vibration 74
 - 3.6.1 State of Stress 76
 - 3.6.2 Dynamic Equilibrium Equations 76
 - 3.6.3 Strain–Displacement Relations 77
 - 3.6.4 Moment–Displacement Relations 79
 - 3.6.5 Equation of Motion in Terms of Displacement 79
 - 3.6.6 Initial and Boundary Conditions 80
- 3.7 Additional Contributions 81

References 81

Problems 82

4 Derivation of Equations: Variational Approach 87

4.1	Introduction	87
4.2	Calculus of a Single Variable	87
4.3	Calculus of Variations	88
4.4	Variation Operator	91
4.5	Functional with Higher-Order Derivatives	93
4.6	Functional with Several Dependent Variables	95
4.7	Functional with Several Independent Variables	96
4.8	Extremization of a Functional with Constraints	98
4.9	Boundary Conditions	102
4.10	Variational Methods in Solid Mechanics	106
4.10.1	Principle of Minimum Potential Energy	106
4.10.2	Principle of Minimum Complementary Energy	107
4.10.3	Principle of Stationary Reissner Energy	108
4.10.4	Hamilton's Principle	109
4.11	Applications of Hamilton's Principle	116
4.11.1	Equation of Motion for Torsional Vibration of a Shaft (Free Vibration)	116
4.11.2	Transverse Vibration of a Thin Beam	118
4.12	Recent Contributions	121
Notes		121
References		122
Problems		122

5 Derivation of Equations: Integral Equation Approach 125

5.1	Introduction	125
5.2	Classification of Integral Equations	125
5.2.1	Classification Based on the Nonlinear Appearance of $\phi(t)$	125
5.2.2	Classification Based on the Location of Unknown Function $\phi(t)$	126

5.2.3	Classification Based on the Limits of Integration	126
5.2.4	Classification Based on the Proper Nature of an Integral	127
5.3	Derivation of Integral Equations	127
5.3.1	Direct Method	127
5.3.2	Derivation from the Differential Equation of Motion	129
5.4	General Formulation of the Eigenvalue Problem	132
5.4.1	One-Dimensional Systems	132
5.4.2	General Continuous Systems	134
5.4.3	Orthogonality of Eigenfunctions	135
5.5	Solution of Integral Equations	135
5.5.1	Method of Undetermined Coefficients	136
5.5.2	Iterative Method	136
5.5.3	Rayleigh–Ritz Method	141
5.5.4	Galerkin Method	145
5.5.5	Collocation Method	146
5.5.6	Numerical Integration Method	148
5.6	Recent Contributions	149
References		150
Problems		151

6 Solution Procedure: Eigenvalue and Modal Analysis Approach 153

6.1	Introduction	153
6.2	General Problem	153
6.3	Solution of Homogeneous Equations: Separation-of-Variables Technique	155
6.4	Sturm–Liouville Problem	156
6.4.1	Classification of Sturm–Liouville Problems	157
6.4.2	Properties of Eigenvalues and Eigenfunctions	162
6.5	General Eigenvalue Problem	165
6.5.1	Self-Adjoint Eigenvalue Problem	165
6.5.2	Orthogonality of Eigenfunctions	167
6.5.3	Expansion Theorem	168

6.6	Solution of Nonhomogeneous Equations	169
6.7	Forced Response of Viscously Damped Systems	171
6.8	Recent Contributions	173
	References	174
	Problems	175

7 Solution Procedure: Integral Transform Methods 177

7.1	Introduction	177
7.2	Fourier Transforms	178
7.2.1	Fourier Series	178
7.2.2	Fourier Transforms	179
7.2.3	Fourier Transform of Derivatives of Functions	181
7.2.4	Finite Sine and Cosine Fourier Transforms	181
7.3	Free Vibration of a Finite String	184
7.4	Forced Vibration of a Finite String	186
7.5	Free Vibration of a Beam	188
7.6	Laplace Transforms	191
7.6.1	Properties of Laplace Transforms	192
7.6.2	Partial Fraction Method	194
7.6.3	Inverse Transformation	196
7.7	Free Vibration of a String of Finite Length	197
7.8	Free Vibration of a Beam of Finite Length	200
7.9	Forced Vibration of a Beam of Finite Length	201
7.10	Recent Contributions	204
	References	205
	Problems	206

8 Transverse Vibration of Strings 209

8.1	Introduction	209
8.2	Equation of Motion	209
8.2.1	Equilibrium Approach	209
8.2.2	Variational Approach	211
8.3	Initial and Boundary Conditions	213
8.4	Free Vibration of an Infinite String	215
8.4.1	Traveling-Wave Solution	215

8.4.2	Fourier Transform-Based Solution	217
8.4.3	Laplace Transform-Based Solution	219
8.5	Free Vibration of a String of Finite Length	221
8.5.1	Free Vibration of a String with Both Ends Fixed	222
8.6	Forced Vibration	231
8.7	Recent Contributions	235
	Note	236
	References	236
	Problems	237

