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Editors

Stochastic Analysis with Financial Applications

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Preface

The Workshop on Stochastic Analysis and Finance took place at City University of Hong Kong from June 29 to July 03, 2009. The goal of this workshop was to present a broad overview of the range of applications of stochastic analysis and its recent theoretical developments, while giving some weight to the research being carried out in the East Asia region. The topics of the talks given in the conference ranged from mathematical aspects of the theory of stochastic processes, to their applications to finance. This is reflected in the organization of the volume which is split into two sections on stochastic analysis and on financial applications.

In recent times the applications of stochastic analysis to finance and insurance have bloomed exponentially, and for this reason we have devoted to them a significant attention. Stochastic analysis has also a variety of other applications to biological systems, physical and engineering problems, requiring the development of advanced techniques, a representative sample of which is also included here.

A large number of articles in this volume deal with stochastic equations, and in particular stochastic (partial) differential equations and stochastic delay equations which arise naturally in physical systems depending on time and space. Contributions dealing with the numerical simulation and error analysis of these stochastic systems, which are an obligatory step before carrying out the actual applications, are also included and can also be of crucial importance to finance. The important and difficult topics of statistical estimation of parameters in these models, as well as their control and robustness, are also treated in this volume.

The conference has also covered two areas that are growing rapidly. First the area of backward stochastic differential equations and all its variants that have deep connections with non-linear partial differential equations. Secondly, the recent developments of (non-linear) G-Brownian motion and its potential uses in risk analysis, which are also opening a new venue of development for stochastic analysis. From a technical point of view, the existence of densities of random variables associated with stochastic differential equations is an important matter, for which a quite complete basic theory is already available for continuous diffusion processes. The case of jump processes, which has already been the object of many important advances, is still in need of many developments that are motivated by applications, as shown in this volume. Concerning the applications to finance, many of the articles deal with a topic that has taken by storm our current society, which is how to deal with the valuation and hedging of credit risk in various forms. The results presented in the financial applications section cover in particular pricing and hedging in credit risk and jump models, including recent results on markets with frictions such as transaction costs, and Lévy driven market models.

The articles contained in these proceedings are survey articles and original research papers which have been peer reviewed, and we take this opportunity to thank the colleagues who have largely contributed with their time as referees. We also thank the contributors for answering our requests to improve the presentation and results in order to produce a high quality volume, and all workshop participants for lively discussions. The participants and organizers are also grateful to the Lee Hysan Foundation, the Hong Kong Pei Hua Education Foundation, and the Department of Mathematics at City University of Hong Kong, for their generous financial support and for providing the conference venue. Last but not least, we acknowledge Ms Lonn Chan of the Mathematics General Office, whose highly efficient organizational skills ensured the complete success of the event.

October 2010

Arturo Kohatsu-Higa
Nicolas Privault
Shuenn-jyi Sheu

List of Speakers

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Corcuera, J.M.	University of Barcelona, Spain
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Duan, J.	Illinois Institute of Technology, USA
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Part I

Stochastic Analysis

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Dirichlet Forms for Poisson Measures and Lévy Processes: The Lent Particle Method

Nicolas Bouleau and Laurent Denis

Abstract. We present a new approach to absolute continuity of laws of Poisson functionals. The theoretical framework is that of local Dirichlet forms as a tool for studying probability spaces. The argument gives rise to a new explicit calculus that we present first on some simple examples: it consists in adding a particle and taking it back after computing the gradient. Then we apply the method to SDE's driven by Poisson measure.

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Keywords. Stochastic differential equation, Poisson functional, Dirichlet form, energy image density, Lévy processes, gradient, carré du champ.

1. Introduction

In order to situate the method it is worth to emphasize some features of the Dirichlet forms approach with comparison to the Malliavin calculus which is generally better known among probabilists.

First the arguments hold under only Lipschitz hypotheses: for example the method applies to a stochastic differential equation with Lipschitz coefficients. Second a general criterion exists, (EID) the Energy Image Density property, (proved on the Wiener space for the Ornstein-Uhlenbeck form, still a conjecture in general cf. Bouleau-Hirsch [7] but established in the case of random Poisson measures with natural hypotheses) which ensures the existence of a density for a $\mathbb{R}^d$-valued random variable. Third, Dirichlet forms are easy to construct in the infinite-dimensional frameworks encountered in probability theory and this yields a theory of errors propagation through the stochastic calculus, especially for finance and physics cf. Bouleau [2], but also for numerical analysis of PDE's and SPDE's cf. Scotti [18].

