ICME-13 Monographs

Peter Liljedahl
Manuel Santos-Trigo *Editors*

Mathematical Problem Solving

Current Themes, Trends, and Research

ICME 13
Hamburg 2016

Springer

ICME-13 Monographs

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ISSN 2520-8322 ISSN 2520-8330 (electronic)
ICME-13 Monographs
ISBN 978-3-030-10471-9 ISBN 978-3-030-10472-6 (eBook)
https://doi.org/10.1007/978-3-030-10472-6

Library of Congress Control Number: 2018964929

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

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Introduction

Mathematical problem solving has long been seen as an important aspect of mathematics, the teaching of mathematics, and the learning of mathematics. It has infused mathematics curricula around the world with calls for the teaching of problem solving as well as the teaching of mathematics through problem solving. And as such, it has been of interest to mathematics education researchers for as long as our field has existed. In July of 2016, over 80 researchers gathered at ICME-13 to expand on this important topic through the presentation of research, critical reflection, and discourse. The coming together of researchers within TSG 19: Problem Solving in Mathematics Education resulted in the presentation of 13 extended papers, 28 oral communications, and 18 posters organized on a wide variety of topic focused on, and stemming from, research into the problem solving. From the richness of the interaction over those 7 days in Hamburg emerged this book—consisting of the extended versions of 15 invited papers on a wide variety of topics, results, and perspectives on mathematical problem solving.

In Part I "Problem Solving Heuristics", Tjoe revisited Pólya's framework, characterizing problem solving phases that appear in individuals' attempts to solve problems and focuses on looking backstage as an approach to encourage high school students to look for different ways to solve problems. Using a mathematics test as well as interviews, he explores and discusses the extent to which students were familiar, fluent and flexible in using multiple solution methods. An interesting finding in Tjoe's study was that students showed little interest in finding other solution methods in addition to those that they reported in the test. Tjoe recommends that students explicitly discuss throughout instruction the importance of finding multiple solution methods to approach any type of problems and teachers should value and encourage their students to looking back and find different approaches to solve the same problem.

Likewise, Maciejewski's contribution invites us to reconceptualise the mathematical problem solving processes to include, what he calls, mathematical foresight and the importance of future thinking when approaching a problem solving situation. Maciejewski grounds these ideas in the literature, where he illustrates the relationships between mathematicians' work, problem solving (Schoenfeld, Pólya), and select psychological work and posits mathematical foresight as a possible lens to analyse students' future-oriented thinking and actions to deal with mathematical situations. In contrasting the mathematicians' foresight models and that of students approaches, Maciejewski reports that while mathematicians see two interrelated components—the sphere of finding the solution and the resolution process or trajectory—students often only see one of these components. That is, students either see a possible solution to a task without seeing the process or path necessary to reach that solution, or they see the beginnings of a trajectory without seeing where this will lead them.

Part II "Problem Solving and Technology" begins with Carreira and Jacinto who investigate how a middle-grade student engages in a web-based mathematics competition. Drawing on the notion of humans-with-media they emphasize the interaction between the solver and the tool in problem solving activities. To document the student's processes, they use a blending framework that accounts for the problem solving phases (read, analysis, exploration, planning and implementation, and verification) as well as the explicit students' use of technology affordances throughout all phases. Based on the analysis of one case the authors report that the use of technology affords the student the possibility to engage in different forms of reasoning, including exploration, manipulation, observation, conjecture, formulation, explanation, and validation.

Similarly, Santos-Trigo also presents a framework for characterizing reasoning that a problem solver might develop as a result of using digital technology to solve mathematical problems. In so doing, he illustrates how the affordances of technology can shape the reconstruction of figures that often are embedded in problem statements, the transformation of textbook or routine problems into an investigation task, the graphical representation and exploration of a variation phenomenon or problem, and the construction and exploration of dynamic configurations to formulate conjectures and ways to support them. Santos-Trigo uses these four problem types to discuss the importance of building dynamic models of problems, the role of controlled movement of certain objects, the search and exploration of loci of points to analyse some variation phenomena, and the use of sliders to visualize patterns and relationships.

Finally, Amado, Carreira, and Nobre look at ways in which the use of spreadsheets provides affordances for students to represent and solve word problems. The chapter begins by addressing both the difficulties that students experience with algebraic representation and the affordances of spreadsheets to make sense and represent key information associated with problem statements. The cases presented in the chapter illustrate different models that students used to solve a word problem. They conclude that the use of spreadsheets allowed middle school students to think of a variety of approaches that involves formulas and tables to identify and explore relations between variables.

Part III "Inquiry and Problem Posing in Mathematics Education" includes two chapters and begins with Hersant and Choquet's use of inquiry-based approaches to engage elementary students in problem solving activities. The chapter includes a review of how inquiry-based learning and teaching has been interpreted and used in both science and mathematics in Europe and elsewhere. They argue that this approach can be characterized as a student-centred way of teaching in which students are encouraged to formulate questions as a way to delve into concepts and solve problems. In the chapter, they present two case studies, framed through an inquiry-based approach, that encourages elementary students to pose and discuss questions during the process of solving specific tasks. Through these cases the authors point out that the role of the teacher in such an environment will either foster or limit what students can achieve in this type of approach.

The second paper, by Malaspina, Torres, and Rubio, presents results from a study that looks closely at problem posing activities during a workshop with 15 high school teachers. The participants were asked to pose a problem at two different stages of the workshop (pre-problem and post-problem) and these were used to analyse the teachers' didactic and problem posing competencies. The authors relied on what they call an onto-semiotic framework to analyse the posed problems via epistemic and cognitive configurations. This analysis led the authors to characterize the participants' didactic competencies by contrasting the mathematical structures between the given problem and those they proposed and discussed. The authors also report on the difficulties participants experienced during the development of the problem-posing sessions.

Part IV "Assessment of and Through Problem Solving" is comprised of four chapters beginning with Loh and Lee's study on grade 7 students use of metacognitive strategies while solving mathematical tasks. The research design involves the use of both quantitative and qualitative methods to gather information about the participants' metacognitive behaviours. Results identify different students' frequency use of metacognitive strategies with an emphasis on surface strategies. However, the analysis of the students' written self-report and interview led the authors to identify students' robust use of metacognitive strategies. The authors suggest that the use of both quantitative and qualitative instruments provided important insights into the students' metacognitive behaviours.