9 Longitudinal Vibration of Bars 239

9.1	Introduction	239
9.2	Equation of Motion Using Simple Theory	239
9.2.1	Using Newton's Second Law of Motion	239
9.2.2	Using Hamilton's Principle	240
9.3	Free Vibration Solution and Natural Frequencies	241
9.3.1	Solution Using Separation of Variables	242
9.3.2	Orthogonality of Eigenfunctions	251
9.3.3	Free Vibration Response due to Initial Excitation	254
9.4	Forced Vibration	259
9.5	Response of a Bar Subjected to Longitudinal Support Motion	262
9.6	Rayleigh Theory	263
9.6.1	Equation of Motion	263
9.6.2	Natural Frequencies and Mode Shapes	264
9.7	Bishop's Theory	265
9.7.1	Equation of Motion	265
9.7.2	Natural Frequencies and Mode Shapes	267
9.7.3	Forced Vibration Using Modal Analysis	269
9.8	Recent Contributions	272
	References	273
	Problems	273

10 Torsional Vibration of Shafts 277

10.1 Introduction 277

10.2 Elementary Theory: Equation of Motion 277

10.2.1 Equilibrium Approach 277

10.2.2 Variational Approach 278

10.3 Free Vibration of Uniform Shafts 282

10.3.1 Natural Frequencies of a Shaft with Both Ends Fixed 283

10.3.2 Natural Frequencies of a Shaft with Both Ends Free 284

10.3.3 Natural Frequencies of a Shaft Fixed at One End and Attached to a Torsional Spring at the Other 285

10.4 Free Vibration Response due to Initial Conditions: Modal Analysis 295

10.5 Forced Vibration of a Uniform Shaft: Modal Analysis 298

10.6 Torsional Vibration of Noncircular Shafts: Saint-Venant’s Theory 301

10.7 Torsional Vibration of Noncircular Shafts, Including Axial Inertia 305

10.8 Torsional Vibration of Noncircular Shafts: The Timoshenko–Gere Theory 306

10.9 Torsional Rigidity of Noncircular Shafts 309

10.10 Prandtl’s Membrane Analogy 314

10.11 Recent Contributions 319

References 320

Problems 321

11 Transverse Vibration of Beams 323

11.1 Introduction 323

11.2 Equation of Motion: The Euler–Bernoulli Theory 323

11.3 Free Vibration Equations 331

11.4 Free Vibration Solution 331

11.5 Frequencies and Mode Shapes of Uniform Beams 332

11.5.1 Beam Simply Supported at Both Ends 333

11.5.2 Beam Fixed at Both Ends 335

11.5.3 Beam Free at Both Ends 336

11.5.4 Beam with One End Fixed and the Other Simply Supported 338

11.5.5 Beam Fixed at One End and Free at the Other 340

11.6 Orthogonality of Normal Modes 345

11.7 Free Vibration Response due to Initial Conditions 347

11.8 Forced Vibration 350

11.9 Response of Beams under Moving Loads 356

11.10 Transverse Vibration of Beams Subjected to Axial Force 358

11.10.1 Derivation of Equations 358

11.10.2 Free Vibration of a Uniform Beam 361

11.11 Vibration of a Rotating Beam 363

11.12 Natural Frequencies of Continuous Beams on Many Supports 365

11.13 Beam on an Elastic Foundation 370

11.13.1 Free Vibration 370

11.13.2 Forced Vibration 372

11.13.3 Beam on an Elastic Foundation Subjected to a Moving Load 373

11.14 Rayleigh’s Theory 375

11.15 Timoshenko’s Theory 377

11.15.1 Equations of Motion 377

11.15.2 Equations for a Uniform Beam 382

11.15.3 Natural Frequencies of Vibration 383

11.16 Coupled Bending–Torsional Vibration of Beams 386

11.16.1 Equations of Motion 387

11.16.2 Natural Frequencies of Vibration 389

11.17 Transform Methods: Free Vibration of an Infinite Beam 391

11.18 Recent Contributions 393

References 395

Problems 396

12 Vibration of Circular Rings and Curved Beams 399

12.1	Introduction	399
12.2	Equations of Motion of a Circular Ring	399
12.2.1	Three-Dimensional Vibrations of a Circular Thin Ring	399
12.2.2	Axial Force and Moments in Terms of Displacements	401
12.2.3	Summary of Equations and Classification of Vibrations	403
12.3	In-Plane Flexural Vibrations of Rings	404
12.3.1	Classical Equations of Motion	404
12.3.2	Equations of Motion that Include Effects of Rotary Inertia and Shear Deformation	405
12.4	Flexural Vibrations at Right Angles to the Plane of a Ring	408
12.4.1	Classical Equations of Motion	408
12.4.2	Equations of Motion that Include Effects of Rotary Inertia and Shear Deformation	409
12.5	Torsional Vibrations	413
12.6	Extensional Vibrations	413
12.7	Vibration of a Curved Beam with Variable Curvature	414
12.7.1	Thin Curved Beam	414
12.7.2	Curved Beam Analysis, Including the Effect of Shear Deformation	420
12.8	Recent Contributions	423
	References	424
	Problems	425

13 Vibration of Membranes 427

13.1	Introduction	427
13.2	Equation of Motion	427
13.2.1	Equilibrium Approach	427
13.2.2	Variational Approach	430

13.3	Wave Solution	432
13.4	Free Vibration of Rectangular Membranes	433
13.4.1	Membrane with Clamped Boundaries	434
13.4.2	Mode Shapes	438
13.5	Forced Vibration of Rectangular Membranes	444
13.5.1	Modal Analysis Approach	444
13.5.2	Fourier Transform Approach	448
13.6	Free Vibration of Circular Membranes	450
13.6.1	Equation of Motion	450
13.6.2	Membrane with a Clamped Boundary	452
13.6.3	Mode Shapes	454
13.7	Forced Vibration of Circular Membranes	454
13.8	Membranes with Irregular Shapes	459
13.9	Partial Circular Membranes	459
13.10	Recent Contributions	460
	Notes	461
	References	462
	Problems	463

