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*This book is dedicated to my wife Guo-Hua Sun and lovely children Bo Dong and Jazmin Yue Dong Sun.*

# Preface

This work will introduce the wave equations in higher dimensions at an advanced level addressing students of physics, mathematics and chemistry. The aim is to put the mathematical and physical concepts and techniques like the wave equations, group theory, generalized hypervirial theorem, the Levinson theorem, exact and proper quantization rules related to the higher dimensions at the reader's disposal. For this purpose, we attempt to provide a comprehensive description of the wave equations including the non-relativistic Schrödinger equation, relativistic Dirac and Klein-Gordon equations in higher dimensions and their wide applications in quantum mechanics which complements the traditional coverage found in the existing quantum mechanics textbooks. Related to this field are the quantum mechanics and group theory. In fact, the author's driving force has been his desire to provide a comprehensive review volume that includes some new and significant results about the wave equations in higher dimensions drawn from the teaching and research experience of the author since the literature is inundated with scattered articles in this field and to pave the reader's way into this territory as rapidly as possible. We have made the effort to present the clear and understandable derivations and include the necessary mathematical steps so that the intelligent and diligent reader is able to follow the text with relative ease, in particular, when mathematically difficult material is presented. The author also embraces enthusiastically the potential of the LaTeX typesetting language to enrich the presentation of the formulas as to make the logical pattern behind the mathematics more transparent. In addition, any suggestions and criticism to improve the text are most welcome. It should be pointed out that the main effort to follow the text and master the material is left to the reader even though this book makes an effort to serve the reader as much as was possible for the author.

This book starts out in Chap. 1 with a comprehensive review for the wave equations in higher dimensions and builds on this to introduce in Chap. 2 the fundamental theory about the SO($N$) group to be used in the successive Chaps. 3–5 including the non-relativistic Schrödinger equation, relativistic Dirac and Klein-Gordon equations. As important applications in non-relativistic quantum mechanics, from Chap. 6 to Chap. 12, we shall apply the theories proposed in Part II to study some important quantum systems such as the harmonic oscillator, Coulomb potential, the Levinson theorem, generalized hypervirial theorem, exact and proper quantization rules and Langer modification, the Schrödinger equation with position-dependent mass and others. We shall illustrate two important applications in relativistic Dirac and Klein-Gordon equations with the Coulomb potential in Chaps. 13 and 14. As crucial generalized applications of Dirac equation in higher dimensions, we shall study the Levinson theorem, generalized hypervirial theorem and Kaluza-Klein theory in Chaps. 15–17. Some conclusions and outlooks are given in Chap. 18. Some useful reference materials such as group theory, group representations, fundamental properties of Lie groups and Lie algebras, the angular momentum theory and the confluent hypergeometric functions are sketched in Appendices A–E.

This book is in a stage of continuing development, various chapters, e.g., on the quantum gravity, on the Kaluza-Klein theory, on the supersymmetry and string theory, on the high dimensional brane will be added to the extent that the respective topics expand. At the present stage, however, the work presented for such topics should be complete enough to serve the reader.

This book shall give the theoretical physicists and researchers a fresh outlook and new ways of handling some important and interesting quantum systems in several branches of physics. This book can be used by graduate students and young researchers in physics, especially theoretical and mathematical physics. It is also suitable for some graduate students in theoretical chemistry.

Mexico city, Mexico Shi-Hai Dong

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Shi-Hai Dong

# About the Authors

Prof. Shi-Hai Dong got his PhD at Institute of High Energy Physics, Chinese Academy of Sciences, China in 1999. After that he joined the Physical Theoretical Chemistry Laboratory of University of Oxford as a Visiting Professor, the Department of Physics of Kansas State University as a Postdoc fellow, the National Autonomous University of Mexico as a Visiting Professor, the Mexican Institute of Petroleum as a Postdoc fellow and Distinguished Visiting Professor, the Superior School of Physics and Mathematics, National Polytechnic Institute as a full-time Professor through Program of Excellent Position and now has a permanent position of Professor at that Institute. He is the Editor of Central European Journal of Physics and Editorial Board Members of The Open Nuclear & Particle Physics Journal and Advances in High Energy Physics. He has authored more than 140 papers, one book Chapter and one book with over 1500 citations excluding self-citation. His research interests include algebraic method and quantum physics.