Chanudet's chapter looks at the use of an assessment tool in a problem solving course that fosters an inquiry approach to learn mathematics. It includes a review of what an inquiry and problem solving approach might entail and the importance of designing a tool to assess problem solving competencies. The first part of the study focuses on the nature of the tasks that participant teachers use to assess students' problem solving. The second part of the study delves deeper into assessment and involves first working collaboratively with teachers to design an assessment tool that involved both summative and formative assessment, and then testing this tool through an exploratory study into one of these teacher's practice. Results indicate that this teacher relied on classroom conversations to assess her students throughout the course.

Meanwhile, Di Martino and Signorini look at assessment of problem solving through the use of standardized assessments such as PISA or national tests. The authors discuss several cases in which students' answers to specific test items, although well-supported within the students' reasoning, do not necessarily lead them to choose the *right* answer. The authors also showed that the time limitation to complete the test becomes an obstacle for students to show what is behind their answers and they argue that teachers and researchers should pay attention to the students' process involved in working on these types of questions.

The final chapter in this part, by Mendoza Álvarez, Rhoads, and Campbell, is centred on a quest to develop an efficient tool to assess the mathematical problem solving abilities necessary for a student to leverage pre-requisite knowledge to be successful in the STEM fields. Grounded in literature, the authors develop and test Likert items that link a student's mathematical problem solving capacity to five key problem solving domains (sense-making, representing and connecting, reviewing, justifying, and challenge) and do not require content knowledge beyond secondary school level algebra.

Part V "The Problem Solving Environment" begins with Koichu and Keller's report on the development of online forums to engage students in problem solving activities. In their chapter, they include examples of problems, the interaction among three communities (two classroom communities and the research group), and a narrative on how these communities behave and interact throughout the development of the forums. The authors characterize how online problem solving discussions became a routine practice in one community in which its members valued and engaged in meaningful discussions beyond classroom problem solving activities. The second community did not activate the use of the forum; but the interaction of this community with the research group led the participants to enhance their peer's interaction within the classroom. The authors also argued that all three communities evolved, and they characterize stages on how this evolution took place including the identification of boundaries that appear during the community interactions.

Meanwhile, Liljedahl's chapter aims to characterize what a thinking classroom involves in terms of the type of tasks used to engage students in problem solving activities, the way teachers give and structure the tasks development, how the students work in groups including work surfaces (vertical non-permanent surfaces), how questions are answered, and the assessment of students' problem solving performances. Throughout the chapter, the author describes a series of studies that led him to identify and categorize students learning behaviours in different classroom environments. He proposes an inventory of classroom norms and practices to examine how classroom activities are developed; indeed, the inventory is expressed in terms of 11 questions that researchers/teachers can use to analyse not only what and how students learn, but also the quality of that learning. Those questions include: What type of tasks are used, and when and how they are used? Where, and on what surfaces, do students work on tasks? How the room is organized, both in general and when students work on tasks? When and how is assessment carried out, both in general and when students work on tasks? etc. Addressing these questions provides useful information for researchers/teachers to construct powerful and cohesive learning environments that foster students' thinking as well as powerful and cohesive professional development environments for teachers to explore and question their practice.

In the same part, Felmer, Perdomo-Díaz and Reyes present initial results from a research and professional development program (Activating Problem Solving in Classrooms, known as ARPA in Spanish) that aims to introduce a problem solving approach into regular teachers' instructional practices. The chapter provides a context to explain the project rationale to focus on problem solving approaches to help teachers improve their practices and their students' mathematical competencies. The program includes a series of workshops in which teachers have an opportunity to work on problems and to think of ways to implement them into regular classrooms. After 3 years of implementation, the authors report that teachers have begun to question their practices, to change their beliefs about teaching and ways to introduce a problem solving approach in their classrooms.

Finally, Ho, Yap, Tay, Leong, Toh, Quek, Toh, and Dindyal present and discuss results from a project whose aim is to implement a mathematical problem solving approach in all classrooms in Singapore. They identify the factors that contribute to, and explain, the success or failure of a school to implement the project. To this end, they focus on analysing factors such as programs and school levels in terms of outcomes, inputs, resources, constrains, strategies, and feedback and evaluations. The authors argue that the sustainability of introducing and maintaining a problem solving approach in schools can be achieved through the infusion and diffusion of a school culture that fosters integration between curriculum and school problem solving practices.

Manuel Santos Trigo
Peter Liljedahl

Part I
Problem Solving Heuristics

Chapter 1
"Looking Back" to Solve Differently: Familiarity, Fluency, and Flexibility

Hartono Tjoe

Problem solving clearly plays an important role in mathematics (Duncker, 1945; Kaiser & Schwarz, 2006; Lesh, 1985; Mason, Burton, & Stacey, 1982), and its role in mathematics education is equally prominent (Common Core State Standards Initiative, 2010; NCTM, 2000). Apart from solving unsolved problems, the professional practice of research mathematicians also often involves solving, through different approaches, problems that have been previously solved (Davis & Hersh, 1981; Liljedahl & Sriraman, 2006; Thurston, 1994). A comparable pursuit of multiple solutions in the classroom experience of K-12 students, however, has seldom been researched (Santos-Trigo, 1996; Silver, Ghousseini, Gosen, Charalambous, & Font Strawhun, 2005).

The present study focuses on the second part of Pólya's (1945) fourth step of problem solving, namely, "looking back" in order to solve a problem differently. In particular, it examines the extent to which the practice of "looking back" to solve differently has been integrated into mathematics instruction in the United States, and thus, whether this practice is familiar to American students. Mathematical interconnectedness was analyzed through student fluency and flexibility in supplying different solution methods. An assessment involving multiple mathematics concepts was utilized to explore the relationship between students' mathematical understanding and their awareness of mathematical interconnections.

The following three research questions guided the present study: (a) Based on a mathematics problem-solving test and interview results, to what extent were students familiar with the practice of problem solving using multiple solution methods? (b) Given their familiarity or unfamiliarity with the practice of solving problems using multiple solution methods, to what extent were the students fluent in understanding, reproducing, and identifying a particular mathematics topic related to the various solution methods? and (c) Given their fluency or non-fluency in such a range of

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P. Liljedahl and M. Santos-Trigo (eds.), *Mathematical Problem Solving*,
ICME-13 Monographs, https://doi.org/10.1007/978-3-030-10472-6_1

mathematics topics, to what extent were the students flexible in making mathematical connections among the different solution methods and in adjusting to these different methods?

In the context of the present study, “familiarity” refers to the quality of a topic being well-known or generally recognizable based on prior mathematical experience; “fluency” refers to the ability to formulate, demonstrate, and communicate strong mathematical ideas effortlessly and articulately; and “flexibility” refers to the willingness to forgo one’s familiar solution method in favor of a novel or unfamiliar method either generated by oneself or presented by others (Leikin, 2009; Silver, 1997; Sriraman, 2009; Star & Rittle-Johnson, 2008; Torrance, 1966).

