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# Progress in Industrial Mathematics at ECMI 2006

With 361 Figures, 52 in color and 53 Tables

 Springer

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To our beloved ones

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## Preface

Increasing computing power in the last decades has given mathematical modeling an ever greater impulse and made it a very important tool to solve problems coming from industry. The European Consortium for Mathematics in Industry (ECMI) was founded 20 years ago by mathematicians from ten European universities to foster the use of mathematics to help European industry and commerce to pose and solve their problems. The aims of ECMI are to (a) promote the use of mathematical models and mathematics in industry, (b) form applied mathematicians capable of working effectively in industry and (c) work for these goals at the European scale. Efficient problem solving often requires the use of results in different mathematical fields, yet no single applied mathematician may be able to cover the whole subject. By providing a European research network, ECMI can bring together experts from a wide geographical range.

Since 1986, ECMI has incorporated many more institutions and industries throughout Europe and it has been consolidated as a brand name for Industrial Mathematics. Twenty years later, the biannual ECMI conference was celebrated for the first time in Spain, at the Universidad Carlos III de Madrid. This is a young university created in 1989. Technological studies and departments are located at the Leganés campus where the conference was held. Moreover, University Carlos III participates in the Leganés Scientific and Technological Park, together with the Autonomous Region of Madrid and the city of Leganés. They contribute to place Madrid at the forefront of research and development in Spain.

The scientific program covered a wide variety of topics related to technological sectors (aerospace and automotive industry, materials and electronics, information and telecommunication technologies, energy and environment, biology, biotechnology, life sciences, imaging) and to finances and economics. The different origin of participants helped making the conference multidisciplinary. Active participation of industry was intended, with reasonable success. The present volume includes a part of the contributions to the conference, selected after a refereeing process. It is a pleasure to see that six

plenary speakers have submitted papers for this volume. Vincenzo Capasso in his “Alan Tayler” lecture, besides presenting his scientific work on statistical geometric measure applied to medicine and materials science, recalls some of the challenges for Mathematics in Industry listed in the first ECMI brochure produced by Alan Tayler and himself in 1994, relates them to the present situation of an enlarged Europe, and tells us how these challenges remain important and pressing for us today. Antonio Barrero (Seville), Alfredo Bermúdez (Santiago), Russel Caffisch (UCLA), Luis Campos (Lisbon) and Pierre Degond (Toulouse) illustrate with their contributions the breadth of applications and variety of techniques that are embraced by ECMI. ECMI’s commitment to educating students in Industrial Mathematics is reflected in the fact that many papers were given by students. The Wacker Prize, offered for a Master’s Level thesis on an industrial problem was awarded to Filippo Terragni, in line with the tradition of excellent work by previous winners. Many of the minisymposia and special sessions included the activities of ECMI Special Interest Groups. Of the 35 minisymposia organized for the conference, many are gathered in this book, usually preceded by a short explanation about their contents. A number of contributed papers complete the volume. I hope that these proceedings will contribute both to show interesting and relevant mathematical problems and methods, and to strengthen cooperation between academia and industry, the absence of which is a major weakness of the European Science-Technology system.

As President of ECMI and on behalf of the ECMI Council, I wish to thank all those who have contributed to the success of the Conference. Among them the participants, the speakers, the International Scientific Committee and the National and Local Organizing Committees. Organizing this meeting has been possible thanks to the efforts of many people both at the Spanish national and local level to whom we are very grateful. In particular all the members of the Modeling, Simulation and Industrial Mathematics Group at Universidad Carlos III worked hard to run a smooth and successful conference which would not have been possible without their help. The dedication of our university congress bureau, Congrega, was also essential for the conference success. Ms. Bárbara Tapiador’s help was very important to process the manuscripts that are gathered in the present book. I am grateful to my co-editors, Gloria Platero, Miguel Moscoso and José Manuel Vega for their invaluable help.

Lastly, the support of our sponsors is gratefully acknowledged: Ministerio de Educación y Ciencia (grant MTM-2005-24569-E), Comunidad de Madrid (grant S-0505/ENE/0229), Universidad Carlos III de Madrid, Universidad Politécnica de Madrid, Consejo Superior de Investigaciones Científicas (CSIC), Instituto Tecnológico de Química y Materiales “Álvaro Alonso Barba”, Ayuntamiento de Leganés and Springer.