Our aim is to extend, thanks to Dirichlet forms, the Malliavin calculus applied to the case of Poisson measures and SDE's with jumps. Let us recall that in the case of jumps, there are several ways for applying the ideas of Malliavin calculus. The works are based either on the chaos decomposition (Nualart-Vives [14]) and provide tools in analogy with the Malliavin calculus on Wiener space, but non-local (Picard [15], Ishikawa-Kunita [12]) or dealing with local operators acting on the size of the jumps using the expression of the generator on a sufficiently rich class and closing the structure, for instance by Friedrichs' argument (cf. especially Bichteler-Gravereaux-Jacod [1], Coquio [8] and Ma-Röckner [13]).

We follow a way close to this last one. We will first expose the method from a practical point of view, in order to show how it acts on concrete cases. Then in a separate part we shall give the main elements of the proof of the main theorem on the lent particle formula. Eventually we will display several examples where the method improves known results. Then, in the last section, we shall apply the lent particle method to SDE's driven by a Poisson measure or a Lévy process. Complete details of the proofs and hypotheses for getting (EID) are published in [3] and [4].

2. The lent particle method

Consider a random Poisson measure as a distribution of points, and let us see a Lévy process as defined by a Poisson random measure, that is let us think on the *configuration space*. We suppose the particles live in a space (called the bottom space) equipped with a local Dirichlet form with carré du champ and gradient. This makes it possible to construct a local Dirichlet form with carré du champ on the configuration space (called the upper space). To calculate for some functional the Malliavin matrix – which in the framework of Dirichlet forms becomes the carré du champ matrix – the method consists first in adding a particle to the system. The functional then possesses a new argument which is due to this new particle. We can compute the bottom-gradient of the functional with respect to this argument and as well its bottom carré du champ. Then taking back the particle we have added does not erase the new argument of the obtained functional. We can integrate the new argument with respect to the Poisson measure and this gives the upper carré du champ matrix – that is the Malliavin matrix. This is the exact summary of the method.

2.1. Let us give more details and notation

Let $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ be a local symmetric Dirichlet structure which admits a carré du champ operator. This means that $(X, \mathcal{X}, \nu)$ is a measured space, ν is σ-finite and the bilinear form $e[f,g] = \frac{1}{2}\int \gamma[f,g]\, d\nu$ is a local Dirichlet form with domain $\mathbf{d} \subset L^2(\nu)$ and carré du champ γ (cf. Fukushima-Oshima-Takeda [10] in the locally compact case and Bouleau-Hirsch [7] in a general setting). $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ is called the bottom space.

Consider a Poisson random measure N on $[0, +\infty[\times X$ with intensity measure $dt \times \nu$. A Dirichlet structure may be constructed canonically on the probability space of this Poisson measure that we denote $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1, \mathbb{D}, \Gamma)$. We call this space the upper space.

$\mathbb{D}$ is a set of functions in the domain of Γ, in other words a set of random variables which are functionals of the random distribution of points. The main result is the following formula:

For all $F \in \mathbb{D}$

$$\Gamma[F] = \int_0^{+\infty} \int_X \varepsilon^-(\gamma[\varepsilon^+ F])\, dN$$

in which ε^+ and ε^- are the creation and annihilation operators.

Let us explain the meaning and the use of this formula on an example.

2.2. First example

Let Y_t be a centered Lévy process with Lévy measure ν integrating x^2. We assume that ν is such that a local Dirichlet structure may be constructed on $\mathbb{R}\backslash\{0\}$ with carré du champ $\gamma[f] = x^2 f'^2(x)$.

The notion of gradient in the sense of Dirichlet forms is explained in [7] Chapter V. It is a linear operator with values in an auxiliary Hilbert space giving the carré du champ by taking the square of the Hilbert norm. It is convenient to choose as Hilbert space a space L^2 of a probability space.