14 Transverse Vibration of Plates 465

14.1	Introduction	465
14.2	Equation of Motion: Classical Plate Theory	465
14.2.1	Equilibrium Approach	465
14.2.2	Variational Approach	466
14.3	Boundary Conditions	473
14.4	Free Vibration of Rectangular Plates	479
14.4.1	Solution for a Simply Supported Plate	481
14.4.2	Solution for Plates with Other Boundary Conditions	482
14.5	Forced Vibration of Rectangular Plates	489
14.6	Circular Plates	493
14.6.1	Equation of Motion	493
14.6.2	Transformation of Relations	494

14.6.3	Moment and Force Resultants	496	15.1.3	Distance between Points Anywhere in the Thickness of a Shell before Deformation	555
14.6.4	Boundary Conditions	497	15.1.4	Distance between Points Anywhere in the Thickness of a Shell after Deformation	557
14.7	Free Vibration of Circular Plates	498	15.2	Strain–Displacement Relations	560
14.7.1	Solution for a Clamped Plate	500	15.3	Love’s Approximations	564
14.7.2	Solution for a Plate with a Free Edge	501	15.4	Stress–Strain Relations	570
14.8	Forced Vibration of Circular Plates	503	15.5	Force and Moment Resultants	571
14.8.1	Harmonic Forcing Function	504	15.6	Strain Energy, Kinetic Energy, and Work Done by External Forces	579
14.8.2	General Forcing Function	505	15.6.1	Strain Energy	579
14.9	Effects of Rotary Inertia and Shear Deformation	507	15.6.2	Kinetic Energy	581
14.9.1	Equilibrium Approach	507	15.6.3	Work Done by External Forces	581
14.9.2	Variational Approach	513	15.7	Equations of Motion from Hamilton’s Principle	582
14.9.3	Free Vibration Solution	519	15.7.1	Variation of Kinetic Energy	583
14.9.4	Plate Simply Supported on All Four Edges	521	15.7.2	Variation of Strain Energy	584
14.9.5	Circular Plates	523	15.7.3	Variation of Work Done by External Forces	585
14.9.6	Natural Frequencies of a Clamped Circular Plate	528	15.7.4	Equations of Motion	585
14.10	Plate on an Elastic Foundation	529	15.7.5	Boundary Conditions	587
14.11	Transverse Vibration of Plates Subjected to In-Plane Loads	531	15.8	Circular Cylindrical Shells	590
14.11.1	Equation of Motion	531	15.8.1	Equations of Motion	591
14.11.2	Free Vibration	536	15.8.2	Donnell–Mushtari–Vlasov Theory	592
14.11.3	Solution for a Simply Supported Plate	536	15.8.3	Natural Frequencies of Vibration According to DMV Theory	592
14.12	Vibration of Plates with Variable Thickness	537	15.8.4	Natural Frequencies of Transverse Vibration According to DMV Theory	594
14.12.1	Rectangular Plates	537	15.8.5	Natural Frequencies of Vibration According to Love’s Theory	595
14.12.2	Circular Plates	539	15.9	Equations of Motion of Conical and Spherical Shells	599
14.12.3	Free Vibration Solution	541	15.9.1	Circular Conical Shells	599
14.13	Recent Contributions	543	15.9.2	Spherical Shells	599
References		545	15.10	Effect of Rotary Inertia and Shear Deformation	600
Problems		547	15.10.1	Displacement Components	600
15	Vibration of Shells	549			
15.1	Introduction and Shell Coordinates	549			
15.1.1	Theory of Surfaces	549			
15.1.2	Distance between Points in the Middle Surface before Deformation	550			