1.1 Conceptual Framework

1.1.1 Problem-Solving Process

Literature in mathematics education indicates that problem solving was one of the most highly researched topics in the field for several decades (Kilpatrick, 1985; Lester, 1994; Schoenfeld, 1985). More recently, many issues regarding problem solving have been discussed in connection with other emerging topics in mathematics education (Felmer, Pehkonen, & Kilpatrick, 2016; Schoenfeld, 2008; Singer, Ellerton, & Cai, 2015).

The important place of problem solving in school mathematics is natural given its strategic role in teaching and learning mathematics (Liljedahl, 2016; Owen & Sweller, 1985). A number of pedagogical approaches have been proposed to incorporate the problem-solving experience into everyday mathematics classrooms (Pressley, Forrest-Pressley, Elliott-Faust, & Miller, 1985). The topic draws considerable interest and attention not only from school teachers and educators, but also from research mathematicians.

Pólya (1945) enumerated four distinct steps in the process of mathematical problem solving: (a) understanding the problem, (b) devising a plan, (c) carrying out the plan, and (d) “looking back.”

The first step, understanding the problem, begins with the identification of what is posed by the problem; that is, problem solvers must determine the nature of the question being asked (Michener, 1978). To this end, it is important to recognize all available data in the problem, and to determine and differentiate necessary, sufficient, relevant, redundant, and contradictory conditions amongst the given information. Additional facts may be further derived from drawing appropriate figures or introducing suitable notation.

The second step is devising a plan. A well-devised plan makes the most straightforward connection between the data and the unknowns. In addition, it builds on comparable problem-solving experiences from the past. It is therefore important to consider analogous problems, some of which may vary in appearance from the problem under consideration in several ways, from the structure of the data they present to the construction of the unknowns (Gick & Holyoak, 1980). Particular techniques and established results employed in the course of past problem solving may inform the restatement of problems presently at hand.

Pólya discussed many heuristic strategies for solving mathematics problems (Schoenfeld, 1979a, b), including drawing pictures, solving simpler, analogous problems, considering special cases to find general patterns, working backward, and adopting different points of view.

The third step is to carry out the plan. It is critical to execute each step of the plan carefully (Garofalo & Lester, 1985), and to verify that each step follows logically.

The fourth step is “looking back.” Arrival at a solution does not necessarily mean that the process of problem solving has ended. In the first part of Pólya’s fourth step, problem solvers examine the obtained solution of a problem by checking the argument along the way, ascertaining in particular an absence of errors in reasoning (Silver, Leung, & Cai, 1995).

In the second part of Pólya’s fourth step, problem solvers review the solution to find alternative approaches to solving the same problem. Deriving the obtained result through the use of alternative approaches can be valuable for future problem solving (Silver et al., 2005).

Pólya devoted much time to illustrating his model of problem solving with concrete exemplars. The model, as a result, gained many enthusiasts from a large audience. He convinced his readers that the problem-solving processes he analyzed were not only accessible to research mathematicians, but could also be utilized by broader audiences.

1.1.2 Problem Solving Using Multiple Solution Methods

Many researchers in mathematics education have comprehensively and systematically examined Pólya’s model. A review of prior literature reveals, however, that much of this attention has focused specifically on the first three steps. In fact, many researchers were particularly attracted by the second step, devising a plan (Schoenfeld, 1985)—and understandably so, as this is what most classroom practitioners expect their students to develop and implement while learning mathematics. This was, after all, the principal reason the model was constructed in the first place. Nonetheless, Pólya’s (1945) model of problem solving does not end at the third step.

Only a limited number of studies in mathematics education have examined students’ use of alternative approaches in problem solving. Despite its importance, Pólya’s fourth step has received less attention in mathematics education community than the other three steps from the empirical point of view (Schoenfeld, 1985; Silver, 1985; Tjoe, 2014).

Some researchers in this field have been particularly successful in exploring the use of mathematical tasks requiring students to solve a single problem via several different approaches. These researchers investigated the presence of multiple mathematics concepts through the solution of non-standard problems via different but related solution methods (Leikin & Lev, 2007), through the transformation of standard problems into non-standard problems (Santos-Trigo, 1998), and through the recognition of specific attributes within standard problems (Tjoe & de la Torre, 2014).

An understanding of interconnections among different mathematical concepts is recognized by many mathematicians as a driving force in the appreciation of mathematical beauty (Borwein, Liljedahl, & Zhai, 2014; Davis & Hersh, 1981; Hadamard, 1945; Poincare, 1946). In turn, mathematics teachers, educators, and practitioners in general agree that knowing how and why mathematics works—and in understanding in particular the connections among many different solutions to a problem as opposed to superficial memorization of solution procedures—should be viewed as fundamental to students' development of mathematical reasoning (Eisenhart et al., 1993; Hiebert & Lefevre, 1986; Rittle-Johnson & Siegler, 1998).

Clearly, the fourth step of Pólya's (1945) problem-solving process plays a critical role in prompting the discovery of a variety of different solution methods. In particular, the idea of "looking back" to solve differently is closely related to the qualities of familiarity, fluency, and flexibility.

In the absence of familiarity with problem solving using multiple solution methods, problem solvers may be less inclined to reflect on the solution process and to seek more than a single solution method. Without considerable fluency in a range of mathematical subjects, "looking back" to solve differently is far less likely to be effective or successful. Similarly, lack of flexibility in switching between different solution methods may lead to an unfavorable attitude toward finding alternative approaches to solve the same problem. The analysis of familiarity, fluency and flexibility might therefore be considered necessary for the fourth step of Pólya's (1945) problem-solving process to materialize in an optimal manner.

Many earlier discussions of problem solving via multiple solution methods focus on a variety of potential benefits of the practice. Silver et al. (2005), for instance, maintain that students "can learn more from solving one problem in many different ways than [they] can from solving many different problems, each in only one way" (p. 288). They particularly advise students interested in mathematics to obtain more experience in solving problems via multiple solution methods. Silver and colleagues regard such experience as having "the potential advantage of providing students with access to a range of representations and solution strategies in a particular instance that can be useful in future problem-solving encounters" (p. 288). They also consider the use of multiple solution methods in order to "facilitate connection of a problem at hand to different elements of knowledge with which a student may be familiar, thereby strengthening networks of related ideas" (p. 288).