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# Introduction

## 1 Basic Review

The exact solutions of wave equations with a spherically symmetric potential have become an important subject in quantum mechanics [1–6]. It should be noticed that many works along this line have been carried out in the usual three dimensional space. However, what extra dimensions could there possibly be if we never see them? It turns out that we do not really know yet how many dimensions our world has. Nevertheless, all that our current observations tell us is that the world around us is at least $(3+1)$ dimensional space-time as illustrated in general relativity.

The idea of extra dimensions has a rich history, dating back at least as far as the middle of 1910s and earlier 1920s when the Nordstrom-Kaluza-Klein theory[1]—usually named as the Kaluza-Klein theory—was proposed [8–11]. This theory is a physical model that seeks to unify two fundamental forces of gravitation and electromagnetism. More precisely, the idea of additional spatial dimensions is from string theory, the only self-consistent quantum theory of gravity so far. For a consistent description of gravity, scientist needs more than $(3+1)$ dimensions, and the world could have up to 11 or more spatial dimensions. The reason why we do not feel these additional spatial dimensions in our life is because they are very different from the

[1]The theory was first proposed by the Finnish physicist Gunnar Nordström in 1914. Before Einstein's general relativity theory was presented, Nordström proposed a relativistic theory for gravity. He unified his gravity theory with Maxwell's electromagnetism through introducing a 5-vector gauge field where the first four components are identified with Maxwell's vector potential $A^{\mu}$ and the 5th component with the scalar gravity field. After that in 1919 a German mathematician Theodor Kaluza performed similar calculations but with Einstein's gravity theory and Maxwell's electromagnetism. In terms of a circular extra dimension Kaluza obtained a 4-dimensional action from a 5-dimensional one. The 4-action contained a graviton, an Abelian gauge boson identified as the photon and a scalar field that Kaluza put to be constant. The resulting equations can be separated out into further sets of equations, one of which is equivalent to Einstein's field equations, another set equivalent to Maxwell's equations for electromagnetic field and the final part an extra scalar field now termed the "radian". In 1926, it was Swedish physicist Oskar Klein who focused on the resulting higher modes of the particles and the size of the extra dimension [7].

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three dimensions. It is possible that our world is "pinned" to a three-dimensional so-called brane located in a higher dimensional space. We could be restrained to a usual three-dimensional world, which is in fact a part of a more complicated multi-dimensional universe.[2] Perhaps, we could feel these extra dimensions through their effect on gravity. While the forces such as the electromagnetic, weak, and strong interactions that hold our world together are constrained to the $(3+1)$ dimensions, the gravitational interaction always occupies the entire universe, thus allowing it to feel the effects of extra dimensions. Unfortunately, since gravity is a very weak force and the radius of extra dimensions is tiny and as large as 1 mm so that the gravitational interaction between them becomes very weak. Until now, no evidence for extra dimensions was found from the high-energy particle accelerators experiments, but we cannot say that they do not exist at all. The search for extra dimensions is not over yet. On the contrary, it has only just started. Scientists have been looking for the effects of extra dimensions in collisions that produce different types of particles, such as quarks and searching events where gravitons are produced in the collisions and then leave our three-dimensional world, traveling off into one of the other dimensions [12].

We have noticed that almost all works about higher dimensional wave equations addressed the generalized orbit angular momentum [13–15], in which Louck studied the harmonic oscillator potential as an exactly solvable model. In fact, such a generalization should go back to the earlier works by Appel, Fock, Bargmann, Sommerfeld *et al.* [16–19], the notes left by Bateman edited by Erdélyi in 1950s [20] and others [21]. Most of them paid more attention to the harmonic oscillator [13–15, 22, 23] than hydrogen atom [24–29]. Following Louck's work, de Broglie and his collaborators [30] proposed the generating bases as the hyperspherical harmonics to analyze the higher dimensional harmonic oscillator and molecular vibration. They considered the rotator model of elementary particles as relativistic extended structures in Minkowski space under the assumption that elementary particles are not pointlike, but are rather, extended structures in Minkowski space. Two years later, Granzow presented orthogonal polar coordinate systems in $N$ dimensional space and showed explicit representations for total orbital angular momentum operator [31]. He also proved that the transformation from polar coordinates to Cartesian ones has a unique form $x^n = Rf^n(\theta)$, $n \in Z$; $\theta = (\theta^1, \theta^2, \ldots, \theta^{N-1})$, where $x^n$ could be interpreted as the wavefunction in quantum system. Based on the generalized orbital angular momentum theory, Bergmann and Frishman established the relation between the hydrogen atom and multidimensional harmonic oscillator by performing simple transformations on wave equations and wavefunctions [32]. Following this, Čížek and Paldus presented a relation between them for the special case of even dimensions [33]. Kostelecky, Nieto and Truax obtained a more general