15.10.2	Strain–Displacement Relations	601	16.8	Laminated Composite Structures	641
15.10.3	Stress–Strain Relations	602	16.8.1	Computation of the Stiffness Matrices of the Individual Laminas	642
15.10.4	Force and Moment Resultants	602	16.8.2	Strain–Displacement Relations of the Laminate	643
15.10.5	Equations of Motion	603	16.8.3	Computation of Mid-Plane Strains for Each Lamina	649
15.10.6	Boundary Conditions	604	16.8.4	Computation of Mid-Plane Stresses Induced in Each Lamina	650
15.10.7	Vibration of Cylindrical Shells	605	16.8.5	Computation of Mid-Plane Strains and Curvatures	656
15.10.8	Natural Frequencies of Vibration of Cylindrical Shells	606	16.8.6	Computation of Stresses in the Laminate	657
15.10.9	Axisymmetric Modes	609	16.9	Vibration Analysis of Laminated Composite Plates	659
15.11	Recent Contributions	611	16.10	Vibration Analysis of Laminated Composite Beams	663
Notes		612	16.11	Recent Contributions	666
References		612	References		667
Problems		614	Problems		668
16	Vibration of Composite Structures	617	17	Approximate Analytical Methods	671
16.1	Introduction	617	17.1	Introduction	671
16.2	Characterization of a Unidirectional Lamina with Loading Parallel to the Fibers	617	17.2	Rayleigh’s Quotient	672
16.3	Different Types of Material Behavior	619	17.3	Rayleigh’s Method	674
16.4	Constitutive Equations or Stress–Strain Relations	620	17.4	Rayleigh–Ritz Method	685
16.4.1	Anisotropic Materials	620	17.5	Assumed Modes Method	695
16.4.2	Orthotropic Materials	623	17.6	Weighted Residual Methods	697
16.4.3	Isotropic Materials	624	17.7	Galerkin’s Method	698
16.5	Coordinate Transformations for Stresses and Strains	626	17.8	Collocation Method	704
16.5.1	Coordinate Transformation Relations for Stresses in Plane Stress State	626	17.9	Subdomain Method	709
16.5.2	Coordinate Transformation Relations for Strains in Plane Strain State	628	17.10	Least Squares Method	711
16.6	Lamina with Fibers Oriented at an Angle	632	17.11	Recent Contributions	718
16.6.1	Transformation of Stiffness and Compliance Matrices for a Plane Stress Problem	632	References		719
16.6.2	Transformation of Stiffness and Compliance Matrices for an Orthotropic Material in Three Dimensions	633	Problems		721
16.7	Composite Lamina in Plane Stress	634	18	Numerical Methods: Finite Element Method	725
			18.1	Introduction	725
			18.2	Finite Element Procedure	725
			18.2.1	Description of the Method	727
			18.2.2	Shape Functions	728

18.3	Element Matrices of Different Structural Problems	739
18.3.1	Bar Elements for Structures Subjected to Axial Force	739
18.3.2	Beam Elements for Beams Subjected to Bending Moment	743
18.3.3	Constant Strain Triangle (CST) Element for Plates Undergoing In-plane Deformation	747
18.4	Dynamic Response Using the Finite Element Method	753
18.4.1	Uncoupling the Equations of Motion of an Undamped System	754
18.5	Additional and Recent Contributions	760
Note		763

References	763
Problems	765

A Basic Equations of Elasticity **769**

A.1	Stress	769
A.2	Strain–Displacement Relations	769
A.3	Rotations	771
A.4	Stress–Strain Relations	772
A.5	Equations of Motion in Terms of Stresses	774
A.6	Equations of Motion in Terms of Displacements	774

B Laplace and Fourier Transforms **777**

Index **783**

Preface

This book presents the analytical and numerical methods of vibration analysis of continuous structural systems, including strings, bars, shafts, beams, circular rings and curved beams, membranes, plates, shells, and composite structures. The objectives of the book are (1) to make a methodical and comprehensive presentation of the vibration of various types of structural elements, (2) to present the exact analytical, approximate analytical, and approximate numerical methods of analysis, and (3) to present the basic concepts in a simple manner with illustrative examples. Favorable reactions and encouragement from professors, students and other users of the book have provided me with the impetus to prepare this second edition of the book.

The following changes have been made from the first edition:

- Some sections were rewritten for better clarity.
- Some new problems are added.
- The errors noted in the first edition have been corrected.
- Some sections have been expanded. The chapter on “Elastic Wave Propagation” has been deleted.
- A new chapter on “Vibration of Composite Structures” has been added.
- A new chapter entitled, “Approximate Numerical Methods: Finite Element Method,” is added to complement the existing chapter on “Approximate Analytical Methods.”

Continuous structural elements and systems are encountered in many branches of engineering, such as aerospace, architectural, chemical, civil, ocean, and mechanical engineering. The design of many structural and mechanical devices and systems requires an accurate prediction of their vibration and dynamic performance characteristics. The methods presented in the book can be used in these applications. The book is intended to serve as a textbook for a dual-level or first graduate degree-level course on vibrations or structural dynamics. More than enough material is included for a one-semester course. The chapters are made as independent and self-contained as possible so that a course can be taught by selecting appropriate chapters or through equivalent self-study. A successful vibration analysis of continuous structural elements and systems requires a knowledge of mechanics of materials, structural mechanics, ordinary and partial differential equations, matrix methods, variational calculus, and integral equations. Applications of these techniques are presented throughout. The selection, arrangement, and presentation of the material have been made based on the lecture notes for a course taught by the author. The contents of the book permit instructors to emphasize a variety of topics, such as basic mathematical approaches with simple applications, bars and beams, beams and plates, or plates and shells. The book will also be useful as a reference book for practicing engineers, designers, and vibration analysts involved in the dynamic analysis and design of continuous systems.

Organization of the Book

The book is organized into 18 chapters and two appendices. The basic concepts and terminology used in vibration analysis are introduced in Chapter 1. The importance, origin, and a brief history of vibration of continuous systems are presented. The difference between discrete and continuous systems, types of excitations, description of harmonic functions, and basic definitions used in the theory of vibrations and representation of periodic functions in terms of Fourier series and the Fourier integral are discussed. Chapter 2 provides a brief review of the theory and techniques used in the vibration analysis of discrete systems. Free and forced vibration of single- and multidegree-of-freedom systems are outlined. The eigenvalue problem and its role in the modal analysis used in the free and forced vibration analysis of discrete systems are discussed.