Madrid, May 2007

*Luis L. Bonilla, President of ECMI*

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## **Part I**

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### **Plenary Lectures**

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# On the Mean Geometric Densities of Random Closed Sets, and Their Estimation: Application to the Estimation of the Mean Density of Inhomogeneous Fibre Processes

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**Dedicated to Alan Tayler**

## Preface [VC]

It has been a great honour for me to deliver the “Alan Tayler Lecture” in this ECMI Conference, to honour one of the leading founders and Presidents of ECMI. I have collaborated with Alan for many years, especially during my term as Chairman of the Educational Committee, and later during the first ECMI-HCM Project. While he was already very ill, he found the way to participate (even though only for a couple of days) in a workshop in Milan, opening ECMI to the Italian academic and industrial community, and highly supported the birth of MIRIAM (the Milan Research Centre for Industrial and Applied Mathematics).

I had a rewarding experience around the early 1990s producing, in a strict collaboration with Alan, the first ECMI Brochure [CT94] (see the ECMI web site) in order to advertise the specific role of ECMI within academia and industry in Europe.

It was clear to me that he had a vision of how to establish in Europe a co-operative action by the most active groups in the applications of mathematics to real world problems; I wish to remind the key issues stated in the brochure, since I may claim that these are still update.

“Realising the need of interaction between universities and research groups in industry, the European Consortium for Mathematics in Industry (ECMI) was founded in 1986 by mathematicians from ten European universities.

...

Mathematics, as the language of the sciences, has always played an important role in technology, and now is applied also to a variety of problems in commerce and the environment.

European industry is increasingly becoming dependent on high technology and the need for mathematical expertise in both research and development can only grow.

...

These new demands on mathematics have stimulated academic interest in Industrial Mathematics and many mathematical groups world-wide are committed to interaction with industry as part of their research activities.

In 1986 ten of these groups in Europe founded ECMI with the intention of offering their collective knowledge and expertise to European Industry.

The experience of ECMI members is that similar technical problems are encountered by different companies in different countries. It is also true that the same mathematical expertise may often be used in differing industrial applications.

If European industry is to compete in world markets it should take advantage of the competitive edge which may be gained from using European mathematical expertise.

**No single European country is likely to have sufficient expertise of mathematical knowledge whereas ECMI can provide a comprehensive coverage of mathematical skills and their diverse applications.” [CT94]**

We are now facing the challenge of a larger European Union.

Alan had anticipated this by promoting an ECMI “patronage”, financially supported by the EU, of those countries usually called “Central Europe”, such as Āekia, Hungary, Poland, Romania, Slovakia.

I am sure that he would have liked to participate in the process of complete integration of all the new entries in the ECMI system.

Going back to the ECMI Brochure, a major scope of ECMI was identified as follows.

#### “C. TO OPERATE ON A EUROPEAN SCALE

Academic resources in Mathematics for Industry are also scarce and distributed across Europe; industrial needs are widely spread. Exchange and interactive programmes are necessary in training, research and industrial collaboration if there is to be an effective transfer of knowledge and skills. The EC is encouraging ECMI to involve relevant groups in Eastern Europe as Associate members.”

As part of this encouragement, the EC provided funds to ECMI for organising a series of workshops in those countries, in collaboration with recognised colleagues at the local level. Thus anticipating the enlargement of the political Europe.

In my opinion, having the EC approved a significant enlargement of Europe towards East, listing soon 27 member states, ECMI, as an enlarged Consortium, should find new ways to exploit the best of the scientific resources of the old and the new member states together, to actively participate in the building up of a common competitive Europe. As far as scientific competence is concerned, there are excellencies in all regions of Europe, some of them well

identifiable also in the new member states; a genuine will to sustain competence of Europe should go through ways to exploit all of them, with the usual ECMI cooperative attitude.

Another anticipation envisaged by Alan has been the shift of meaning of the key word “Industry” in the ECMI system.

**“This collaboration may also be extended to developing mathematical models for the environment, earth sciences, biology and finance.”** [CT94]

We have already achieved the inclusion of what we call **Economathematics**, and today we are facing a further shift of attention towards **Medicine and Biotechnology**.

All over the world leading experts of **Mathematics for/in Industry**, are participating actively in the development of **Mathematics for/in Medicine**, thus undertaking the further challenge of contributing to the development of innovative methods for diagnosis and treatment of relevant diseases, from cancer to infectious diseases.

My own presentation here is aimed to showing an example of how mathematics, originally developed for mining industry or more in general for material science and chemical industry, is now moving to deal with problems of interest in medicine.

