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# Contents

# Preface

*Nanotechnology* is a key technology of the 21st century. It investigates very small structures of the size of a few nanometers up to several 100 nanometers. Thus, these structures are often smaller then the wave length of light. In this book, we concentrate on the mathematics of photonic crystals, which form an important class of physical structures investigated in nanotechnology.

The investigation of these structures by mathematical methods is highly important for the following reasons.

- Since the physical behaviour in the nanoscale is very difficult and expensive to measure in real experiments, numerical simulations play a fundamental role in understanding such processes. In many cases theses structures are fully three dimensional, and process on very different scales in space and time, so that constructing efficient and reliable simulation methods is a true mathematical challenge.
- Often ill-posed problems arise in a natural way, e. g., in the reconstruction problem of nanostructures from measurements. Here, methods of the theory of inverse problems must be developed and applied.
- Often it is not possible, e. g. due to very large differences in the underlying scales, to consider the full basic physical equations directly in a numerical simulation. Then, reduced and simplified models have to be constructed, and their analytical properties and approximation qualities have to be investigated.
- The numerical simulation can study only specific configurations. For a qualitative understanding of the behaviour of the underlying system, the mathematical analysis of the underlying equations is indispensable.

In the mathematical analysis and the numerical appromixation of the partial differential equations describing nanostructures, several mathematical difficulties arise, e. g., the appropriate treatment of nonlinearities, simultaneous occurrence of continuous and discrete spectrum, multiple scales in space and time, and the ill-posedness of these problems.

Photonic crystals are materials which are composed of two or more different dielectrics or metals, and which exhibit a spatially periodic structure, typically at the length scale of hundred nanometers. Photonic crystals can be fabricated using processes such as photolithography or vertical deposition methods. They also occur in nature, e. g. in the microscopic structure of certain bird feathers, butterfly wings, or beetle shells.

A characteristic feature of photonic crystals is that they strongly affect the propagation of light waves at certain optical frequencies. This is due to the fact that the optical density inside a photonic crystal varies periodically at the length scale of about 400 to 800 nanometers, i. e., precisely at the scale of the wavelengths of optical light waves. Light waves that penetrate a photonic crystal are

therefore subject to periodic, multiple diffraction, which leads to coherent wave interference inside the crystal. Depending on the frequency of the incident light wave this interference can either be constructive or destructive. In the latter case the light wave is not able to propagate inside the photonic crystal. Typically, this phenomenon only occurs for bounded ranges of optical wave frequencies, if it does occur at all. Such a range of inhibited wave frequencies is called a photonic band gap. Light waves with frequencies inside a photonic band gap are totally reflected by the photonic crystal. It is this effect which causes, e. g., the iridescent colours of peacock feathers.

In the mathematical modeling of photonic crystals by Maxwell's equations with periodic permittivity, such photonic band gaps are described as gaps in the spectrum of a selfadjoint operator with periodic coefficients, while the frequency ranges where constructive interference takes place form the spectrum (which is arranged in bands) of this selfadjoint operator.

In this proceedings volume we collect a series of lectures which introduce into the mathematical background needed for the modeling and simulation of light, in particular in periodic media, and for its applications in optical devices. We start with an introduction to Maxwell's equations, which build the basis for the mathematical description of all electro-magnetic phenomena, and thus in particular of optical waves. Next, we focus on explicit methods for the numerical computation of photonic band gaps. Furthermore, a general introduction to the so-called Floquet–Bloch theory is given, which provides analytical tools to investigate the spectrum of periodic differential operators and provides the aforementioned band-gap structure of selfadjoint operators with periodic coefficients, such as they occur for Maxwell's equations in a photonic crystal. In the rest of this volume we consider two applications. In the first application the theory of direct and inverse scattering is introduced and applied to periodic media, and the second application investigates nonlinear optical effects in wave guides which can be described by the nonlinear Schrödinger equation.

We greatly appreciate the opportunity to give this course in the framework of the Oberwolfach seminars, and we would like to thank the Oberwolfach institute for their kind hospitality. Further, we want to thank for the kind assistance of Birkhäuser in the realisation of these lecture notes.

# Chapter 1

# Introduction

*by Willy Dörfler*

In this introduction we will present both the main physical concepts of electromagnetism (Section 1.1.1) and mathematical basic tools (Section 1.2) that are needed for the topics discussed here.