[2]This is just like an insect crawling on a sheet of paper. For this insect, the universe is pretty much two-dimensional since it cannot leave the surface of that paper. As a result, the insect only knows the surface, but up and down does not make any sense as long as it has to stay on the sheet of that paper. These extra spatial dimensions, if they really exist, are thought to be curled-up, or "compactified".

mapping for arbitrary $d$ and even $D$ that involves a free parameter along with the corresponding mappings to the supersymmetric partners of these systems [34], in which they adopted the results about the $D$-dimensional oscillator with spin-orbit coupling obtained by Balantekin [35]. One decade later, Kostelecky and his collaborator Russell restudied this topic, but following the supersymmetry-based quantum defect theory [36]. Among the special cases is an injection from bound states of the three-dimensional radial Coulomb system into a three-dimensional radial isotropic oscillator where one of two systems has an analytical quantum defect. Also, they considered the issue of mapping the continuum states [36]. It should be pointed out that most of contributions about the relationship between the hydrogen atom and harmonic oscillator in $D$ dimensions are based on the transformation of the radial equations.

Closely related to this, however, Zeng, Su and Li have made use of algebraic method, i.e., an su$(1,1)$ algebra as a bridge to establish a most general and simplest relationship between their energy levels and eigenstates [37]. Similar to this, Lévai, Kónya and Papp proposed a unified su$(1,1)$ algebraic treatment to the Coulomb and harmonic oscillator potentials in $D$ dimensions by using Green's operator calculated from a Hilbert basis and the generalized Coulomb-Sturmian basis [38]. Except for these relations, it is noticed that there exist the degeneracies between the hydrogen atom and harmonic oscillator. For example, Shea and Aravind studied the degeneracies of the spherical well, harmonic oscillator and hydrogen atom in arbitrary dimensions from the view point of group theory [39]. In a similar way, Jafarizadeh, Kirchberg and their coauthors investigated the degeneracies of the Coulomb potential in higher dimensions $d$ by using the irreducible representations of the group SO$(d+1)$ [40, 41]. The reason why the harmonic oscillator and hydrogen atom are taken as typically soluble models is because their study represents an interesting field of mathematical physics in itself, but more importantly results from them are essential for the description of realistic physical problems.

Obviously, there are no more essential advances on the higher dimensional wave equations in 1970s. On the contrary, the study on this field has revived and attracted much attention to many authors in 1980s, e.g., the eigenvalues of the Schrödinger equation for spherically symmetric states for various types of potentials in $N$ dimensions by using perturbative and non-perturbative methods [42], the $1/N$ expansion technique for the Schrödinger equation [43–53], the generalized $D$-dimensional oscillator [54]. It should be noticed that the special case about the $1/N$ method was extended by Papp [55], who dealt with the $q$-deformed radial Schrödinger equation in $N$ dimensions through the underlying SO$(N)$ group realized in Refs. [56, 57] and opened a new way to derive $q$-deformed $1/N$-energy formulas for arbitrary spherically symmetrical potentials such as the harmonic oscillator and the Coulomb potential.

Except for these, the higher dimensional Schrödinger equation are also concerned with the following scattered fields such as the position and momentum information entropies of the $D$-dimensional harmonic oscillator and hydrogen atom [58], the Fermi pseudo-potential in arbitrary dimensions [59], the uncertainty relation for Fisher information of $D$-dimensional single-particle systems with central potentials [60], the dimensional expansion for the Ising limit of quantum field theory [61], the scalar Casimir effect for an $N$-dimensional sphere [62], the multidimensional extension of a WKB improvement for the spherical quantum billiard zeta functions [63], the study of bound states in continuous $D$ dimensions [64], the suppersymmetry and relationship between a class of singular potentials in arbitrary dimensions [65], the bound states and resonances for "sombrero" potential in arbitrary dimensions [66], the renormalization of the inverse squared potential in $D$ dimensions [67], the generalized coherent states for the $d$-dimensional Coulomb problem [68], the quantum particles trapped in a position-dependent mass barrier [69, 70], the harmonic oscillator in arbitrary dimensions with minimal length uncertainty relations [71], the stable hydrogen atom in higher dimensions [72], the relation between dimension and angular momentum for radially symmetric potential in $D$-dimensional space [73], the $D$-dimensional hydrogenic systems in position and momentum spaces [74], the first-order intertwining operators and position-dependent mass Schrödinger equation in $d$ dimensions [75], intertwined isospectral potentials in arbitrary dimensions [76], convergent iterative solutions for a sombrero-shaped potential in any space dimension and arbitrary angular momentum [77].