Leikin and Levav-Waynberg (2007) were interested in surveying teachers for their thoughts about alternative solution methods in problem solving. They interviewed several high school mathematics teachers in a comparative study of teachers' beliefs. Their findings reveal positive attitudes toward the use of multiple solution methods in problem solving. Most teachers in the study by Leikin and Levav-Waynberg considered the use of these methods beneficial to fostering student success in problem solving. They believed that working with many different approaches accommodated the learning experiences of students who had pronounced preferences in learning style. In turn, they reasoned that struggling students could benefit from the presentation of various approaches, especially with regard to problems having a high level of difficulty. Such presentations should be applied to problems with complex approaches requiring sophisticated mathematical knowledge yet which are solvable using elementary techniques. As one teacher remarked, when presented with different solution methods, students should be able to choose the solution method "that is easiest [for them] to understand" (Leikin & Levav-Waynberg, 2007, p. 363).

Other teachers in the study by Leikin and Levav-Waynberg (2007) valued in particular the students' development of mathematical thinking and reasoning as integral to the establishment of a solid foundation for future academic success. Several teachers acknowledged the significance of students' awareness of connections between mathematics topics. Mathematics should be viewed "as a whole"—that is, as a collection of connected, rather than disjoint, ideas (Leikin & Levav-Waynberg, 2007, p. 363). In general, Leikin and Levav-Waynberg (2007) concluded that these teachers evinced favorable views of the use of multiple solution methods.

In addition to mathematics education researchers, a number of cognitive psychologists interested in educational psychology with applications to learning and cognition have also endorsed employing multiple solution strategies in problem solving. Collins, Brown, and Newman (1989) discuss the use of multiple perspectives by means of their "cognitive apprenticeship" approach to instructional method. In their model, students' learning processes were considered in light of five teaching methods: modeling, coaching, scaffolding, reflection, and articulation. The role of the teacher in supporting the students' learning experience gradually decreased as the students felt more confident in communicating their understanding of the problem-solving solutions.

Collins et al. (1989) argue that the more approaches and perspectives students explore, the more effective the implementation of this cognitive-based learning method will be. Some benefits of this method they found included improved "apprenticeship" through the use of real-world activities and assessments (Collins et al., 1989). The method also enhanced students' motivation and engagement in overall learning (Collins, 1991), greater transfer and retention rates (Resnick, 1989), and higher-order reasoning (Hogan & Tudge, 1999).

Spiro, Feltovich, Jacobson, and Coulson originated the "cognitive flexibility theory" (Spiro, Feltovich, Jacobson, & Coulson, 1991). Spiro et al. (1991) maintain that restructuring knowledge through changes in approach makes learning new concepts more effective. Such adaptations are based on the notion that the human mind can be trained to be flexible enough to accommodate different situations (Spiro & Jehng, 1990). New information and experiences are processed via the transfer of knowledge and skills, and further constructed to develop new meaning and understanding. In other words, Spiro and Jehng (1990) assert that learning through different perspectives associated with different situations deepens students' understanding and learning experiences.

Tabachneck, Koedinger, and Nathan (1994) also recognized the purpose of adopting many different solution methods in problem solving. They argue that on its own, each solution method entails certain disadvantages and weaknesses. In order to overcome these, Tabachneck et al. (1994) recommend students employ a combination of different solution methods instead of relying on only one. More specifically, they emphasize that students could benefit from employing this learning style in mathematical problem solving.

In addition to advocating the use of many different solution methods, many cognitive psychologists encourage teaching a coherent interrelation among those solution methods (Bodemer, Plötzner, Feuerlein, & Spada, 2004; de Jong et al., 1998; Skemp, 1987; Van Someren, Boshuizen, de Jong, & Reimann, 1998). Equally important, Reeves and Weisberg (1994) suggest showing students many analogical problems or examples concurrently.

On the whole, cognitive psychologists have taken a positive stance on problem solving using multiple solution methods, as have mathematics education researchers. Despite the benefits of implementing this learning style, some of these discussions were not without uncertainties.

A few teachers in the study by Silver et al. (2005) discussed issues and concerns in teaching problem solving via multiple approaches. They included the constraints of instructional time, limitations involving instructors' perceptions of student abilities, the selection and presentation order of solution methods, and uncertainty about the advantages and disadvantages of reviewing incorrect approaches to problems.

Some teachers in the study by Leikin and Levav-Waynberg (2007) showed genuine concern about students' learning experiences. They worried that students might confuse "whether the object of study is to solve the problem, the fact that there is more than one solution to the problem, or the principles behind the solutions and the connections between them" (p. 366).

Despite these constraints and concerns, many researchers still felt firmly confident in their recommendations for teaching problem solving using many different approaches. Silver et al. (2005) nonetheless point out the possibility that teachers may possess inadequate mathematical knowledge to effectively employ this instruction technique. They hypothesized that this might constitute a significant limiting factor in its overall success as an instructional strategy.

Several empirical findings have been presented to demonstrate students' learning outcomes as a result of approaches teaching multiple solution strategies. Große and Renkl (2006) examined the effects of teaching problem solving using many different solution methods presented in the form of worked-out examples. Their experiment involved combinatorics lessons for university-level students. The authors found that exposing students to the presentation of many different solution methods did in fact improve their procedural and conceptual understanding.

Rittle-Johnson and Star (2007) analyzed the effect of comparing many different solution methods upon students' learning experience. Their experiment involved algebra lessons for seventh grade students. The researchers found that exposing students to the practice of comparing and contrasting different solution methods in a simultaneous manner improved their procedural understanding more than their conceptual understanding.

In general, experimental studies, along with their pedagogical recommendations described earlier, showed that the benefits and potential opportunities of problem solving using multiple solution methods outweigh the concerns and challenges associated with the actual teaching of these methods. The present study examines the extent to which the practice of problem solving using multiple solution methods might be effectively presented in an existing classroom routine.

1.2 Methodology

The present study involved nine students (4 female, 5 male, aged 16–18, in grades 11–12) in a highly regarded urban northeastern American high school which has graduated notable scientists in the past. It is one of the highest ranking among public high schools with an academic specialization in mathematics and sciences (Vogeli, 2015).

The nine students who participated in the present study received strong recommendations from their mathematics teachers. These students were carefully selected to be part of the present study with an expectation that they might be significantly more capable than their peers of not only solving the problems involved in the study, but also of supplying more than one solution method for each problem.

At the time of the study, these students were enrolled in an Advanced Placement (AP) Calculus course, a university-level calculus course with topics in differential and integral calculus typically taken by high school students in the United States seeking university credit or placement in a university calculus course. These students volunteered to take a paper-and-pencil test consisting of three non-standard mathematics problems (Problems 1, 2, and 3; Tjoe, 2015). The researcher identified beforehand, as part of the careful selection process of the problems included in the test, 15 different solution methods associated with the three non-standard mathematics problems: four solution methods for Problem 1 (P1S1, P1S2, P1S3, and P1S4), eight solution methods for Problem 2 (P2S1, P2S2, P2S3, P2S4, P2S5, P2S6, P2S7, P2S8), and three solution methods for Problem 3 (P3S1, P3S2, P3S3; Tjoe, 2015).