## 1.1 The Maxwell Equations

The foundations of electromagnetism were laid by James Clerk Maxwell [1] in the years 1861–1865. It not only provided insight into the nature of electromagnetism, but also gave the theory a clear mathematical structure. It consists of a system of partial differential equations and some constitutive laws that describe the interaction of the fields with matter. Here, we mainly consider waves of a fixed frequency (in the range of light) or pulses with concentrated frequencies. This leads to the simplification of the time-harmonic equations. On idealised periodic material arrangements one ends up with linear (and nonlinear) eigenvalue equations. Using results from spectral theory it is possible to understand the structure of the (linear) eigenvalue problems qualitatively.

### 1.1.1 The partial differential equations

We seek vector fields $\boldsymbol{D}, \boldsymbol{E}, \boldsymbol{B}, \boldsymbol{H} : \Omega \to \mathbb{R}^3$ such that *Maxwell equations*

$$\partial_t \boldsymbol{D} - \boldsymbol{\nabla} \times \boldsymbol{H} = -\boldsymbol{J} \qquad \textit{(Ampère's circuital law)}, \tag{1.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \varrho \qquad \textit{(Gauss's law)}, \tag{1.2}$$

$$\partial_t \boldsymbol{B} + \boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0} \qquad \textit{(Faraday's law of induction)}, \tag{1.3}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad \textit{(Gauss's law for magnetism)}, \tag{1.4}$$

[1] 13 June 1831 – 5 November 1879

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hold for given $\boldsymbol{J} : \Omega \to \mathbb{R}^3$ and $\varrho : \Omega \to \mathbb{R}$. We call $\boldsymbol{E}$ *electric field*, $\boldsymbol{D}$ *electric displacement field* (or *electric flux density*), $\boldsymbol{H}$ *magnetic field intensity*, $\boldsymbol{B}$ *magnetic induction*, and $\boldsymbol{J}$ and $\varrho$ *electric current intensity* and *electric charge density*, respectively. Furthermore, there are *constitutive relations* $\boldsymbol{D}(\boldsymbol{E}, \boldsymbol{H})$ and $\boldsymbol{B}(\boldsymbol{E}, \boldsymbol{H})$ that describe the interaction of the fields with matter. Examples are given in Section 1.1.3. This yields 14 scalar equations for 12 unknown functions. From the above equations we can derive the compatibility condition

$$\partial_t \varrho + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0,$$

which expresses *conservation of charge*. For a physical interpretation of equations (1.1)–(1.4) see [17, Ch. 1].

### 1.1.2 The integral formulation

Let $M$ be a smooth oriented two-dimensional manifold in $\mathbb{R}^3$ and $V \subset \mathbb{R}^3$ a bounded domain with smooth boundary. Using the well-known integral relations for sufficiently smooth vector fields $\boldsymbol{A} : V \to \mathbb{R}^3$, resp. $\boldsymbol{A} : M \to \mathbb{R}^3$,

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{A} = \int_{\partial V} \boldsymbol{A} \cdot \boldsymbol{n}_V \quad (\textit{Gauss' theorem}),$$
$$\int_M (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \boldsymbol{n}_M = \oint_{\partial M} \boldsymbol{A} \cdot \boldsymbol{t}_M \quad (\textit{Stokes' theorem}),$$

we can derive the following identities from (1.1)–(1.4)

$$\oint_{\partial M} \boldsymbol{H} \cdot \boldsymbol{t}_M = \int_M \big(\partial_t \boldsymbol{D} + \boldsymbol{J}\big) \cdot \boldsymbol{n}_M, \qquad \oint_{\partial V} \boldsymbol{D} \cdot \boldsymbol{n}_V = \int_V \varrho, \tag{1.5}$$
$$\oint_{\partial M} \boldsymbol{E} \cdot \boldsymbol{t}_M = -\int_M \partial_t \boldsymbol{B} \cdot \boldsymbol{n}_M, \qquad \oint_{\partial V} \boldsymbol{B} \cdot \boldsymbol{n}_V = 0. \tag{1.6}$$

Here, $\boldsymbol{n}_M$ is the normal vector field on $M$, $\boldsymbol{t}_M$ is the tangential vector field on $\partial M$ (oriented counter-clockwise if one looks into the direction of $\boldsymbol{n}_M$), and $\boldsymbol{n}_V$ the exterior normal to $V$.