Here we define a gradient $\flat$ associated with γ by choosing ξ such that

$$\int_0^1 \xi(r)dr = 0 \quad \text{and} \quad \int_0^1 \xi^2(r)dr = 1$$

and putting

$$f^\flat = xf'(x)\xi(r).$$

Practically $\flat$ acts as a derivation with the chain rule $(\varphi(f))^\flat = \varphi'(f).f^\flat$ (for $\varphi \in \mathcal{C}^1 \cap \mathrm{Lip}$ or even only Lipschitz).

N is the Poisson random measure associated with Y with intensity $dt \times \sigma$ such that $\int_0^t h(s)\, dY_s = \int \mathbf{1}_{[0,t]}(s)h(s)x\tilde{N}(dsdx)$ for $h \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$.

We study the regularity of

$$V = \int_0^t \varphi(Y_{s-})dY_s$$

where φ is Lipschitz and $\mathcal{C}^1$.

1) We add a particle (α, x), i.e., a jump to Y at time α with size x what gives

$$\varepsilon^+ V = V + \varphi(Y_{\alpha-})x + \int_{]\alpha}^t (\varphi(Y_{s-} + x) - \varphi(Y_{s-}))dY_s.$$

2) $V^\flat = 0$ since V does not depend on x, and

$$(\varepsilon^+ V)^\flat = \left(\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-} + x)xdY_s \right) \xi(r)$$

because $x^\flat = x\xi(r)$.

3) We compute

$$\gamma[\varepsilon^+ V] = \int (\varepsilon^+ V)^{\flat 2} dr = (\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-}+x)x dY_s)^2.$$

4) We take back the particle what gives

$$\varepsilon^-\gamma[\varepsilon^+ V] = (\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-})x dY_s)^2$$

and compute $\Gamma[V] = \int \varepsilon^-\gamma[\varepsilon^+ V] dN$ (lent particle formula)

$$\Gamma[V] = \int \left(\varphi(Y_{\alpha-}) + \int_{]\alpha}^t \varphi'(Y_{s-}) dY_s\right)^2 x^2 \; N(d\alpha dx)$$
$$= \sum_{\alpha \leqslant t} \Delta Y_\alpha^2 (\int_{]\alpha}^t \varphi'(Y_{s-}) dY_s + \varphi(Y_{\alpha-}))^2$$

where $\Delta Y_\alpha = Y_\alpha - Y_{\alpha-}$.

For real functional, the condition (EID) is always fulfilled: V possesses a density as soon as $\Gamma[V] > 0$. Then the above expression may be used to discuss the strict positivity of $\Gamma[V]$ depending on the finite or infinite mass of ν cf. [4] Example 5.2.

Before giving a typical set of assumptions that the Lévy measure ν has to fulfill, let us explicit the (EID) property.

2.3. Energy Image Density property (EID)

A Dirichlet form on $L^2(\Lambda)$ (Λ σ-finite) with carré du champ γ satisfies (EID) if, for any d and all U with values in $\mathbb{R}^d$ whose components are in the domain of the form, the image by U of the measure with density with respect to Λ the determinant of the carré du champ matrix is absolutely continuous with respect to the Lebesgue measure, i.e.,

$$U_*[(\det\gamma[U, U^t]) \cdot \Lambda] \ll \lambda^d.$$

This property is true for the Ornstein-Uhlenbeck form on the Wiener space, and in several other cases cf. Bouleau-Hirsch [7]. It was conjectured in 1986 that it were always true. It is still a conjecture.

It is therefore necessary to prove this property in the context of Poisson random measures. With natural hypotheses, cf. [4] Parts 2 and 4, as soon as (EID) is true for the bottom space, (EID) is true for the upper space. Our proof uses a result of Shiqi Song [19].

2.4. Main example of bottom structure in $\mathbb{R}^d$

Let $(Y_t)_{t \geqslant 0}$ be a centered d-dimensional Lévy process without gaussian part, with Lévy measure $\nu = k dx$. Under standard hypotheses, we have the following representation:

$$Y_t = \int_0^t \int_{\mathbb{R}^d} x \tilde{N}(ds, dx),$$

where $\tilde{N}$ is a compensated Poisson measure with intensity $dt \times kdx$. In this case, the idea is to introduce an ad-hoc Dirichlet structure on $\mathbb{R}^d$.