Various methods of formulating vibration problems associated with continuous systems are presented in Chapters 3, 4, and 5. The equilibrium approach is presented in Chapter 3. Use of Newton's second law of motion and D'Alembert's principle is outlined, with application to different types of continuous elements. Use of the variational approach in deriving equations of motion and associated boundary conditions is described in Chapter 4. The basic concepts of calculus of variations and their application to extreme value problems are outlined. The variational methods of solid mechanics, including the principles of minimum potential energy, minimum complementary energy, stationary Reissner energy, and Hamilton's principle, are presented. The use of Hamilton's principle in the formulation of continuous systems is illustrated with torsional vibration of a shaft and transverse vibration of a thin beam. The integral equation approach for the formulation of vibration problems is presented in Chapter 5. A brief outline of integral equations and their classification, and the derivation of integral equations, are given together with examples. The solution of integral equations using iterative, Rayleigh–Ritz, Galerkin, collocation, and numerical integration methods is also discussed in this chapter.

The common solution procedure based on eigenvalue and modal analyses for the vibration analysis of continuous systems is outlined in Chapter 6. The orthogonality of eigenfunctions and the role of the expansion theorem in modal analysis are discussed. The forced vibration response of viscously damped systems are also considered in this chapter. Chapter 7 covers the solution of problems of vibration of continuous systems using integral transform methods. Both Laplace and Fourier transform techniques are outlined together with illustrative applications.

The transverse vibration of strings is presented in Chapter 8. This problem finds application in guy wires, electric transmission lines, ropes and belts used in machinery, and the manufacture of thread. The governing equation is derived using equilibrium and variational approaches. The traveling-wave solution and separation of variables solution are outlined. The free and forced vibration of strings are considered in this chapter. The longitudinal vibration of bars is the topic of Chapter 9. Equations of motion based on simple theory are derived using the equilibrium approach as well as Hamilton's principle. The natural frequencies of vibration are determined for bars with different end conditions. Free vibration response due to initial excitation and forced vibration of bars are both presented, as is response using modal analysis. Free and forced vibration of bars using Rayleigh and Bishop theories are also outlined in Chapter 9. The torsional vibration of shafts plays an important role in mechanical transmission of power in prime movers and other high-speed machinery. The torsional vibration of uniform and nonuniform rods with

both circular and noncircular cross-sections is described in Chapter 10. The equations of motion and free and forced vibration of shafts with circular cross-section are discussed using the elementary theory. The Saint-Venant and Timoshenko–Gere theories are considered in deriving the equations of motion of shafts with noncircular cross-sections. Methods of determining the torsional rigidity of noncircular shafts are presented using the Prandtl stress function and Prandtl membrane analogy.

Chapter 11 deals with the transverse vibration of beams. Starting with the equation of motion based on Euler–Bernoulli or thin beam theory, natural frequencies and mode shapes of beams with different boundary conditions are determined. The free vibration response due to initial conditions, forced vibration under fixed and moving loads, response under axial loading, rotating beams, continuous beams, and beams on an elastic foundation are presented using the Euler–Bernoulli theory. The effects of rotary inertia (Rayleigh theory) and rotary inertia and shear deformation (Timoshenko theory) on the transverse vibration of beams are also considered. The coupled bending-torsional vibration of beams is discussed. Finally, the use of transform methods for finding the free and forced vibration problems is illustrated toward the end of Chapter 11. In-plane flexural and coupled twist-bending vibration of circular rings and curved beams is considered in Chapter 12. The equations of motion and free vibration solutions are presented first using a simple theory. Then the effects of rotary inertia and shear deformation are considered. The vibration of rings finds application in the study of the vibration of ring-stiffened shells used in aerospace applications, gears, and stators of electrical machines.

The transverse vibration of membranes is the topic of Chapter 13. Membranes find application in drums and microphone condensers. The equation of motion of membranes is derived using both the equilibrium and variational approaches. The free and forced vibration of rectangular and circular membranes are both discussed in this chapter. Chapter 14 covers the transverse vibration of plates. The equation of motion and the free and forced vibration of both rectangular and circular plates are presented. The vibration of plates subjected to in-plane forces, plates on elastic foundation, and plates with variable thickness is also discussed. Finally, the effect of rotary inertia and shear deformation on the vibration of plates according to Mindlin’s theory is outlined. The vibration of shells is the topic of Chapter 15. First, the theory of surfaces is presented using shell coordinates. Then the strain-displacement relations according to Love’s approximations, stress–strain, and force and moment resultants are given. Then the equations of motion are derived from Hamilton’s principle. The equations of motion of circular cylindrical shells and their natural frequencies are considered using Donnell–Mushtari–Vlasov and Love’s theories. Finally, the effect of rotary inertia and shear deformation on the vibration of shells is considered.

Chapter 16 presents vibration of fiber-reinforced composite structures and structural members. The composite material mechanics of laminates including constitutive relations, stress analysis under in-plane and transverse loads as well as free vibration analysis of rectangular plates and beams are presented in this chapter. Chapter 17 is devoted to the approximate analytical methods useful for vibration analysis. The computational details of the Rayleigh, Rayleigh–Ritz, assumed modes, weighted residual, Galerkin, collocation, subdomain collocation, and least squares methods are presented along with numerical examples. Finally, the numerical methods, based on the finite element method, for the vibration analysis of continuous structural elements and systems

are outlined in Chapter 18. The displacement approach is used in deriving the element stiffness and mass matrices of bar, beam, and linear triangle (constant strain triangle or CST) elements. Numerical examples are presented to illustrate the application of the finite element method for the solution of simple vibration problems.