### 1.1.3 Constitutive relations

In vacuum there hold the relations

$$\boldsymbol{D}(t, \boldsymbol{x}) = \varepsilon_0 \boldsymbol{E}(t, \boldsymbol{x}), \quad \boldsymbol{B}(t, \boldsymbol{x}) = \mu_0 \boldsymbol{H}(t, \boldsymbol{x})$$

with the *permittivity of free space* $\varepsilon_0$ and the *permeability of free space* $\mu_0$. In matter, however, the fields have to be interpreted in a macroscopic way as a mean field. An electric field $\boldsymbol{E}$ induces local dipoles in nonconducting media by

dislocation of charges (as sketched in Figure 1.1). This gives rise to a *polarisation field* $\boldsymbol{P}$, that superposes the electric field and results in a total field

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}.$$

The connection between $\boldsymbol{E}$ und $\boldsymbol{P}$ can be described by a material model. For a *linear medium* one makes the ansatz

$$\boldsymbol{P}(t, \boldsymbol{x}) = \varepsilon_0 (\boldsymbol{G} * \boldsymbol{E})(t, \boldsymbol{x}) := \varepsilon_0 \int_{-\infty}^{\infty} \boldsymbol{G}(t - \tau, \boldsymbol{x}) \boldsymbol{E}(\tau, \boldsymbol{x})\, d\tau. \tag{1.7}$$


Figure 1.1: Dislocation of local charges by an exterior electric field $\boldsymbol{E}$ and the polarisation field $\boldsymbol{P}$ (without (left) and with (right) exterior electrical field).

The *transfer function* $\boldsymbol{G}(\cdot, \boldsymbol{x}) : \mathbb{R} \to \mathbb{R}^{3,3}$ is assumed to be *causal*, which means that $\boldsymbol{P}(t, .)$ can only depend on $\boldsymbol{E}(\tau, .)$ on past times $\tau \leq t$. This is expressed by the requirement

$$\boldsymbol{G}(s, \boldsymbol{x}) = \boldsymbol{0} \qquad \text{for all } s < 0 \text{ and all } \boldsymbol{x} \in \mathbb{R}^3.$$

Special cases are *homogeneous* media, where $\boldsymbol{G}$ does not depend on $\boldsymbol{x}$ and *isotropic* media, where $\boldsymbol{G}$ is real valued. If we perform a *Fourier transformation* in time, e. g.,

$$\widehat{\boldsymbol{D}}(\omega, \boldsymbol{x}) = \int_{\mathbb{R}^3} \boldsymbol{D}(t, \boldsymbol{x}) e^{-\mathrm{i}\omega t}\, dt$$

(and likewise for $\boldsymbol{E}, \boldsymbol{P}$), the convolution will turn into

$$\widehat{\boldsymbol{P}}(\omega, \boldsymbol{x}) = \varepsilon_0 \boldsymbol{\chi}_{\mathrm{E}}(\omega, \boldsymbol{x}) \widehat{\boldsymbol{E}}(\omega, \boldsymbol{x})$$

with the *electric susceptibility*

$$\boldsymbol{\chi}_{\mathrm{E}}(\omega, \boldsymbol{x}) = \frac{1}{\epsilon_0} \widehat{\boldsymbol{G}}(\omega, \boldsymbol{x}).$$

Thus we obtain the relation

$$\begin{aligned} \widehat{\boldsymbol{D}}(\omega, \boldsymbol{x}) &= \varepsilon_0 \big(1 + \boldsymbol{\chi}_{\mathrm{E}}(\omega, \boldsymbol{x})\big) \widehat{\boldsymbol{E}}(\omega, \boldsymbol{x}) \\ &\equiv \varepsilon_0 \boldsymbol{\varepsilon}_{\mathrm{r}}(\omega, \boldsymbol{x}) \widehat{\boldsymbol{E}}(\omega, \boldsymbol{x}). \end{aligned} \tag{1.8}$$