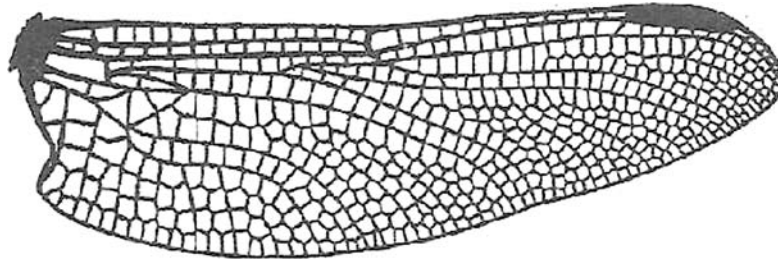
At first this research was motivated by polymer industry in Europe, and constitutes one of the most important success stories of collaborative research within ECMI, that was supported within the first HCM Project coordinated by Alan Tayler. As a documentation of the cooperation between different research teams in Europe within the ECMI Special Interest Group on “Polymers”, the volume “Mathematical Modelling for Polymer Processing. Polymerization, Crystallization, Manufacturing”, edited by myself, was published as Volume 2 in the **ECMI Series on Mathematics in Industry by Springer-Verlag, Heidelberg 2002**, showing an additional success story of ECMI: the start of the **Springer Series on Mathematics in Industry**.

## 1 Introduction

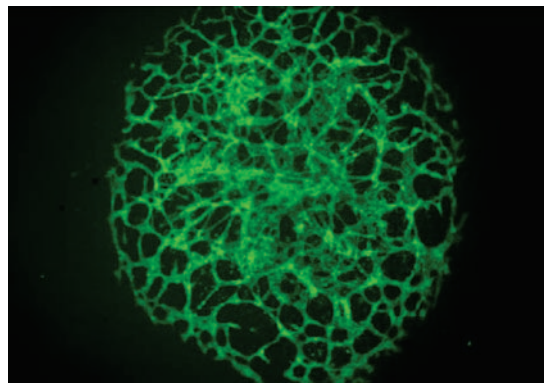
Many processes of biomedical or material science interest may be modelled as birth-and-growth processes (germ–grain models), which are composed of two processes, birth (nucleation, branching, etc.) and subsequent growth of spatial structures (cells, vessel networks, etc.), which, in general, are both stochastic in time and space. These structures induce a random division of the relevant spatial region, known as random tessellation (see Fig. 1). A quantitative description of the spatial structure of a tessellation can be given, in terms of the mean densities of interfaces ( $n$ -facets).

In applications to material science a main industrial interest is controlling the quality of the relevant final product in terms of its mechanical properties; as shown, e.g. in [FC98], these are strictly related to the final morphology





**Fig. 1.** The spatial tessellation generated by vessels in a dragonfly wing



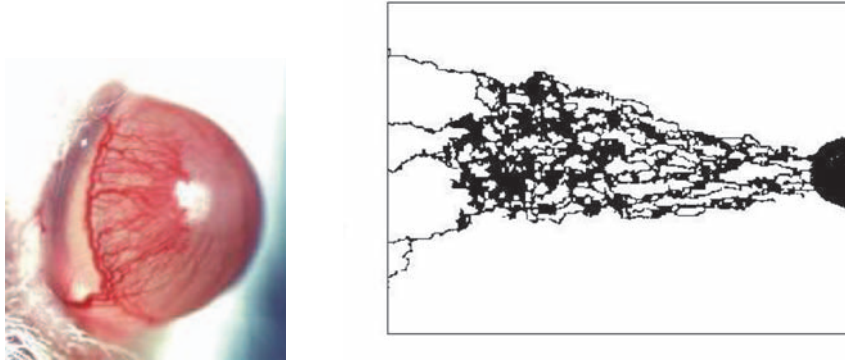
**Fig. 2.** Vascularization of an allantoid [Credit: Dejana et al. 2005]

of the solidified material, so that quality control in this case means optimal control of the final morphology.

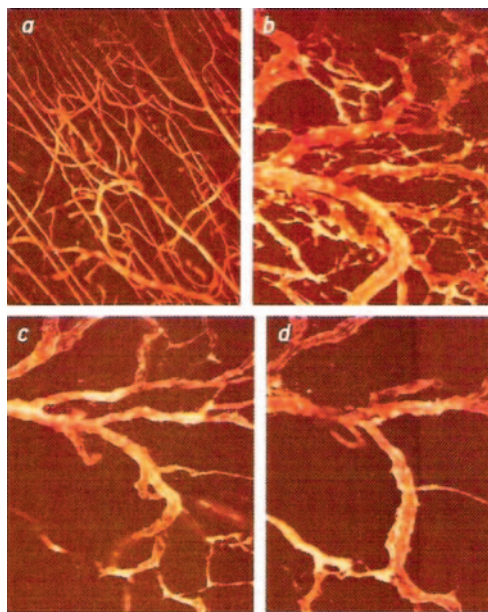
In medicine, an important area of application of birth-and-growth processes and other models of stochastic geometry is tumour-induced angiogenesis. It can be modelled as a fibre process of Hausdorff dimension 1 in the relevant 2D or 3D space.

