Jean-Pierre Aubin · Alexandre M. Bayen
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Viability Theory

New Directions

Second Edition

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Chapter 1
Overview and Organization

Viability theory designs and develops mathematical and algorithmic methods for investigating the *adaptation to viability constraints of evolutions governed by complex systems under uncertainty* that are found in many domains involving living beings, from biological evolution to economics, from environmental sciences to financial markets, from control theory and robotics to cognitive sciences. It involves interdisciplinary investigations spanning fields that have traditionally developed in isolation.

The purpose of this book is to present an initiation to applications of viability theory, explaining and motivating the main concepts and illustrating them with numerous numerical examples taken from various fields.

Viability Theory. New Directions plays the role of a second edition of *Viability Theory*, [18, Aubin] (1991), presenting advances occurred in set-valued analysis and viability theory during the two decades following the publication of the series of monographs: *Differential Inclusions. Set-Valued Maps and Viability Theory*, [25, Aubin & Cellina] (1984), *Set-valued Analysis*, [27, Aubin & Frankowska] (1990), *Analyse qualitative*, [85, Dordan] (1995), *Neural Networks and Qualitative Physics: A Viability Approach*, [21, Aubin] (1996), *Dynamic Economic Theory: A Viability Approach*, [22, Aubin] (1997), *Mutational, Morphological Analysis: Tools for Shape Regulation and Morphogenesis*, [23, Aubin] (2000), *Mutational Analysis*, [150, Lorenz] (2010) and *Sustainable Management of Natural Resources*, [77, De Lara & Doyen] (2008).

The monograph *La mort du devin, l'émergence du démiurge. Essai sur la contingence et la viabilité des systèmes*, [24, Aubin] (2010), divulges vernacularly the motivations, concepts, theorems and applications found in this book. Its English version, *The Demise of the Seer, the Rise of the Demiurge. Essay on contingency, viability and inertia of systems*, is under preparation.

However, several issues presented in the first edition of *Viability Theory*, [18, Aubin] are not covered in this second edition for lack of room. They concern Haddad's viability theorems for functional differential inclusions where both the dynamics and the constraints depend on the history (or path) of

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the evolution and the Shi Shuzhong viability theorems dealing with partial differential evolution equation (of parabolic type) in Sobolev spaces, as well as fuzzy control systems and constraints, and, above all, differential (or dynamic) games. A sizable monograph on tychastic and stochastic viability and, for instance, their applications to finance, would be needed to deal with uncertainty issues where the actor has no power on the choice of the uncertain parameters, taking over the problems treated in this book in the worst case (tychastic approach) or in average (stochastic approach).

We have chosen an outline, which is increasing with respect to mathematical technical difficulty, relegating to the end the proofs of the main Viability and Invariance Theorems (see Chap. 19, p.769).

The proofs of the theorems presented in *Set-valued analysis* [27, Aubin & Frankowska] (1990) and in convex analysis (see *Optima and Equilibria*, [19, Aubin]), are not duplicated but referred to. An appendix, *Set-Valued Analysis at a Glance* (18, p. 713) provides without proofs the statements of the main results of set-valued analysis used in these monographs. The notations used in this book are summarised in its Sect. 18.1, p. 713.

1.1 Motivations

1.1.1 Chance and Necessity

The purpose of viability "theory" (in the sense of a sequence [*theôria*, procession] of mathematical tools sharing a common background, and not necessarily an attempt to explain something [*theôrein*, to observe]) is to attempt to answer directly the question of dynamic adaptation of uncertain evolutionary systems to environments defined by constraints, that we called viability constraints for obvious reasons. Hence the name of this body of mathematical results developed since the end of the 1970s that needed to forge a differential calculus of set-valued maps (set-valued analysis), differential inclusions and differential calculus in metric spaces (mutational analysis). These results, how imperfect they might be to answer this challenge, have at least been motivated by social and biological sciences, even though constrained and shaped by the mathematical training of their authors.

It is by now a consensus that the evolution of many variables describing systems, organizations, networks arising in biology and human and social sciences do not evolve in a deterministic way, not even always in a stochastic way as it is usually understood, but evolve with a Darwinian flavor.