Note that a relation $\boldsymbol{D} = \varepsilon_0 \boldsymbol{\varepsilon}_\mathrm{r} \boldsymbol{E}$ will only follow if the material parameter $\boldsymbol{\varepsilon}_\mathrm{r}$ is frequency independent, or, if the relation is considered at a single frequency $\omega_0$. To see this, multiply (1.8) by the Dirac measure $\delta_{\omega_0}$ and apply the inverse Fourier transform. This independency is not given in reality, however, it often holds approximately on large frequency intervals. Linear relations $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$, $\boldsymbol{M}$ the *magnetisation*, and $\boldsymbol{B} = \mu_0 \boldsymbol{\mu}_\mathrm{r} \boldsymbol{H}$ follows from analogous reasoning.

Usually, $\boldsymbol{\varepsilon}_\mathrm{r}$ is taken to be real. This can only be an approximation, since $\boldsymbol{\varepsilon}_\mathrm{r}$ *must* have an imaginary part, except in the vacuum case, by the *Kramers–Kronig relation*, that relates $\Im(\boldsymbol{\varepsilon}_\mathrm{r})$ to $\Re(\boldsymbol{\varepsilon}_\mathrm{r}) - 1$ [17, Ch. 7.10].

In the following, we will use the linear constitutive relations

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} = \varepsilon_0 \varepsilon_\mathrm{r} \boldsymbol{E}, \tag{1.9}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} = \mu_0 \mu_\mathrm{r} \boldsymbol{H} \tag{1.10}$$

(here with possibly space-dependent but time-independent scalar functions $\varepsilon, \mu$). $\varepsilon, \mu$ ($\varepsilon_\mathrm{r}, \mu_\mathrm{r}$) are called *(relative) permittivity* and *(relative) permeability*, respectively.

The interaction of electromagnetic fields with media is a rich and still developing topic. We mention only a few models.

**The homogeneous, isotropic one–resonance model**

Here it is assumed that the mentioned local dipoles are oscillating systems at resonance frequency $\omega_0$, that oscillate in the direction of the electric field. In this model the equation for $\boldsymbol{P}$ is that of a *damped harmonic oscillator* and can be solved explicitly. As a result one obtains the *Drude–Lorentz model*

$$\chi_\mathrm{E}(\omega) = \frac{1}{\epsilon_0} \frac{\omega_\mathrm{p}^2}{\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega},$$

where $\omega_\mathrm{p}^2$ and $\gamma$ are material constants named *plasma frequency* and *damping factor*, respectively [17, Ch. 7.5]. The dependence of $\epsilon$ on $\omega$ is especially important for metallic materials. Here, one often uses the expression with $\omega_0 := 0$ (*Drude model*).

**A quantum mechanical model**

In a medium or an a scale where quantum effects become apparent the polarisation will depend on a *density matrix* $\boldsymbol{\varrho} \in \mathbb{C}^{N,N}$ with a number $N$ of energy levels. $\boldsymbol{\varrho}$ is hermitian and nonnegative with positive diagonal entries. Then one finds a dependence $\boldsymbol{P}(\boldsymbol{E}, \boldsymbol{\varrho})$ and $\boldsymbol{\varrho}$ satisfies an equation of the form $\partial_t \boldsymbol{\varrho} = \boldsymbol{L}(\boldsymbol{E}, \boldsymbol{\varrho})$, where $\boldsymbol{L}$ is affine linear in both arguments. The final set of (nonlinear) equations is called *Maxwell–Bloch equations*. In case of $N = 2$ (*two level model*) one arrives

for example at equations

$$\partial_t^2 \boldsymbol{P} + \alpha_1 \partial_t \boldsymbol{P} + \omega^2 \boldsymbol{P} = \alpha_2 \varrho \boldsymbol{E},$$
$$\partial_t \varrho + \alpha_3(\varrho - \varrho_0) = -\alpha_4 \partial_t \boldsymbol{P} \cdot \boldsymbol{E}.$$

$\alpha_1, \ldots, \alpha_4$ are positive constants and $\varrho$ is the difference in number between excited states and ground states per unit volume [22]. For mathematical investigations see [8] [9].

**Effective medium approximations**

These are methods to describe the effective permittivity or effective permeability of composed materials when the material structures are very small, like the *Bruggeman model* [4] or the *Maxwell–Garnett model* [15]. Especially newly designed materials at nano-scale with electric and magnetic effects can result in effective media with unusual optical properties (*meta materials*) [23].