Tumour-induced angiogenesis is believed to occur when normal tissue vasculature is no longer able to support growth of an avascular tumour. At this stage the tumour cells, lacking nutrients and oxygen, become hypoxic. This is assumed to trigger cellular release of tumour angiogenic factors (TAFs) which start to diffuse into the surrounding tissue and approach endothelial cells (ECs) of nearby blood vessels. ECs subsequently respond to the TAF concentration gradients by forming sprouts, dividing, and migrating towards the tumour. A summary of these mechanisms can be found in the recent paper by Carmeliet [JK01] (see also Figs. 2–4 where examples of real or simulated vascular networks are depicted).

Initially, the sprouts arising from a parent vessel grow essentially parallel to each other. It is observed that once the finger-like capillary sprouts have



**Fig. 3.** *Left:* Angiogenesis on a rat cornea [Credit: Dejana et al. 2005]. The white spot is a pellet implanted in the cornea containing an angiogenic substance, emulating the effect of a tumour. *Right:* A simulation of an angiogenesis due to a localized tumour mass (*black region on the right*) (from [CA99])



**Fig. 4.** Response of a vascular network to an antiangiogenic treatment (from [JK01])

reached a certain distance from the parent vessel, they tend to incline towards each other, leading to fusions called anastomoses. Such fusions lead to a network of vessels. On the other hand the sprout branching dramatically increases while approaching the tumour mass, eventually resulting in vascularization.

The coupling of the branching and growth process to the underlying chemical gradients is limited by the local density of the existing capillary network, thus leading to a mathematical strong coupling of this density and the kinetic parameters of the branching and growth process.

The study of angiogenesis has such potential for providing new therapies that it has received enthusiastic interest from the pharmaceutical and biotechnology industries. Indeed, dozens of companies are now pursuing angiogenesis-related therapies, and approximately 20 compounds that either induce or block vessel formation are being tested in humans. Although such drugs can potentially treat a broad range of disorders, many of the compounds now under investigation inhibit angiogenesis and target cancer. Intriguingly, animal tests show that inhibitors of vessel growth can boost the effectiveness of traditional cancer treatments (chemotherapy and radiation). Preliminary studies also hint that the agents might one day be delivered as a preventive measure to block malignancies from arising in the first place in people at risk for cancer.

In developing mathematical models of angiogenesis, the hope is to be able to provide a deeper insight into the underlying mechanisms which cause the process. It is therefore essential that predictive mathematical models are developed, capable of producing precise quantitative morphological features of developing blood vessels. Such models might be used for predicting the evolution of tumours (prognosis), and identifying optimal control strategies (medical treatment).

Unfortunately, a satisfactory modelling of angiogenesis requires a theory of stochastic fibre processes, evolving in time, and strongly coupled with underlying fields. In this case the theory of birth-and-growth processes (or branching-and-growth processes), developed for volume growth, cannot be applied to analyse realistic models, due to intrinsic mathematical difficulties, coming from the dependence of the kinetic parameters from the geometric spatial densities of the existing tumour, or capillary network itself [CM05, McDou06].

All these aspects induce stochastic time and space heterogeneities, thus motivating a more general analysis of the stochastic geometry of the process. The formulation of an exhaustive evolution model which relates all the relevant features of a real phenomenon dealing with different scales, and a stochastic domain decomposition at different Hausdorff dimensions, is a problem of high complexity, both analytical and computational.

Anyway statistical methods for the estimation of geometric densities may offer significant tools for diagnosis and dose/response analysis in medical treatments.

In the modelling of the above-mentioned systems it is of great importance to handle random closed sets of different (even though integer) Hausdorff dimensions. Following a standard approach in geometric measure theory, such sets may be described in terms of suitable measures. For a random closed set of lower dimension with respect to the environment space, the relevant measures induced by its realizations are singular with respect to the Lebesgue measure, and so their usual Radon–Nikodym derivatives are zero almost everywhere.