On the surface, these three problems appear to depend only on the three most common elementary mathematics topics, namely arithmetic, algebra, and geometry. At a deeper level, they incorporate multiple access points to more advanced mathematics topics such as trigonometry, calculus, linear algebra, and real analysis. Overall, the three problems were carefully selected to allow accessibility for average students in a typical American high school that has adopted the national curriculum in mathematics (Common Core State Standards Initiative, 2010; NCTM, 2000). For instance, the approaches involved in P1S4, P2S1, and P3S1 can be readily comprehended by students in regular high school arithmetic, algebra, and geometry courses, respectively, and not exclusively by more advanced students in the specialized high schools as described by Vogeli (2015).

The nine students were instructed to creatively solve the three problems using as many different solution methods as they could without the aid of a calculator and without any time limitations, and they were specifically instructed to solve the problems using multiple methods. While this methodology was deliberately and specifically adopted in order to assess students' familiarity with the practice of problem solving using multiple solution methods (Leikin & Lev, 2007), it was well noted that it departed from the normal assessment procedure with respect to the role of didactical contract (Hersant, 2011).

After their written responses were checked for accuracy, the students were presented with their work and the 15 solution methods, and were interviewed individually. A video recorder was utilized to capture the students' problem solving processes as presented in written responses as well as during the individual interviews.

Students' solution methods were evaluated on the basis of a simple acceptability scoring system. An acceptability score of 1 indicated that a student successfully supplied a correct answer by using an approach in a logical manner to solve the problem; otherwise, an acceptability score of 0 was given. Students' solution methods were also classified based upon the list of 15 different solution methods identified by the researcher beforehand.

Follow-up interviews were conducted with the nine students who had previously taken the paper-and-pencil test. The interview was designed to elicit the students' explanations for their particular solution methods. In addition to questions about their mathematical background, each of the nine students was asked (a) whether they were familiar with the practice of "looking back" to solve differently, (b) whether they understood each of the 15 solution methods, (c) whether they had learned the content involved in each of the 15 solution methods in their previous mathematics coursework, and (d) whether in the future they might solve similar problems to the three tested using any of the 15 solution methods they had considered in reviewing the test.

The first question assessed students' familiarity with the practice of "looking back" to solve problems differently. The second and third questions assessed the students' fluency in diverse mathematical knowledge. The fourth question assessed students' flexibility in accepting solution methods other than their own. In analyzing these four questions, the researcher coded the nine students' responses with the following scoring system: a score of 1 indicating familiarity with the practice of "looking back" to solve differently, understanding of a particular solution method, recognition of the relation of a particular solution method to mathematics courses previously taken, and likelihood of supplying a different solution method in the future; otherwise, a score of 0 was assessed.

The results of the test and the student interviews were analyzed to detect similarities or differences in the justifications provided by the other students regarding their supply of particular solution methods. The responses to the interview questions were analyzed to determine the students' familiarity, fluency and flexibility regarding problem solving using many different solutions.

1.3 Findings

Nine students participated in the present study. Eight of these students were enrolled in Grade 12, and one was enrolled in Grade 11. The nine students reported an average SAT Math Section score of 754, SAT Subject-Math I score of 750, and SAT Subject-Math II score of 790. The national average scores of SAT Math Section, SAT Subject-Math Level I, and SAT Subject-Math Level II were 516, 605, and 649, all of which were out of a possible maximum score of 800 (*The College Board*, 2011a). One student reported an American Mathematics Contest 12 (AMC-12) score of 94.5. The SAT is a standardized test that universities in the United States generally use in admission criteria to measure college readiness of prospective students (*The College Board*, 2011b), whereas the AMC is a series of mathematics competitions generally used to determine participants' eligibility for the International Mathematical Olympiad (*Mathematical Association of America*, 2011).

Because they were all recruited from the same high school and because the school utilized a relatively uniform mathematics curriculum (with the exception of honors courses), all of the nine students were found to have received formal courses in algebra, geometry, trigonometry, pre-calculus, calculus, and linear algebra throughout their mathematics education in this particular, specialized high school.

Although they were reminded several times of the unlimited time to solve the problems using numerous methods, the students generally finished the test in less than one hour. Six, three, and seven students successfully solved Problems 1, 2, and 3, respectively.

Table 1.1 summarizes the mathematical background of the nine students as well as the problems each successfully solved. (If a student did not report taking the SAT Math Section, SAT Subject-Math I, SAT Subject-Math II, or AMC-12, "n/a" is recorded in Table 1.1 to indicate that the score is not available.)

Table 1.1 Summary of students' mathematical background and test results

Student	Grade level	SAT Math Section	SAT Subject-Math I	SAT Subject-Math II	AMC-12	Solved problems
1	12	770	n/a	800	94.5	1, 2, 3
2	12	780	n/a	800	n/a	1, 2
3	12	770	750	770	n/a	1, 3
4	12	740	n/a	770	n/a	1, 3
5	12	640	n/a	n/a	n/a	1, 3
6	11	n/a	n/a	n/a	n/a	1, 3
7	12	770	n/a	800	n/a	2
8	12	800	n/a	800	n/a	3
9	12	760	n/a	n/a	n/a	3

1.3.1 Familiarity

Only one student (Student 1) solved a problem (Problem 3) using more than one solution method; the other eight students either failed to solve certain problems entirely or solved them using only one solution method. Based upon the interview responses, the nine students were not at all familiar with the practice of "looking back" to solve differently. There were nine scores of 0 for the first question in the interview.

The impulsive manner in which the nine students were eager to find the answers to the three problems suggests, to a certain degree, that they were more accustomed to contently solving problems using a single, familiar method than they were to persistently and purposefully looking for alternative solutions. Obtaining a correct answer to a problem appeared more important to these students than searching for more efficient or enlightening solution methods. It did not appear to occur to most of the nine students that problem solving in mathematics might be a recurrent process, or that exploring alternative solution methods might be beneficial.

When asked whether they could relate the practice of "looking back" to solve problems differently to their past experiences in learning mathematics, many of them highlighted their algebra class. Specifically, they referred to the topic of solving systems of simultaneous linear equations using graphical, substitution, and elimination methods, among others approaches. Yet, they expressed that tests in this topic, like any other tests in their mathematics classes, specified explicitly which solution methods were expected in addressing particular problems. There was not much liberty provided by their instructors with regard to choosing any viable solution method, including those that students might devise on their own, in solving test problems. That being said, some students mentioned that their mathematics teachers were generally more amenable to student-invented solutions in a classroom discussion than during formal examinations.