**Nonlinear polarisation**

The previous approach (1.7) can be generalised to higher polynomial dependence on $\boldsymbol{E}$ [1]. Of special importance is the cubic (*Kerr–*) nonlinearity

$$\boldsymbol{P} = \varepsilon_0(\varepsilon_\mathrm{r} + \alpha|\boldsymbol{E}|^2)\boldsymbol{E}$$

(with some parameter $\alpha \in \mathbb{R}$). Equations with cubic nonlinearities will be studied in Chapter 5.

### 1.1.4 The wave equation

We consider the case of an electromagnetic field in the absence of charges and currents ($\varrho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$) and in the presence of a material with constitutive relations $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$ and (1.10). From the Maxwell equations (1.1) and (1.3) we get

$$\partial_t^2(\varepsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \partial_t^2 \boldsymbol{D} = \partial_t(\boldsymbol{\nabla} \times \boldsymbol{H}) = -\boldsymbol{\nabla} \times \Big(\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}\Big)$$

or

$$\varepsilon_0 \partial_t^2 \Big(\boldsymbol{E} + \frac{1}{\varepsilon_0}\boldsymbol{P}\Big) + \boldsymbol{\nabla} \times \Big(\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}\Big) = \boldsymbol{0}.$$

Using the linear relations (1.8) and (1.9), we find $\boldsymbol{E} + 1/\varepsilon_0 \, \boldsymbol{P} = \varepsilon \boldsymbol{E}$ and therefore

$$\varepsilon \partial_t^2 \boldsymbol{E} + \boldsymbol{\nabla} \times \Big(\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}\Big) = \boldsymbol{0}.$$

In vacuum, we have $\varepsilon = \varepsilon_0$, $\mu = \mu_0$. With $0 = \boldsymbol{\nabla} \cdot (\varepsilon_0 \boldsymbol{E}) = \varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}$ we observe that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \Delta \boldsymbol{E} = -\Delta \boldsymbol{E}$$

and thus obtain the *wave equation*

$$\frac{1}{c^2}\partial_t^2 \boldsymbol{E} - \Delta \boldsymbol{E} = \boldsymbol{0}, \tag{1.11}$$

where $c^2 = 1/(\varepsilon_0\mu_0)$ is the *speed of light*. An analogous equation holds for $\boldsymbol{H}$. Note, that the wave equation has solutions of the form $\boldsymbol{E}(t,\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x} \pm ct\boldsymbol{n})$ for some suitable function $\boldsymbol{A} : \mathbb{R}^3 \to \mathbb{R}^3$ and a unit vector $\boldsymbol{n}$. This models the propagation of light in vacuum.

### 1.1.5 Time–harmonic Maxwell equations

In case of a monochromatic wave we can assume that all fields are of the form $\boldsymbol{E}(\boldsymbol{x})e^{\mathrm{i}\omega t}$, etc. With $\varrho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$ and the linear relations $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ and $\boldsymbol{B} = \mu\boldsymbol{H}$, the Maxwell equations (1.1)–(1.4) turn into the *time-harmonic Maxwell equations*

$$\mathrm{i}\omega\varepsilon\boldsymbol{E} - \boldsymbol{\nabla}\times\boldsymbol{H} = \boldsymbol{0}, \tag{1.12}$$
$$\boldsymbol{\nabla}\cdot(\varepsilon\boldsymbol{E}) = 0, \tag{1.13}$$
$$\mathrm{i}\omega\mu\boldsymbol{H} + \boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0}, \tag{1.14}$$
$$\boldsymbol{\nabla}\cdot(\mu\boldsymbol{H}) = 0. \tag{1.15}$$

Elimination of $\boldsymbol{E}$ yields

$$\boldsymbol{\nabla}\times\Big(\frac{1}{\varepsilon}\boldsymbol{\nabla}\times\boldsymbol{H}\Big) = \boldsymbol{\nabla}\times(\mathrm{i}\omega\boldsymbol{E}) = \mathrm{i}\omega(-\mathrm{i}\omega)\mu\boldsymbol{H} = \omega^2\mu\boldsymbol{H}, \tag{1.16}$$

while elimination of $\boldsymbol{H}$ gives analogously

$$\boldsymbol{\nabla}\times\Big(\frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{E}\Big) = \omega^2\varepsilon\boldsymbol{E}. \tag{1.17}$$