In Sect. 2 an original approach is reported, recently proposed by the research group of the authors, who have suggested to cope with these difficulties by introducing generalized densities (distributions) *à la Dirac–Schwartz*, for both the deterministic case and the stochastic case. In this last one, mean generalized densities are of interest.

These instruments may then help to formulate stochastic models (that is solving direct problems) for the over-mentioned applications; they also suggest methods for the solution of the related inverse problems, including methods of statistical analysis for the estimation of geometric densities of a stochastic fibre process that characterize the morphology of a real system. We apply such methods to real data, taken from the literature, and to simulated data, obtained by existing computational models of tumour-induced angiogenesis.

These methods can be used for validating computational models, and for monitoring the efficacy of possible medical treatment.

### 1.1 Nomenclature

We remind that a *random closed set (RACS)*  $\Xi$  in  $\mathbb{R}^d$  is a measurable map

$$\Xi : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}}),$$

where  $\mathbb{F}$  denotes the class of the closed subsets in  $\mathbb{R}^d$ , and  $\sigma_{\mathbb{F}}$  is the  $\sigma$ -algebra generated by the so-called *hit-or-miss topology* (see [Mat75]).

The theory of Choquet–Matheron shows that it is possible to assign a unique probability law associated with a *RACS*  $\Xi$  in  $\mathbb{R}^d$  on the measurable space  $(\mathbb{F}, \sigma_{\mathbb{F}})$  by assigning its *hitting functional*  $T_{\Xi}$ .

This is defined as

$$T_{\Xi} : K \in \mathcal{K} \longmapsto P(\Xi \cap K \neq \emptyset),$$

where  $\mathcal{K}$  denotes the family of compact sets in  $\mathbb{R}^d$ .

Actually we may consider, equivalently, the restriction of  $T_{\Xi}$  to the family of closed balls  $\{B_{\varepsilon}(x); x \in \mathbb{R}^d, \varepsilon \in \mathbb{R}_+ - \{0\}\}$ .

In dependence of its regularity, a random closed set  $\Theta_n$  with Hausdorff dimension  $n$  (i.e.  $\dim_{\mathcal{H}}\Theta_n(\omega) = n$  for a.e.  $\omega \in \Omega$ ), may induce a random Radon measure

$$\mu_{\Theta_n}(\cdot) := \mathcal{H}^n(\Theta_n \cap \cdot)$$

on  $\mathbb{R}^d$  ( $\mathcal{H}^n$  is the  $n$ -dimensional Hausdorff measure), and, as a consequence, an *expected measure*

$$\mathbb{E}[\mu_{\Theta_n}](\cdot) := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap \cdot)]$$

(for a discussion about measurability of  $\mathcal{H}^n(\Theta_n)$  we refer to [BM97, Z82]).

In several real applications, it is of interest to study the density (said *mean density*) of the measure  $\mathbb{E}[\mu_{\Theta_n}]$  [BR04], and, in the dynamical case, its evolution in time [Mol92, Mol94]. Here we present a synthesis of a theory of

random distributions as generalized densities of random measures, and mean geometric densities as expected values of random generalized densities, as proposed in [CV06c]. In particular we introduce a *Delta formalism*, á la Dirac–Schwartz, for the description of random measures associated with random closed sets of lower dimensions, such that the well known usual Dirac delta at a point follows as a particular case (see, for instance, [Jones82, KF70, Vlad79]).

In dealing with mean densities, a concept of *absolutely continuous random closed set* arises in a natural way in terms of the expected measure; indeed, an interesting property of a random set in  $\mathbb{R}^d$  is whether the expected measure induced by the random set is absolutely continuous or not with respect to the  $d$ -dimensional Lebesgue measure  $\nu^d$ . Thus, it is of interest to distinguish between random closed sets which induce an absolutely continuous expected measure, and random closed sets which induce a singular one. To this aim we introduce definitions of *discrete*, *continuous*, and *absolutely continuous* random closed set, coherently with the classical 0-dimensional case, in order to propose an extension of the standard definition of discrete, continuous, and absolutely continuous random variable, respectively (see also [CV06a, CV06b]).

## 2 Generalized Densities

In the sequel we will refer to a class of sufficiently regular random closed sets in the Euclidean space  $\mathbb{R}^d$ , of integer dimension  $n$ .