Viability theory started in 1976 by translating mathematically the title

$$\begin{array}{ccc} \textit{Chance} & \textit{and} & \textit{Necessity} \\ \Updownarrow & & \Updownarrow \\ x'(t) \in F(x(t)) & \& & x(t) \in K \end{array}$$

of the famous 1973 book by *Jacques Monod*, *Chance and Necessity* (see [163, Monod]), taken from an (apocryphical?) quotation of Democritus who held that *"the whole universe is but the fruit of two qualities, chance and necessity"*.

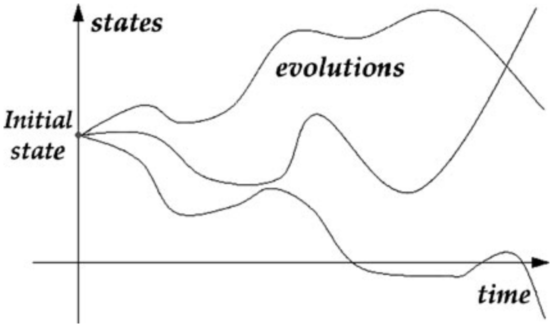

Fig. 1.1 The mathematical translation of "chance".
*The mathematical translation of "**chance**" is the differential inclusion $x'(t) \in F(x(t))$, which is a type of evolutionary engine (called an* evolutionary system*) associating with any initial state x the subset $\mathcal{S}(x)$ of evolutions starting at x and governed by the differential inclusion above. The figure displays evolutions starting from a give initial state, which are functions from time (in abscissas) to the state space (ordinates).*

The system is said to be *deterministic* if for any initial state x, $\mathcal{S}(x)$ is made of one and only one evolution, whereas "contingent uncertainty" happens when the subset $\mathcal{S}(x)$ of evolutions contains more than one evolution for at least one initial state. "Contingence is a non-necessity, it is a characteristic attribute of freedom", wrote *Gottfried Leibniz*.

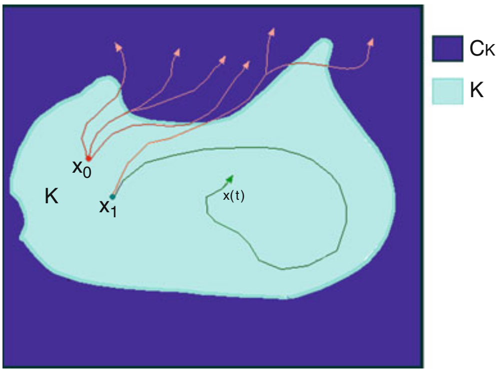

Fig. 1.2 The mathematical translation of "necessity".
*The mathematical translation of "**necessity**" is the requirement that for all $t \geq 0$, $x(t) \in K$, meaning that at each instant, "viability constraints" are*

satisfied by the state of the system. The figure represents the state space as the plane, and the environment defined as a subset. It shows two initial states, one, x_0 from which all evolutions violate the constraints in finite time, the other one x_1, from which starts one viable evolution and another one which is not viable.

One purpose of viability theory is to attempt to answer directly the question that some economists, biologists or engineers ask: *"Complex organizations, systems and networks, yes, but for what purpose?"* The answer we suggest: *"to adapt to the environment."*

This is the case in economics when we have to adapt to scarcity constraints, balances between supply and demand, and many other constraints.

This is also the case in biology, since Claude Bernard's "constance du milieu intérieur" and Walter Cannon's "homeostasis". This is naturally the case in ecology and environmental studies.

This is equally the case in control theory and, in particular, in robotics, when the state of the system must evolve while avoiding obstacles forever or until they reach a target.

In summary, *the environment is described by viability constraints* of various types, a word encompassing polysemous concepts as *stability, confinement, homeostasis, adaptation*, etc., expressing the idea that some variables must obey some constraints (representing physical, social, biological and economic constraints, etc.) that can never be violated. So, viability theory started as *the confrontation of evolutionary systems governing evolutions and viability constraints* that such evolutions must obey.

At the same time, controls, subsets of controls, in engineering, regulons (regulatory controls) such as prices, messages, coalitions of actors, connectionnist operators in biological and social sciences, which parameterize evolutionary systems, do evolve: *Their evolution must be consistent with the constraints, and the targets or objectives they must reach in finite or prescribed time.* The aim of viability theory is to provide the "regulation maps" associating with any state the (possibly empty) subset of controls or regulons governing viable evolutions.