Appendix A presents the basic equations of elasticity. Laplace and Fourier transform pairs associated with some simple and commonly used functions are summarized in Appendix B.

Acknowledgments

- I would like to thank the many graduate students who offered constructive suggestions when drafts of this book were used as class notes.
- I am grateful to the following people for offering their comments and suggestions, and for pointing out some of the errors in the first edition of the book:

Professor Isaac Elishakoff, Florida Atlantic University, Boca Raton, Florida, USA

Dr. Ronald R. Merritt, NAVAIRWARCEN, China Lake, California, USA

Professor Barbaros Celinkol, University of New Hampshire, Connecticut, USA

Dr. Alastair Graves, University of Oxford, Oxford, UK

Professor John S. Allen III, University of Hawaii, Hawaii, USA

Professor Vadrevu R. Murthy, Syracuse University, New York, USA

Dr. Poorya Hosseini, Iran University of Science and Technology, Tehran, Iran

Professor Orner Civalek, Akdeniz University, Antalya, Turkey

- Finally, I would also like to express my gratitude to my wife, Kamala, for her infinite patience, encouragement, and moral support in completing this book. She shares with me the fun and pain associated with the writing of the book.

S. S. Rao

srao@miami.edu

About the Author

Dr. Singirisu S. Rao is a Professor in the Department of Mechanical and Aerospace Engineering at the University of Miami, Coral Gables, Florida. He was the Chairman of the Mechanical and Aerospace Engineering Department during 1998–2011 at the University of Miami. Prior to that, he was a Professor in the School of Mechanical Engineering at Purdue University, West Lafayette, Indiana; Professor of Mechanical Engineering at San Diego State University, San Diego, California; and the Indian Institute of Technology, Kanpur, India. He was a visiting research scientist for two years at NASA Langley Research Center, Hampton, Virginia.

Professor Rao is the author of eight textbooks entitled: *The Finite Element Method in Engineering*; *Engineering Optimization*; *Mechanical Vibrations*; *Reliability-Based Design*; *Vibration of Continuous Systems*; *Reliability Engineering*; *Applied Numerical Methods for Engineers and Scientists*; *Optimization Methods: Theory and Applications*. He co-edited a three-volume *Encyclopedia of Vibration*. He has edited four volumes of *Proceedings of the ASME Design Automation and Vibration Conferences*. He has published over 200 journal papers in the areas of multiobjective optimization, structural dynamics and vibration, structural control, uncertainty modeling, analysis, design and optimization using probability, fuzzy, interval, evidence and gray system theories. Some 34 Ph.D. students have received their degrees under the supervision of Professor Rao. In addition, 12 post-doctoral researchers and scholars have conducted research under his guidance.

Professor Rao has received numerous awards for his academic and research achievements. He was awarded the *Vepa Krishnamurti Gold Medal for University First Rank* in all the five years of the BE (Bachelor of Engineering) program among students of all branches of engineering in all the Engineering Colleges of Andhra University. He was awarded the *Lazarus Prize for University First Rank* among students of Mechanical Engineering in all the Engineering Colleges of Andhra University. He received the *First Prize* in the James F. Lincoln Design Contest open to all M.S. and Ph.D. students in the USA and Canada for a paper he wrote on automated optimization of aircraft wing structures from his Ph.D. dissertation. He received the *Eliahu I. Jury Award for Excellence in Research* from the College of Engineering, the University of Miami, in 2002; he was awarded the *Distinguished Probabilistic Methods Educator Award* from the Society of Automotive Engineers (SAE) International for *Demonstrated Excellence in Research Contributions in the Application of Probabilistic Methods to Diversified Fields, Including Aircraft Structures, Building Structures, Machine Tools, Air Conditioning and Refrigeration Systems, and Mechanisms* in 1999; he received the American Society of Mechanical Engineers (ASME) *Design Automation Award for Pioneering Contributions to Design Automation, particularly in Multiobjective Optimization, and Uncertainty Modeling, Analysis and Design Using Probability, Fuzzy, Interval, and Evidence*

Theories in 2012; and he was awarded the *ASME Worcester Reed Warner Medal* in 2013 for *Outstanding Contributions to the Permanent Literature of Engineering*, particularly for his *Many Highly Popular Books and Numerous Trendsetting Research Papers*. In 2018, Dr. Rao received the *Albert Nelson Marquis Lifetime Achievement Award* for demonstrated unwavering excellence in the field of Mechanical Engineering.

**This book is accompanied by a book companion site:
www.wiley.com/go/rao/vibration**

Introduction: Basic Concepts and Terminology

1.1 CONCEPT OF VIBRATION

Any repetitive motion is called *vibration* or *oscillation*. The motion of a guitar string, motion felt by passengers in an automobile traveling over a bumpy road, swaying of tall buildings due to wind or earthquake, and motion of an airplane in turbulence are typical examples of vibration. The theory of vibration deals with the study of oscillatory motion of bodies and the associated forces. The oscillatory motion shown in Fig. 1.1(a) is called *harmonic motion* and is denoted as

$$x(t) = X \cos \omega t \quad (1.1)$$

where X is called the *amplitude of motion*, ω is the *frequency of motion*, and t is the time. The motion shown in Fig. 1.1(b) is called *periodic motion*, and that shown in Fig. 1.1(c) is called *nonperiodic* or *transient motion*. The motion indicated in Fig. 1.1(d) is *random* or *long-duration nonperiodic vibration*.