The following example gives a case of such a structure $(\mathbf{d}, e)$ which satisfies all the required hypotheses and which is flexible enough to encompass many cases:

Lemma 1. *Let $r \in \mathbb{N}^*$, $(X, \mathcal{X}) = (\mathbb{R}^r, \mathcal{B}(\mathbb{R}^r))$ and $\nu = kdx$ where k is non-negative and Borelian. We are given $\xi = (\xi_{i,j})_{1 \leqslant i,j \leqslant r}$ an $\mathbb{R}^{r\times r}$-valued and symmetric Borel function. We assume that there exist an open set $O \subset \mathbb{R}^r$ and a function ψ continuous on O and null on $\mathbb{R}^r \setminus O$ such that*

1) *$k > 0$ on O ν-a.e. and is locally bounded on O.*
2) *ξ is locally bounded and locally elliptic on O.*
3) *$k \geqslant \psi > 0$ ν-a.e. on O.*
4) *for all $i, j \in \{1, \dots, r\}$, $\xi_{i,j}\psi$ belongs to $H^1_{\text{loc}}(O)$.*

We denote by H the subspace of functions $f \in L^2(\nu) \cap L^1(\nu)$ such that the restriction of f to O belongs to $C_c^\infty(O)$. Then, the bilinear form defined by

$$\forall f, g \in H,\ e(f, g) = \sum_{i,j=1}^{r} \int_O \xi_{i,j}(x)\partial_i f(x)\partial_j g(x)\psi(x)\, dx$$

is closable in $L^2(\nu)$. Its closure, $(\mathbf{d}, e)$, is a local Dirichlet form on $L^2(\nu)$ which admits a carré du champ γ:

$$\forall f \in \mathbf{d},\ \gamma(f)(x) = \sum_{i,j=1}^{r} \xi_{i,j}(x)\partial_i f(x)\partial_j f(x)\frac{\psi(x)}{k(x)}.$$

Moreover, it satisfies property (EID).

Remark. In the case of a Lévy process, we will often apply this lemma with ξ the identity mapping. We shall often consider an open domain of the form $O = \{x \in \mathbb{R}^d;\ |x| < \varepsilon\}$ which means that we "differentiate" only w.r.t. small jumps and hypothesis 3. means that we do not need to assume regularity on k but only that k dominates a regular function.

2.5. Multivariate example

Consider as in the previous section, a centered Lévy process without gaussian part Y such that its Lévy measure ν satisfies assumptions of Lemma 1 (which imply $1 + \Delta Y_s \neq 0$ a.s.) with $d = 1$ and $\xi(x) = x^2$.

We want to study the existence of density for the pair $(Y_t, \mathcal{E}xp(Y)_t)$ where $\mathcal{E}xp(Y)$ is the Doléans exponential of Y.

$$\mathcal{E}xp(Y)_t = e^{Y_t} \prod_{s \leqslant t} (1 + \Delta Y_s)e^{-\Delta Y_s}.$$

1) We add a particle (α, y), i.e., a jump to Y at time $\alpha \leqslant t$ with size y:

$$\varepsilon^+_{(\alpha,y)}(\mathcal{E}xp(Y)_t) = e^{Y_t+y} \prod_{s \leqslant t} (1 + \Delta Y_s)e^{-\Delta Y_s}(1 + y)e^{-y} = \mathcal{E}xp(Y)_t(1 + y).$$

2) We compute $\gamma[\varepsilon^+\mathcal{E}xp(Y)_t](y) = (\mathcal{E}xp(Y)_t)^2 y^2 \frac{\psi(y)}{k(y)}$.
3) We take back the particle:

$$\varepsilon^-\gamma[\varepsilon^+\mathcal{E}xp(Y)_t] = \left(\mathcal{E}xp(Y)_t(1+y)^{-1}\right)^2 y^2 \frac{\psi(y)}{k(y)}$$

we integrate w.r.t. N and that gives the upper carré du champ operator (lent particle formula):

$$\begin{aligned}\Gamma[\mathcal{E}xp(Y)_t] &= \int_{[0,t]\times\mathbb{R}} \left(\mathcal{E}xp(Y)_t(1+y)^{-1}\right)^2 y^2 \frac{\psi(y)}{k(y)} N(d\alpha, dy)\\ &= \sum_{\alpha \leqslant t} \left(\mathcal{E}xp(Y)_t(1+\Delta Y_\alpha)^{-1}\right)^2 \frac{\psi(\Delta Y_\alpha)}{k(\Delta Y_\alpha)} \Delta Y_\alpha^2.\end{aligned}$$