We thus finally arrive at *eigenvalue problems* for $\boldsymbol{H}$ and $\boldsymbol{E}$. It means that, except in cases where the frequency $\omega$ (actually, $\omega^2$) is an eigenvalue of this equation, there exists no nontrivial electric field and thus no wave is transmitted. Note that a solution of one of this problems will determine a solution of the other, using the first order equations (1.12) and (1.14) above. Note further that (1.16) and (1.17) have to be accompanied by (1.13) and (1.15), respectively, but formally, the ladder follow as a consequence from the previous by taking the divergence and using $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times = 0$.

In the following we study this problem in special geometric configurations for illustration.

#### 1d–like structures

Assume that we have a layered structure in $x_1$–direction, with material properties $x_1 \mapsto \varepsilon(x_1)$ and $x_1 \mapsto \mu(x_1)$ (see Figure 1.2 (left)). We especially seek an electric

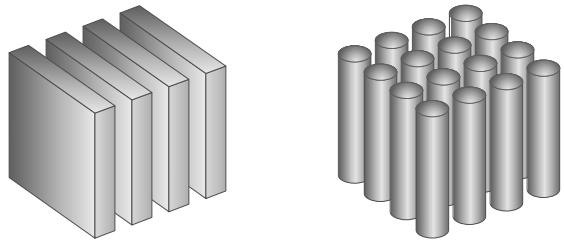

Figure 1.2: Examples of a one-dimensional (left) and two-dimensional (right) structure.

field of the form $x_1 \mapsto \boldsymbol{E}(x_1)$. As an example, we think of an idealisation of a device that consists of plane layers of material with alternating constant permittivity and that we look for nonvanishing standing waves $\boldsymbol{E}(x_1)e^{\mathrm{i}\omega t}$ for $\omega \neq 0$. Note that in this case

$$\boldsymbol{\nabla} \times \boldsymbol{E} = \begin{bmatrix} 0 \\ -\partial_1 E_3 \\ \partial_1 E_2 \end{bmatrix}$$

and therefore

$$\boldsymbol{\nabla} \times \left( \frac{1}{\mu} \begin{bmatrix} 0 \\ -\partial_1 E_3 \\ \partial_1 E_2 \end{bmatrix} \right) = - \begin{bmatrix} 0 \\ \partial_1 \left( \frac{1}{\mu} \partial_1 E_2 \right) \\ \partial_1 \left( \frac{1}{\mu} \partial_1 E_3 \right) \end{bmatrix}.$$

As a consequence we obtain from (1.17) the decoupled system

$$\begin{aligned} E_1 &= 0, \\ -\partial_1 \Big( \frac{1}{\mu} \partial_1 E_j \Big) &= \omega^2 \varepsilon E_j \quad \text{for } j = 2, 3. \end{aligned}$$

The equation for $\boldsymbol{H}$ is analogous with $\varepsilon$ and $\mu$ exchanged. As a result, we obtain standard elliptic eigenvalue problems for $E_2$ and $E_3$.

**2d–like structures (waveguide)**

Consider now a material that consists of infinite columns in $x_3$–direction modeled by a permittivity $(x_1, x_2) \mapsto \varepsilon(x_1, x_2)$ (see Figure 1.2 (right)), and we look for time-harmonic solutions that are also periodic in $x_3$–direction along the structure. Thus we seek fields $(t, x_1, x_2, x_3) \mapsto \boldsymbol{E}(x_1, x_2)e^{\mathrm{i}k_3 x_3}e^{\mathrm{i}\omega t}$ and $(t, x_1, x_2, x_3) \mapsto \boldsymbol{H}(x_1, x_2)e^{\mathrm{i}k_3 x_3}e^{\mathrm{i}\omega t}$ for $\omega \neq 0$. This corresponds to a wave with frequency $\omega$ that travels in $x_3$–direction with wavenumber $k_3$. This structure is called a *waveguide* if nontrivial fields of this kind exist. For such a vector field $\boldsymbol{E}$ we obtain