Other students offered their impressions that mathematical concepts were supposed to be learned sequentially; that is, they felt that topics in mathematics were properly viewed as preconditions to further study rather than as interrelated ideas. They described, for example, the belief that the techniques of algebra are only applicable to classes such as coordinate geometry or calculus when employed in the process of manipulating variables. They did not recall many classroom discussions about connecting topics from different mathematics courses, such as how one might approach calculus problems using concepts from elementary algebra. Essentially, the nine students in the present study considered their mathematics courses as disconnected subjects under the single label of "mathematics."

1.3.2 Fluency

The nine students in the present study had achieved top percentiles in standardized tests and had received a more rigorous mathematics curriculum—including classes in trigonometry, pre-calculus, and calculus—than one could find in typical public high schools in the United States. They were also among the students most highly recommended by their mathematics teachers. As such, they might be considered to have acquired a high level of mathematical training.

Based on the interview responses, the nine students understood all 15 solution methods and recognized all 15 solution methods as being related to specific mathematics courses they had previously taken. There were nine scores of 1 for both the second and third questions in the interview.

In fact, after being presented with the 15 solution methods for the three problems, within a relatively short period of time, all of the students immediately acknowledged that they understood all of the methods. They could each replicate the different solution methods without difficulty during the interview.

They were also able to spontaneously and accurately identify specific mathematics courses in which they were taught content associated with each of the 15 solution methods. Moreover, they mentioned with confidence that there were no concepts involved in the 15 solution methods that they had not previously encountered in their mathematics courses.

They described, for example, how the geometric and algebraic solutions (P1S1 and P1S2, respectively) to Problem 1 were accessible based on the material they learned in their algebra course, how the limit-definition-of-derivative solution (P1S3) was accessible based on material learned in their pre-calculus course, and how the arithmetic solution (P1S4) was accessible based on material learned in their middle school mathematics course. For Problem 2, the students confidently related the geometric solution (P2S1) to their coordinate geometry course, the Cauchy-Schwartz-inequality solution (P2S2) to their pre-calculus course, the contradiction-via-symmetry and quadratic-equation solutions (P2S3 and P2S5, respectively) to their algebra course, the vector-dot-product solution (P2S4) to their linear algebra course, the calculus-in-polar-coordinate and single-variable-calculus solutions (P2S6 and P2S8) to their calculus course, and the angle-sum-trigonometric-identity solution (P2S7) to their trigonometry course. In Problem 3, as in the previous two problems (Problems 1 and 2), the nine students easily recognized distinct mathematics concepts from their geometry course (such as the congruent-diagonals property of a parallelogram, the characteristics of inscribed angles of a circle, and the sum of internal angles of a circle) in the three solutions (P3S1, P3S2, and P3S3, respectively).

It was clear that the students were relatively fluent as regards their knowledge of mathematical content. The results of the interview particularly substantiated the mathematics background they had reported prior to the interview as well as their perceptions of their own mathematics skills. Overall, the nine students in the present study demonstrated an uncommon level of mathematics proficiency compared to typical high school students in the United States.

1.3.3 Flexibility

Despite their fluency, the nine students for the most part failed to supply more than one solution method for each problem contrary to the instructions for the test. Based on the interview responses, the nine students were not at all likely to supply a different solution method aside from their own preferred solution method. There were nine scores of 0 for the fourth question in the interview. One clear indicator was observed in the students' written work for Problem 2 (which is in essence an algebra problem but was perceived by the nine students as being a calculus problem).

All nine students in fact identified Problem 2 as a calculus problem: they immediately operated the differentiation technique to arrive at an answer. One might expect that the students' past mathematical experience (especially given that they were enrolled in an AP Calculus course at the time of the study) had directly influenced their focus on certain solution methods.

Their fixation on a single solution method became more apparent after they were presented with the 15 solution methods for the three problems. The calculus approach that most students supplied was only one of the eight possible solution methods for Problem 2. (The other seven solution methods included topics involving elementary algebra, geometry, trigonometry, and linear algebra.)

The nine students maintained that they would not solve problems similar to Problem 2 in the future using any of the other seven solutions, even though they had no difficulty grasping those seven other solution methods. They argued that their calculus solution was more practical than other solution methods in obtaining the correct answer. This result demonstrates the fixation effect students revealed in their rigid association between particular problems and particular solution methods.

Nevertheless, the one student who solved one of the three problems using more than one solution method might be analyzed differently than the other eight students. Compared to the others, Student 1 had a greater past mathematical experience: he had taken the AMC-12 test, he was an active member of the mathematics team in that particular high school, and he mentioned having seen a mathematical fact similar to that in Problem 3 in the course of reading a number of mathematics books outside the confines of his course requirements.

Furthermore, the test results of Student 1 differed substantially from those of the other eight students both in terms of quantity and quality. Student 1 was the only student who was able to solve all three problems correctly, and he was the only student able to produce more than one solution method to a problem.

Student 1 solved Problem 1, an arithmetic problem, using an algebraic solution (P1S2), whereas the other five students who solved the same problem successfully did so using an arithmetic solution (P1S4). Student 1 solved Problem 2, an algebra problem, using a polar coordinate substitution approach from calculus (P2S6), whereas the other two students who solved the same problem successfully did so using a single variable substitution approach from calculus (P2S8).

Furthermore, Student 1 solved Problem 3, a geometry problem, using two different solution methods: one used the given facts from the sum of internal angles (P3S3), and the other used an extension of the inscribed angle of a circle (P3S2). The former was the only solution method supplied by the other six students who successfully solved Problem 3. Student 1 discussed in the interview how he simply attempted to prove a known fact that he recalled from a mathematics book as he was solving Problem 3, instead of formulating an answer anew.

To the extent that Student 1 demonstrated the capacity to transform his mathematical background into a unique test result, such a positive correlation between fluency and flexibility was nonetheless rather unclear in his consideration of solution methods beyond those he presented in his written responses. Despite his clear understanding of all of the 15 solution methods for the three problems, Student 1 maintained that if he were to take the test again, he would still supply the same solution methods he did previously.

As Student 1 asserted that his solution methods resulted in correct answers and that there was no need for him to consider the other methods, it was clear that the same fixation effect observed in the case of the other eight students emerged in spite of Student 1's distinct combination of mathematical background and test results. In summary, the emphasis on doing well on mathematics assessments, and on ensuring that each problem was solved correctly irrespective of how it might have been solved differently appeared, to a certain extent, pervasive and persistent. Despite how capable the students involved in the present study may be, they nevertheless became desensitized to the directive to use multiple solution methods. It was evident that the nine students somehow overlooked the relationship between their mathematical understanding and their realization of mathematical interconnectedness in the pursuit of academic success in mathematics.