Together with the selection of evolutions governed by teleological objectives, mathematically translated by intertemporal optimality criteria as in optimal control, viability theory offers other selection mechanisms by requiring evolutions to obey several forms of *viability* requirements. In social and biological sciences, intertemporal optimization can be replaced by *myopic, opportunistic, conservative and lazy* selection mechanisms of viable evolutions that involve present knowledge, sometimes the knowledge of the history (or the path) of the evolution, instead of anticipations or knowledge of the future (whenever the evolution of these systems cannot be reproduced experimentally). Other forms of uncertainty do not obey statistical laws, but also take into account unforeseeable rare events (tyches, or perturbations, disturbances) that must be avoided at all costs (precautionary principle[1]). These systems can be regulated by using regulation (or cybernetical) controls that have to be chosen as feedbacks for guaranteeing the viability of constraints and/or the capturability of targets and objectives, possibly against perturbations played by "Nature", which we call *tyches*.

However, there is no reason why collective constraints are satisfied at each instant by evolutions under uncertainty governed by evolutionary systems. This leads us to the study of *how to correct either the dynamics, and/or the constraints* in order to restore viability. This may allow us to provide an explanation of the formation and the evolution of controls and regulons through regulation or adjustment laws that can be designed (and computed) to insure viability, as well as other procedures, such as using *impulses* (evolutions with infinite velocity) governed by other systems, or by regulating the evolution of the environment.

Presented in such an evolutionary perspective, this approach of (complex) evolutionary systems departs from main stream modelling by a direct approach:

1 [**Direct Approach.**] It consists in studying properties of evolutions governed by an evolutionary system: gather the larger number of properties of evolutions starting from each initial state. It may be an information both costly and useless, since our human brains cannot handle simultaneously too many observations and concepts.

Moreover, it may happen that evolutions starting from a given initial state satisfy properties which are lost by evolutions starting from another initial state, even close to it (sensitivity analysis) or governed by (stability analysis).

Viability theory rather uses instead an *inverse approach*:

2 [**Inverse Approach.**] A set of prescribed properties of evolutions being given, study the (possibly empty) subsets of initial states from which

1. starts at least one evolution governed by the evolutionary system satisfying the prescribed properties,
2. all evolutions starting from it satisfy these prescribed properties.

These two subsets coincide whenever the evolutionary system is deterministic.

[1] Stating that one should limit, bound or even forbid potential dangerous actions, without waiting for a scientific proof of their hazardous consequences, whatever the economic cost.

Stationarity, periodicity and asymptotic behavior are examples of classical properties motivated by physical sciences which have been extensively studied.

We thus have to add to this list of classical properties other ones, such as concepts of *viability* of an environment, of *capturability* of a target in finite time, and of other concepts combining properties of this type.

1.1.2 Motivating Applications

For dealing with these issues, one needs *"dedicated"* concepts and formal tools, algorithms and mathematical techniques motivated by complex systems evolving under uncertainty. For instance, and without going into details, we can mention systems sharing common features:

1. ***Systems designed by human brains*** in the sense that agents, actors, decision-makers act on the evolutionary system, as in engineering. ***Control theory and differential games***, conveniently revisited, provide numerous metaphors and tools for grasping viability questions. Problems in *control design, stability, reachability, intertemporal optimality, tracking of evolutions, observability, identification and set-valued estimation*, etc., can be formulated in terms of viability and capturability concepts investigated in this book.
   Some technological systems such as robots of all types, from drones, unmanned underwater vehicles, etc., to animats (artificial animals, a contraction of anima-materials) need "embedded systems" implementations *autonomous* enough to regulate viability/capturability problems by adequate regulation (feedback) control laws. Viability theory provides algorithms for computing the feedback laws by modular and portable software flexible enough for integrating new problems when they appear (hybrid systems, dynamical games, etc.).
2. ***Systems observed by human brains***, are more difficult to understand since human beings did not design or construct them. Human beings live, think, are involved in socio-economic interactions, but struggle for grasping why and how they do it, at least, why. This happens for instance in the following fields:

   - ***economics***, where the viability constraints are the scarcity constraints among many other ones. We can replace the fundamental Walrasian model of resource allocations by decentralized dynamical model in which the role of the controls is played by the prices or other economic decentralizing messages (as well as coalitions of consumers, interest rates, and so forth). The regulation law can be interpreted as the behavior of Adam Smith's invisible hand choosing the prices as a function of allocations of commodities,