The phenomenon of vibration involves an alternating interchange of potential energy to kinetic energy and kinetic energy to potential energy. Hence, any vibrating system must have a component that stores potential energy and a component that stores kinetic energy. The components storing potential and kinetic energies are called a *spring* or *elastic element* and a *mass* or *inertia element*, respectively. The elastic element stores potential energy and gives it up to the inertia element as kinetic energy, and vice versa, in each cycle of motion. The repetitive motion associated with vibration can be explained through the motion of a mass on a smooth surface, as shown in Fig. 1.2. The mass is connected to a linear spring and is assumed to be in equilibrium or rest at position 1. Let the mass m be given an initial displacement to position 2 and released with zero velocity. At position 2, the spring is in a maximum elongated condition, and hence the potential or strain energy of the spring is a maximum and the kinetic energy of the mass will be zero since the initial velocity is assumed to be zero. Because of the tendency of the spring to return to its unstretched condition, there will be a force that causes the mass m to move to the left. The velocity of the mass will gradually increase as it moves from position 2 to position 1. At position 1, the potential energy of the spring is zero because the deformation of the spring is zero. However, the kinetic energy and hence the velocity of the mass will be maximum at position 1 because of the conservation of energy (assuming no dissipation of energy due to damping or friction). Since the velocity

2 Introduction: Basic Concepts and Terminology

is maximum at position 1, the mass will continue to move to the left, but against the resisting force due to compression of the spring. As the mass moves from position 1 to the left, its velocity will gradually decrease until it reaches a value of zero at position 3. At position 3, the velocity and hence the kinetic energy of the mass will be zero and the

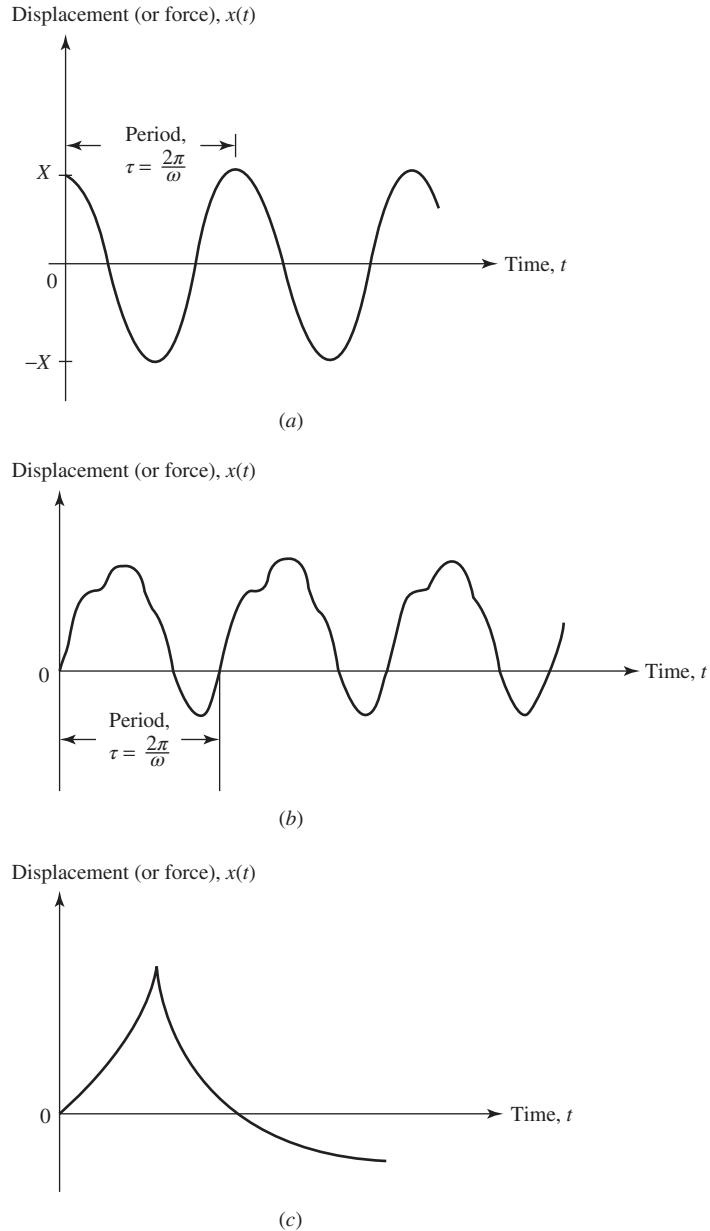
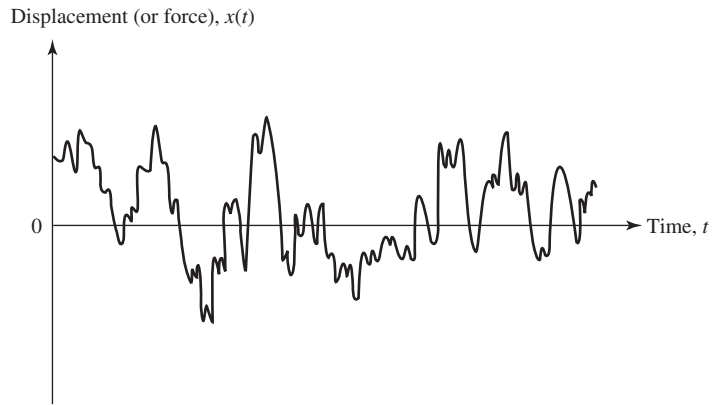