1.4 Conclusions and Discussions

The present study reveals, to some extent, that based on a mathematics problem-solving test and subsequent interview results, the nine students were less familiar with the practice of problem solving using multiple solution methods at the assessment level than in the classroom discussion environment. It suggests for the most part that despite their fluency in understanding, reproducing, and identifying a particular mathematics topic or course related to specific solution methods, the nine students were unfamiliar with the practice of "looking back" to solve problems differently. It also indicates that, regardless of their fluency with a variety of mathematics topics, the nine students were not flexible in making mathematical connections among different solutions or in adjusting to the different solution methods.

The nine students' perceived mastery of particular methods and disinterest in others indicates, to some extent, that pedagogical recommendations or educational policies that underscore fluency in acquired mathematical concepts and procedures might not guarantee flexibility in accepting different solution methods. This condition appears to be exacerbated by the unfamiliar ways in which problem-solving processes might encourage, or even necessitate, students to “look back” to find alternative approaches to solve the same problem.

Given that it is not generally required or part of any curriculum, mathematics teachers cannot expect students to demonstrate the importance of the fourth step of Pólya’s (1945) problem solving process on their own or without additional prompts. Students in the present study pointed out that student-invented strategies usually only make their appearance during classroom discussions, not at the assessment level where it may be more valuable to invite elements such as surprise and creativity. It is evident from the interviews that, regardless of their mathematical background, students need early exposure to and constant opportunities to cultivate the practice of “looking back” to find different solution methods to previously solved problems.

The present study not only identifies that the practice of “looking back” has not been effectively integrated into mathematics classroom instruction in one of the most highly-regarded high schools in the United States, but also demonstrates that non-standard problems have the potential to offer students an appreciation for mathematical interconnections. In relation to earlier studies (Leikin & Lev, 2007; Silver et al., 2005), the present findings show that a more concrete pedagogical framework (Collins et al., 1989; Skemp, 1987; Spiro et al., 1991) is necessary to effectively integrate the practice of “looking back” into the current curriculum and classroom practice in mathematics. Changing the didactical approach to assessing problem solving in the mathematics classroom consequently requires careful consideration of different pedagogical frameworks, from one assessment which did not require multiple solution methods to another that did (Douady & Perrin-Glorian, 1989).

The present study also demonstrates the value of mathematics teachers adept at, and adaptive to, the identification and examination of the appropriateness and effectiveness of student-invented strategies relating to the solution methods introduced in their classroom, and to other related mathematics topics outside their classroom. To this end, it calls attention to the need to train, equip and enable future classroom instructors teaching rigorous and advanced mathematics courses to place an emphasis on illustrating connections between various topics in mathematics. Ill-equipped classroom instructors may be more liable to dismiss student-invented strategies when faced with unfamiliar solution methods (Silver et al., 2005). By accepting accurate solution methods that they did not explicitly teach in class, and by making connections between students’ mathematical backgrounds and the content they are currently teaching, teachers can nurture students’ deeper understanding of mathematics (Michener, 1978).

This understanding should therefore be carefully evaluated not only in terms of how well students might retain their acquired mathematical knowledge, but also in terms of how far students might form mental connections between new knowledge and past knowledge. Students need to appreciate that the whole field of mathematics was not developed in isolation of its parts (Davis & Hersh, 1981) the way it is presently studied in the elementary and secondary schools, but rather presented as a gradual progression of ideas that built one result upon another in a consciously connected manner.

Furthermore, by revealing many different solution methods, teachers can open up the possibility for students to consider the idea that topics in mathematics courses might be viewed on a coherent and interrelated continuum. What happens in algebra class, for instance, does not have to stay in algebra class; what happens in algebra class can and should be carried forward to other mathematics classes such as geometry and calculus. Further studies might be considered to examine a pedagogical framework integrating the need for problem solving using different solution methods within mathematics instruction, especially one incorporating students possessing a wider range of mathematical abilities.

References

Bodemer, D., Plötzner, R., Feuerlein, I., & Spada, H. (2004). The active integration of information during learning with dynamic and interactive visualizations. *Learning and Instruction, 14,* 325–341.

Borwein, P., Liljedahl, P., & Zhai, H. (2014). *Mathematicians on creativity*. Washington, D.C.: Mathematical Association of America.

Collins, A. (1991). Cognitive apprenticeship and instructional technology. In L. Idol & B. F. Jones (Eds.), *Educational values and cognitive instruction: Implication for reform* (pp. 121–138). Hillsdale, NJ: Erlbaum.

Collins, A., Brown, J. S., & Newman, S. E. (1989). Cognitive apprenticeship. Teaching the crafts of reading, writing, and mathematics. In L. B. Resnick (Ed.), *Knowing, learning, and instruction* (pp. 453–493). Hillsdale, NJ: Erlbaum.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

Davis, P. J., & Hersh, R. (1981). *The mathematical experience*. Boston, MA: Birkhauser.

de Jong, T., Ainsworth, S., Dobson, M., van der Hulst, A., Levonen, J., Reimann, P., et al. (1998). Acquiring knowledge in science and mathematics: The use of multiple representations in technology-based learning environments. In M. van Someren, P. Reimann, H. Boshuizen, & T. de Jong (Eds.), *Learning with multiple representations* (pp. 9–41). Oxford, England: Elsevier Sciences.

Douady, R., & Perrin-Glorian, M.-J. (1989). Un processus d'apprentissage du concept d'aire de surface plane. *Educational Studies in Mathematics, 20*(4), 387–423.

Duncker, K. (1945). On problem-solving (L. S. Lees, Trans.). *Psychological Monographs, 58*(5), i-113 (Whole No. 270).

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education, 24*(1), 8–40.

Felmer, P., Pehkonen, E., & Kilpatrick, J. (2016). *Posing and solving mathematical problems: Advances and new perspectives*. New York, NY: Springer.

Garofalo, J., & Lester, F. K. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education, 16*(3), 163–176.

Gick, M. L., & Holyoak, K. J. (1980). Analogical problem solving. *Cognitive Psychology, 12,* 306–355.

Große, C. S., & Renkl, A. (2006). Effects of multiple solution methods in mathematics learning. *Learning and Instruction, 16*(2), 122–138.

Hadamard, J. (1945). *The psychology of invention in the mathematical field*. Princeton, NJ: Princeton University Press.