**Definition 1 ( $n$ -regular set).** *Given an integer  $n \in [0, d]$ , we say that a closed subset  $S$  of  $\mathbb{R}^d$  is  $n$ -regular, if it satisfies the following conditions:*

- (i)  $\mathcal{H}^n(S \cap B_R(0)) < \infty$  for any  $R > 0$
- (ii)  $\lim_{r \rightarrow 0} \frac{\mathcal{H}^n(S \cap B_r(x))}{b_n r^n} = 1$  for  $\mathcal{H}^n$ -a.e.  $x \in S$

Here  $b_n$  denotes the volume of the unit ball in  $\mathbb{R}^n$ .

*Remark 1.* Note that condition (ii) is related to a characterization of the  $\mathcal{H}^n$ -rectifiability of the set  $A$  ([Fal85], p. 256, 267, [AFP00], p. 83).

We may observe that if  $A_n$  is an  $n$ -regular closed set in  $\mathbb{R}^d$ , we have

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^n(A_n \cap B_r(x))}{b_n r^n} = \begin{cases} 1 & \mathcal{H}^n\text{-a.e. } x \in A_n, \\ 0 & \forall x \notin A_n; \end{cases}$$

as a consequence (by assuming  $0 \cdot \infty = 0$ ), for  $0 \leq n < d$  we have

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(A_n \cap B_r(x))}{b_d r^d} &= \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(A_n \cap B_r(x)) b_n r^n}{b_n r^n b_d r^d} \\ &= \begin{cases} \infty & \mathcal{H}^n\text{-a.e. } x \in A_n, \\ 0 & \forall x \notin A_n. \end{cases} \end{aligned}$$

It is well known that every positive Radon measure  $\mu$  on  $\mathbb{R}^d$  can be decomposed as

$$\mu = \mu_{\ll} + \mu_{\mathbb{P}\text{erp}},$$

where  $\mu_{\ll}$  and  $\mu_{\mathbb{P}\text{erp}}$  are the absolutely continuous, and the singular parts of  $\mu$ , respectively, with respect to  $\nu^d$ , the usual Lebesgue measure on  $\mathbb{R}^d$ .

It then follows that  $\mu_{\ll}$  admits a (nontrivial) Radon–Nikodym derivative with respect to  $\nu^d$ , which is known as its density; while the Radon–Nikodym derivative of  $\mu_{\mathbb{P}\text{erp}}$ , with respect to  $\nu^d$ , would be zero  $\nu^d$ - a.e.

Anyhow in analogy with the usual Dirac delta function  $\delta_{x_0}(x)$  associated with a point  $x_0 \in \mathbb{R}^d$  (a 0-regular closed set), a density can be introduced also for  $\mu_{\mathbb{P}\text{erp}}$ , in a generalized sense, according to Definition 2 [KF70].

**Definition 2 (Generalized density).** *We call  $\delta_{\mu_{\mathbb{P}\text{erp}}}$ , the generalized density (or, briefly, the density  $P$ ) of  $\mu_{\mathbb{P}\text{erp}}$ , the quantity*

$$\delta_{\mu_{\mathbb{P}\text{erp}}}(x) := \lim_{r \rightarrow 0} \frac{\mu_{\mathbb{P}\text{erp}}(B_r(x))}{b_d r^d},$$

*finite or not.*

Clearly, if  $A_n$  is an  $n$ -regular closed set in  $\mathbb{R}^d$  with  $n < d$ , then the measure

$$\mu_{A_n}(\cdot) := \mathcal{H}^n(A_n \cap \cdot)$$

is a singular measure with respect to  $\nu^d$ . Based on Definition 1, the quantity

$$\delta_{A_n}(x) := \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(A_n \cap B_r(x))}{b_d r^d},$$

(finite or not), can now be introduced as the (generalized) density associated with  $A_n$ .

With an abuse of notations, we may introduce the linear functional  $\delta_{A_n}$  associated with the measure  $\mu_{A_n}$ , as follows:

$$(\delta_{A_n}, f) := \int_{\mathbb{R}^d} f(x) \mu_{A_n}(dx),$$

for any  $f \in C_c(\mathbb{R}^d, \mathbb{R})$ , having denoted by  $C_c(\mathbb{R}^d, \mathbb{R})$  the space of all continuous functions from  $\mathbb{R}^d$  to  $\mathbb{R}$  with compact support. In accordance with the usual representation of distributions in the theory of generalized functions, we formally write

$$\int_{\mathbb{R}^d} f(x) \delta_{A_n}(x) dx := (\delta_{A_n}, f).$$

Define the function

$$\delta_{A_n}^{(r)}(x) := \frac{\mathcal{H}^n(A_n \cap B_r(x))}{b_d r^d},$$

and correspondingly the associated measure