On the other hand, a number of contributions related to the higher dimensional Schrödinger equation have been carried out in atomic physics. For example, Hosoya investigated the hierarchical structure of the set of atomic orbital wavefunctions of $D$-dimensional atoms by using the set of their rectangular coordinate expressions [78]. In terms of group theory Dunn and Watson developed a formalism for the $N$ electron $D$-dimensional Schwartz expansion and applied it to study the Schrödinger equation for two-electron system [79, 80]. However, their method seems rather complicated. To overcome the difficulty occurred in [79, 80], Ma and his coauthors made use of the group theory method [81] to develop a different formalism to separate the $D$-dimensional rotational degrees of freedom from the internal degrees of freedom. They have studied quantum three-body system [82], interdimensional degeneracies for quantum three-body and $N$-body systems [83, 84], the quantum four-body system [85] and the $D$-dimensional helium atom [86].

As illustrated above, we find that most of contributions have been made to higher dimensional Schrödinger equation. In comparison with the non-relativistic Schrödinger equation case, undoubtedly the studies of relativistic Dirac and Klein-Gordon equations in higher dimensions seem less than those in the Schrödinger equation case. Nevertheless, there are considerable works appearing in the literature. For example, Nieto dealt with the hydrogen atom in arbitrary dimensions $D$ and particularly studied the Klein-Gordon equation case [87]. This might be the earliest contribution to the generalized Klein-Gordon equation, to our best knowledge. In fact, such a generalization can be easily achieved from the Schrödinger equation since the same Laplacian is involved for both equations. On the other hand, Joseph made a great contribution to self-adjoint ladder operators [88–90], in particular he applied this method to study the solutions of the generalized angular momentum problem. This revealed many interesting aspects of this approach to eigenvalue problems and specially its relationship to the addition of angular momentum. In that work, he obtained a complete set of irreducible unitary representations of the underlying algebra so($n$) and calculated the corresponding Clebsch-Gordon coefficients

(CGCs) for the addition of spin and angular momentum in arbitrary dimensions. Without doubt, this shall provide some useful preliminaries including the spin algebra and Dirac operators to study the Dirac equation in higher dimensions [91], which was derived by using the fundamental properties of symmetry group SO($N$). Unfortunately, we have not recognized his work [88–90] before our study [91]. In the middle of 1980s, Bollini and Giambiagi extended the Wess-Zumino model to higher dimensions, which led to a generalized Klein-Gordon equation [92]. In terms of the $1/N$ expansion technique, the relativistic Dirac and Klein-Gordon equations were performed [93–98]. Lin carried out the path integration of a relativistic particle on an $N$-dimensional sphere [99]. Recently, we have studied the Klein-Gordon equation with a Coulomb potential in $N$ dimensions by traditional approach [100]. Others related to the Klein-Gordon equation with the Kratzer and pseudoharmonic potential potentials as well as the comparison theorems for the Klein-Gordon equation [101–103] have also been studied.

For solvable higher dimensional wave equations, since the energy levels depend on the dimension $N$ and then bound state energy levels in different dimensions would be of interest. With this spirit we have studied the effect of dimension $N$ on the energy levels for some interesting and important quantum systems. For example, we have dealt with the higher dimensional Klein-Gordon equation case [100], the Dirac equation with a Coulomb potential [104], the $D$-dimensional relativistic equations with a Coulomb plus a scalar potential [105, 106], the $D$-dimensional Schrödinger equation with the pseudoharmonic potential and the Coulomb plus an inverse squared potential [107, 108]. On the other hand, we have established the Levinson theorem[3] for the Schrödinger equation and Dirac equation in $N$ dimensions [111, 112] and obtained the generalized hypervirial theorem [113, 114].