Hersant, M. (2011). Correspondance entre élèves: conditions d'une activité mathématique « créative » et problématisée à la fin du lycée. *Educational Studies in Mathematics, 78*(3), 343–370.
Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale, NJ: Erlbaum.
Hogan, D. M., & Tudge, J. R. (1999). Implications of Vygotsky's theory for peer learning. In A. M. O'Donnell & A. King (Eds.), *Cognitive perspectives on peer learning* (pp. 39–65). Mahwah, NJ: Erlbaum.
Kaiser, G., & Schwarz, B. (2006). Mathematical modelling as bridge between school and university. *Zentralblatt für Didaktik der Mathematik, 38*(2), 196–208.
Kilpatrick, J. (1985). A retrospective account of the past 25 years of research on teaching mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1–15). Hillsdale, NJ: Erlbaum.
Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129–145). Rotterdam, The Netherlands: Sense Publisher.
Leikin, R., & Lev, M. (2007). Multiple solution tasks as a magnifying glass for observation of mathematical creativity. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 161–168). Seoul, Korea: PME.
Leikin, R., & Levav-Waynberg, A. (2007). Exploring mathematics teacher knowledge to explain the gap between theory-based recommendations and school practice in the use of connecting tasks. *Educational Studies in Mathematics, 66*(3), 349–371.
Lesh, R. (1985). Conceptual analyses of problem solving performance. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 309–330). Hillsdale, NJ: Lawrence Erlbaum.
Lester, F. K. (1994). Musings about mathematical problem-solving research: The first 25 years in the JRME. *Journal for Research in Mathematics Education, 25*(6), 660–675.
Liljedahl, P. (2016). Building thinking classrooms: Conditions for problem-solving. In P. Felmer, E. Pehkonen, & J. Kilpatrick (Eds.), *Posing and solving mathematical problems*. New York, NY: Springer.
Liljedahl, P., & Sriraman, B. (2006). Musings on mathematical creativity. *For the Learning of Mathematics, 26*(1), 20–23.
Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. New York, NY: Addison Wesley.
Mathematical Association of America. (2011). Retrieved from https://www.maa.org/math-competitions/about-amc.
Michener, E. R. (1978). Understanding understanding mathematics. *Cognitive Science, 2,* 361–383.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
Owen, E., & Sweller, J. (1985). What do students learn while solving mathematics problems? *Journal of Educational Psychology, 77,* 272–284.
Poincare, H. (1946). *The foundations of science* (G. B. Halsted, Trans.). Lancaster, PA: Science Press.
Pólya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.
Pressley, M., Forrest-Pressley, D. L., Elliott-Faust, D., & Miller, G. E. (1985). Children's use of cognitive strategies, how to teach strategies, and what to do if they can't be taught. In M. Pressley & C. J. Brainerd (Eds.), *Cognitive processes in memory development*. New York, NY: Springer.
Reeves, L. M., & Weisberg, R. W. (1994). The role of content and abstract information in analogical transfer. *Psychological Bulletin, 115,* 381–400.
Resnick, L. B. (1989). *Knowing, learning, and instruction*. Hillsdale, NJ: Erlbaum.
Rittle-Johnson, B., & Siegler, R. S. (1998). The relation between conceptual and procedural knowledge in learning mathematics: A review. In C. Donlan (Ed.), *The development of mathematical skills* (pp. 75–110). East Sussex, England: Psychology Press.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561–574.

Santos-Trigo, M. (1996). An exploration of strategies used by students to solve problems with multiple ways of solution. *Journal of Mathematical Behavior, 15,* 263–284.

Santos-Trigo, M. (1998). Can routine problems be transformed into non-routine problems? *Teaching Mathematics and Its Applications, 17*(3), 132–135.

Schoenfeld, A. H. (1979a). Can heuristics be taught? In J. Lochhead & J. Clement (Eds.), *Cognitive process instruction* (pp. 315–338). Philadelphia, PA: Franklin Institute.

Schoenfeld, A. H. (1979b). Explicit heuristic training as a variable in problem solving performance. *Journal for Research in Mathematics Education, 10,* 173–187.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (2008). Problem solving in the United States, 1970–2008: Research and theory, practice and politics. *ZDM The International Journal of Mathematics Education, 39*(5–6), 537–551.

Silver, E. A. (1985). Research on teaching mathematical problem solving: Some underrepresented themes and needed directions. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 247–266). Hillsdale, NJ: Erlbaum.

Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM Mathematics Education, 3,* 75–80.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Font Strawhun, B. T. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *Journal of Mathematical Behavior, 24,* 287–301.

Silver, E. A., Leung, S. S., & Cai, J. (1995). Generating multiple solutions for a problem: A comparison of the responses of U.S. and Japanese students. *Educational Studies in Mathematics, 28,* 35–54.

Singer, F. M., Ellerton, N. F., & Cai, J. (2015). *Mathematical problem posing: From research to effective practice*. New York, NY: Springer.

Skemp, R. (1987). *The psychology of learning mathematics*. Mahwah, NJ: Erlbaum.

Spiro, R. J., Feltovich, P. J., Jacobson, M. J., & Coulson, R. L. (1991). Cognitive flexibility, constructivism, and hypertext: Random access instruction for advanced knowledge acquisition in ill-structured domains. In T. Duffy & D. Jonassen (Eds.), *Constructivism and the technology of instruction* (pp. 57–76). Hillsdale, NJ: Erlbaum.

Spiro, R. J., & Jehng, J. C. (1990). Cognitive flexibility and hypertext: Theory and technology for the nonlinear and multidimensional traversal of complex subject matters. In D. Nix & R. J. Spiro (Eds.), *Cognition, education, and multimedia: Exploring ideas in high technology* (pp. 163–205). Hillsdale, NJ: Erlbaum.

Sriraman, B. (2009). *The characteristics of mathematical creativity. ZDM, 41,* 13–27.

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction, 18*(6), 565–579.

Tabachneck, H. J., Koedinger, K. R., & Nathan, M. J. (1994). Toward a theoretical account of strategy use and sense-making in mathematics problem solving. In A. Ram & K. Eiselt (Eds.), *Proceedings of the Sixteenth Annual Conference of the Cognitive Science Society* (pp. 836–841). Hillsdale, NJ: Erlbaum.

The College Board. (2011a). Retrieved from http://professionals.collegeboard.com/data-reports-research/sat/data-tables.

The College Board. (2011b). Retrieved from https://collegereadiness.collegeboard.org/sat.

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society, 20,* 161–177.

Tjoe, H. (2014). When understanding evokes appreciation: The effect of mathematics content knowledge on aesthetic predisposition. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan. (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 249–256). Vancouver, BC: PME.