Figure 1.1 Types of displacements (or forces): (a) periodic, simple harmonic; (b) periodic, nonharmonic; (c) nonperiodic, transient; (d) nonperiodic, random.



(d)

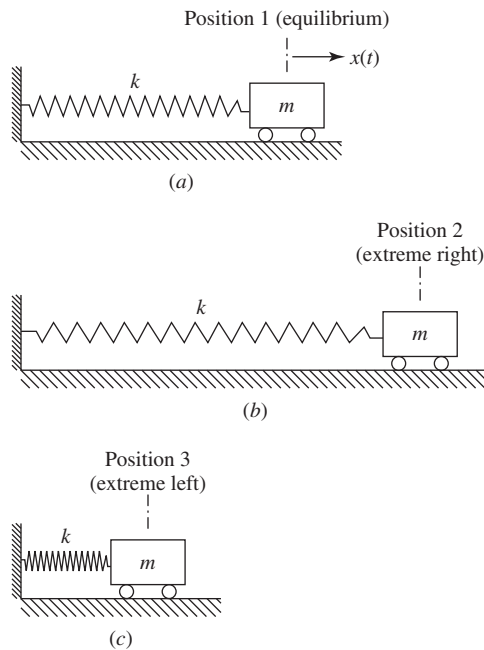
Figure 1.1 (continued)

Figure 1.2 Vibratory motion of a spring–mass system: (a) system in equilibrium (spring undeformed); (b) system in extreme right position (spring stretched); (c) system in extreme left position (spring compressed).

deflection (compression) and hence the potential energy of the spring will be maximum. Again, because of the tendency of the spring to return to its uncompressed condition, there will be a force that causes the mass m to move to the right from position 3. The velocity of the mass will increase gradually as it moves from position 3 to position 1.

At position 1, all the potential energy of the spring has been converted to the kinetic energy of the mass, and hence the velocity of the mass will be maximum. Thus, the mass continues to move to the right against increasing spring resistance until it reaches position 2 with zero velocity. This completes one cycle of motion of the mass, and the process repeats; thus, the mass will have oscillatory motion.

The initial excitation to a vibrating system can be in the form of initial displacement and/or initial velocity of the mass element(s). This amounts to imparting potential and/or kinetic energy to the system. The initial excitation sets the system into oscillatory motion, which can be called *free vibration*. During free vibration, there will be an exchange between the potential and the kinetic energies. If the system is conservative, the sum of the potential energy and the kinetic energy will be a constant at any instant. Thus, the system continues to vibrate forever, at least in theory. In practice, there will be some damping or friction due to the surrounding medium (e.g. air), which will cause a loss of some energy during motion. This causes the total energy of the system to diminish continuously until it reaches a value of zero, at which point the motion stops. If the system is given only an initial excitation, the resulting oscillatory motion eventually will come to rest for all practical systems, and hence the initial excitation is called *transient excitation* and the resulting motion is called *transient motion*. If the vibration of the system is to be maintained in a steady state, an external source must continuously replace the energy dissipated due to damping.

1.2 IMPORTANCE OF VIBRATION

Any body that has mass and elasticity is capable of oscillatory motion. In fact, most human activities, including hearing, seeing, talking, walking, and breathing, also involve oscillatory motion. Hearing involves vibration of the eardrum, seeing is associated with the vibratory motion of light waves, talking requires oscillations of the larynx (tongue), walking involves oscillatory motion of legs and hands, and breathing is based on the periodic motion of the lungs. In engineering, an understanding of the vibratory behavior of mechanical and structural systems is important for the safe design, construction, and operation of a variety of machines and structures.

The failure of most mechanical and structural elements and systems can be associated with vibration. For example, the blade and disk failures in steam and gas turbines and structural failures in aircraft are usually associated with vibration and the resulting fatigue. Vibration in machines leads to rapid wear of parts, such as gears and bearings, to loosening of fasteners, such as nuts and bolts, to poor surface finish during metal cutting, and excessive noise. Excessive vibration in machines causes not only the failure of components and systems but also annoyance to humans. For example, imbalance in diesel engines can cause ground waves powerful enough to create a nuisance in urban areas. Supersonic aircraft create sonic booms that shatter doors and windows. Several spectacular failures of bridges, buildings, and dams are associated with wind-induced vibration, as well as oscillatory ground motion during earthquakes.

In some engineering applications, vibrations serve a useful purpose. For example, in vibratory conveyors, sieves, hoppers, compactors, dentist drills, electric toothbrushes, washing machines, clocks, electric massaging units, pile drivers, vibratory testing of materials, vibratory finishing processes, and materials processing operations, such as casting and forging, vibration is used to improve the efficiency and quality of the process.