$$\boldsymbol{\nabla} \times (\boldsymbol{E} e^{\mathrm{i}k_3 x_3}) = \begin{bmatrix} \boldsymbol{\nabla}_{1,2}^{\perp} E_3 - \mathrm{i}k_3 [E_1; E_2]^{\perp} \\ -\boldsymbol{\nabla}_{1,2}^{\perp} \cdot [E_1; E_2] \end{bmatrix} e^{\mathrm{i}k_3 x_3},$$

where $\boldsymbol{\nabla}_{1,2} := [\partial_1; \partial_2]$, $\boldsymbol{\nabla}_{1,2}^{\perp} := [\partial_2; -\partial_1]$, and $[E_1; E_2]^{\perp} = [E_2; -E_1]$. If we insert this into (1.17), we get a system that can be separated as

$$-\boldsymbol{\nabla}_{1,2}^{\perp}\Big(\frac{1}{\mu}\boldsymbol{\nabla}_{1,2}^{\perp}\cdot\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}\Big) + \mathrm{i}\frac{1}{\mu}k_3\boldsymbol{\nabla}_{1,2}E_3 + \frac{1}{\mu}k_3^2\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \omega^2\varepsilon\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad (1.18)$$

$$-\boldsymbol{\nabla}_{1,2}\cdot\Big(\frac{1}{\mu}\boldsymbol{\nabla}_{1,2}E_3\Big) + \mathrm{i}\frac{1}{\mu}k_3\boldsymbol{\nabla}_{1,2}\cdot\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \omega^2\varepsilon \quad E_3. \quad (1.19)$$

For given $k_3$ this is an eigenvalue problem for $\omega$, while for given $\omega$ this is a quadratic eigenvalue problem for $k_3$ (for such problems see [25]). In case of $k_3 = 0$ (standing wave) this system decouples as

$$-\boldsymbol{\nabla}_{1,2}^{\perp}\left(\frac{1}{\mu}\boldsymbol{\nabla}_{1,2}^{\perp}\cdot\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}\right) = \omega^2\varepsilon\begin{bmatrix} E_1 \\ E_2 \end{bmatrix},$$

$$-\boldsymbol{\nabla}_{1,2}\cdot\Big(\frac{1}{\mu}\boldsymbol{\nabla}_{1,2}E_3\Big) = \omega^2\varepsilon E_3.$$

The second equation is an elliptic eigenvalue problem for $(E_3, \omega)$. Correspondingly, one can also obtain $(H_3, \omega)$ by the same type of equation (but with $\mu$ and $\varepsilon$ interchanged). Having determined in this way $E_3$ and $H_3$, we derive equations for the remaining components from Maxwell's first order equations

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \frac{1}{\mathrm{i}\omega\varepsilon}\boldsymbol{\nabla}_{1,2}^{\perp}H_3, \qquad \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = -\frac{1}{\mathrm{i}\omega\mu}\boldsymbol{\nabla}_{1,2}^{\perp}E_3.$$

Waves that are described by the components $E_1, E_2, H_3$ are called *transverse-electric* (TE) and those described by the components $H_1, H_2, E_3$ are called *transverse-magnetic* (TM). If $\mu$ and $\varepsilon$ are constant, then (1.18)–(1.19) simplifies to the eigenvalue problem

$$-\Delta_{1,2}\boldsymbol{E} = \big(\omega^2\varepsilon\mu - k_3^2\big)\boldsymbol{E}.$$

However, this result could easier be obtained from (1.11).

We have seen that problems with 1D or 2D structures can be solved using an eigenvalue problem for the Laplace-operator. The *off-plane* problem ($k_3 \neq 0$) however leads to the more involved equation (1.18). Also, full 3D problems will in general be of a more complex type.

The construction of waveguides is an important topic in optical communication technology [1], see also Chapter 5.7.3.

### 1.1.6 Boundary and interface conditions

Electromagnetic fields generally exist in the whole space, they are generated by charges and currents ($\varrho$ and $\boldsymbol{J}$ in (1.1) and (1.2)) and influenced by interaction with matter as indicated in Section 1.1.3. In our macroscopic modeling, permittivity and permeability are assumed to be smooth functions that are discontinuous along smooth material interfaces. It is therefore important to consider how this influences the electromagnetic field.