- ***finance***, where shares of assets of a portfolio play the role of controls for guaranteing that the values of the portfolio remains above a given time/price dependent function at each instant until the exercise time (horizon), whatever the prices and their growth rates taken above evolving bounds,
- ***dynamical connectionnist networks and/or dynamical cooperative games***, where coalitions of players may play the role of controls: each coalition acts on the environment by changing it through dynamical systems. The viability constraints are given by the architecture of the network allowed to evolve,
- ***Population genetics***, where the viability constraints are the ecological constraints, the state describes the phenotype and the controls are genotypes or fitness matrices.
- ***sociological sciences***, where a society can be interpreted as a set of individuals subjected to viability constraints. Such constraints correspond to what is necessary for the survival of the social organization. Laws and other cultural codes are then devised to provide each individual with psychological and economical means of survival as well as guidelines for avoiding conflicts. Subsets of cultural codes (regarded as cultures) play the role of regulation parameters.
- ***cognitive sciences***, in which, at least at one level of investigation, the variables describe the sensory-motor activities of the cognitive system, while the controls translate into what could be called conceptual controls (which are the synaptic matrices in neural networks.)

Theoretical results about the ways of thinking described above are useful for the understanding of non teleological evolutions, of the inertia principle, of the emergence of new regulons when viability is at stakes, of the role of different types of uncertainties (contingent, tychastic or stochastic), of the (re)designing of regulatory institutions (regulated markets when political convention must exist for global purpose, mediation or metamediation of all types, including law, social conflicts, institutions for sustainable development, etc.). And progressively, when more data gathered by these institutions will be available, qualitative (and sometimes quantitative) prescriptions of viability theory may be useful.

1.1.3 Motivations of Viability Theory from Living Systems

Are social and biological systems sufficiently similar to systems currently studied in mathematics, physics, computer sciences or engineering? *Eugene Wigner*'s considerations on *the unreasonable effectiveness of mathematics in the natural sciences* [215, Wigner] are even more relevant in life sciences.

For many centuries, human minds used their potential "mathematical capabilities" to describe and share their "mathematical perceptions" of the world. This mathematical capability of human brains is assumed to be analogous to the language capability. Each child coming to this world uses this specific capability in social interaction with other people to join at each instant an (evolving) consensus on the perception of their world by learning their mother tongue (and few others before this capability fades away with age). We suggest the same phenomenon happens with mathematics. They play the "mathematical role" of *metaphors* that language uses for allowing us to understand a new phenomenon by metaphors comparing it with previously "understood phenomena". Before it exploded recently in a Babel skyscraper, this "mathematical father tongue" was quite consensual and perceived as universal. This is this very universality which makes mathematics so fascinating, deriving mathematical theories or tools motivated by one field to apply them to several other ones. However, apparently, because up to now, the mathematical "father tongue" was mainly shaped by "simple" physical problems of the inert part of the environment, letting aside, with few exceptions, the living world. For good reasons. Fundamental simple principles, such as the *Pierre de Fermat*'s "variational principle", including *Isaac Newton*'s law thanks to *Maupertuis*'s least action principle, derived explanations of complex phenomena from simple principles, as *Ockham*'s razor prescribes: This "law of parsimony" states that an explanation of any phenomenon should make as few assumptions as possible, and to choose among competing theories the one that postulates the fewest concepts. This is the result of an "abstraction process", which is the (poor) capability of human brains that select among the perceptions of the world the few ones from which they may derive logically or mathematically many other ones. *Simplifying complexity* should be the purpose of an emerging science of complexity, if such a science will emerge beyond its present fashionable status.

So physics, which could be defined as the part of the cultural and physical environment which is understandable by mathematical metaphors, has not yet, in our opinion, encapsulated the mathematical metaphors of living systems, from organic molecules to social systems, made of human brains controlling social activities. The reason seems to be that the adequate mathematical tongue does not yet exist. And the challenge is that before creating it, the present one has to be forgotten, de-constructed. This is quite impossible because mathematicians have been educated *in the same way* all over the world, depriving mathematics from the Darwinian evolution which has operated on languages. This uniformity is the strength and the weakness of present day mathematics: its universality is partial. The only possibility to mathematically perceive living systems would remain a dream: to gather in secluded convents young children with good mathematical capability, but little training in the present mathematics, under the supervision or guidance of economists or biologists without mathematical training. They possibly could come up with new mathematical languages unknown to us