One of the reasons why the higher dimensional theories have attracted much attention to many authors is that the higher dimensional theories allow us to reduce enormous amounts of information into a concise, elegant fashion that unifies the two great theories of the 20th century: Quantum Theory and Relativity. It is evident to show that the contributions mentioned above are made within the framework of quantum theory. Consequently, it is necessary to review the development of the relativity and gravity in higher dimensional wave equations for completeness. For example, based on our recent work [91] Lin studied the Friedel sum rule, the Levinson theorem and the Atiyah-Singer index [115, 116]. Such method was also generalized to quantum modes of the scalar field on $\text{AdS}_{d+1}$ space-time [117] as well as geometric models of the $(d + 1)$-dimensional relativistic rotating oscillators [118]. More importantly, it should be noted that the generalization of the Dirac equation to higher dimensions might shed light on the solution of the Kaluza-Klein theory in higher dimensions if the extra dimensions are space-like. This theory has become a focus of attention for many particle physicists in past several decades. Its revival stems from the work on the string theory and also from

[3] It was first proposed by Levinson in 1949 [109] and reviewed by Ma [110]. The Levinson theorem establishes the relation between the number of the bound states and the phase shift of the scattering states at the zero momentum.

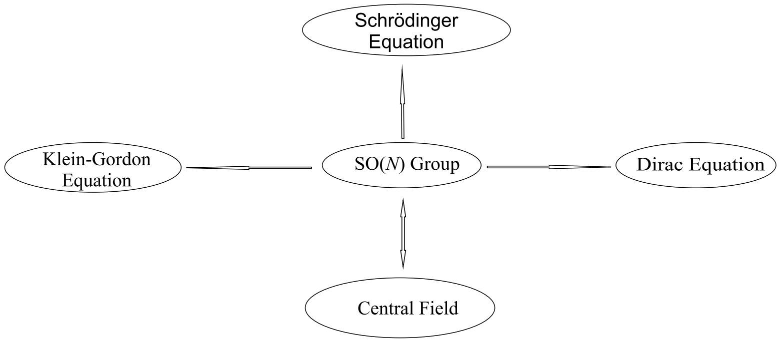


**Fig. 1.1** The relations among the SO($N$) group, central fields, non-relativistic and relativistic equations

the supergravity theory. Until now, the study of gravity theory and other relevant fields has become a main and interesting topic. These contributions can be summarized as follows: the brane models [119], scalar field contribution to rotating black hole entropy [120], brane cosmology [121], $N$-dimensional Vaidya metric with a cosmological constant in double-null coordinates [122], the spherical gravitational collapse in $N$ dimensions [123], the motion of a dipole in a cosmic string background [124], repulsive Casimir effect from extra dimensions and Robin boundary conditions [125], extremal black hole/CFT correspondence in gauged supergravity [126], massive fermion emission from higher dimensional black holes [127], magnetic and electric black holes [128], fermion families from two layer warped extra dimensions [129], quasinormal behavior of the $D$-dimensional Schwarzschild black hole [130], the study of the Schrödinger-Newton equations in $D$ dimensions [131], rotating Einstein-Maxwell-Dilaton black holes in $D$ dimensions [132], the Kaluza-Klein theory in the limit of large number of extra dimensions [133], gauge invariance of the one-loop effective potential in $(d+1)$-dimensional Kaluza-Klein theory [134] and the multicentered solution for maximally charged dilaton black holes in arbitrary dimensions [135].

Heretofore, it should be emphasized that the symmetry group SO($N$) for the symmetrically central fields plays an important role in higher dimensional wave equations. Therefore, we shall outline this group in next Chapter. The relations among those related topics are shown in Fig. 1.1.

## 2 Motivations and Aims

The motivations of this work are as follows. Since the literature related to this field is inundated with scattered articles on this topic we try to give a comprehensive review of the wave equations in higher dimensions and their wide-spread applications in quantum mechanics, which shall fill the gap in the existing quantum mechanics textbooks. In particular, we attempt to make use of fundamental properties of the rotational group SO($N$) to study the higher dimensional wave equations with symmetrically central fields. In this book, we are going to put the mathematical and physical concepts at the reader's disposal and to pave the reader's way into this territory as rapidly as possible.

# Part II
# Theory