By a similar computation the matrix $\underline{\underline{\Gamma}}$ of the pair $(Y_t, \mathcal{E}xp(Y_t))$ is given by

$$\underline{\underline{\Gamma}} = \sum_{\alpha \leqslant t} \begin{pmatrix} 1 & \mathcal{E}xp(Y)_t(1+\Delta Y_\alpha)^{-1} \\ \mathcal{E}xp(Y)_t(1+\Delta Y_\alpha)^{-1} & \left(\mathcal{E}xp(Y)_t(1+\Delta Y_\alpha)^{-1}\right)^2 \end{pmatrix} \frac{\psi(\Delta Y_\alpha)}{k(\Delta Y_\alpha)} \Delta Y_\alpha^2.$$

Hence under hypotheses implying (EID), such as those of Lemma 1, the density of the pair $(Y_t, \mathcal{E}xp(Y_t))$ is yielded by the condition

$$\dim \mathcal{L}\left(\begin{pmatrix} 1 \\ \mathcal{E}xp(Y)_t(1+\Delta Y_\alpha)^{-1} \end{pmatrix} \quad \alpha \in JT\right) = 2$$

where JT denotes the jump times of Y between 0 and t.

Making this in details we obtain

Let Y be a real Lévy process with infinite Lévy measure with density dominating near 0 a positive continuous function, then the pair $(Y_t, \mathcal{E}xp(Y)_t)$ possesses a density on $\mathbb{R}^2$.

3. Demonstration of the lent particle formula

3.1. The construction

Let us recall that $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ is a local Dirichlet structure with carré du champ called the bottom space, ν is σ-finite and the bilinear form $e[f,g] = \frac{1}{2}\int \gamma[f,g]\, d\nu$ is a local Dirichlet form with domain $\mathbf{d} \subset L^2(\nu)$ and with carré du champ γ. We assume $\{x\} \in \mathcal{X}$ for all $x \in X$ and ν is diffuse. The associated generator is denoted a, its domain is $\mathcal{D}(a) \subset \mathbf{d}$.

We consider a random Poisson measure N, on $[0, +\infty[\times X$ with intensity $dt \times \nu$. It is defined on $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ where Ω_1 is the configuration space of countable sums of Dirac masses on $[0, +\infty[\times X$, $\mathcal{A}_1$ is the σ-field generated by N and $\mathbb{P}_1$ is the law of N.

$(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ is called the upper space. The question is to construct a Dirichlet structure on the upper space, induced "canonically" by the Dirichlet structure of the bottom space.

This question is natural by the following interpretation. The bottom structure may be thought as the elements for the description of a single particle moving according to a symmetric Markov process associated with the bottom Dirichlet form. Then considering an infinite family of independent such particles with initial law given by $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ shows that a Dirichlet structure can be canonically considered on the upper space (cf. the introduction of [4] for different ways of tackling this question).

Because of typical formulas on functions of the form $e^{iN(f)}$ related to the Laplace functional, we consider the space of test functions $\mathcal{D}_0$ to be the set of elements in $L^2(\mathbb{P}_1)$ which are the linear combinations of variables of the form $e^{i\tilde{N}(f)}$ where f belongs to $\big(\mathcal{D}(a) \otimes L^2(dt)\big) \bigcap L^1(dt \times \nu)$ and is such that $\gamma[f] \in L^2(dt \times \nu)$, recall that $\tilde{N} = N - dt \times \nu$.

Remark 1. *As we need* $\mathcal{D}_0$ *to be a dense subset, we make what we call a* Bottom core *hypothesis. Namely we assume that there exists a subspace* H *of* $\mathcal{D}(a) \bigcap L^1(\nu)$*, dense in* $L^1(\nu) \cap L^2(\nu)$ *and such that* $\forall f \in H,\ \gamma[f] \in L^2(\nu)$ (*see* [4] *for more details on the technical hypotheses we adopt*).

If $U = \sum_p \lambda_p e^{i\tilde{N}(f_p)}$ belongs to $\mathcal{D}_0$, we put

$$A_0[U] = \sum_p \lambda_p e^{i\tilde{N}(f_p)} \big(i\tilde{N}(a[f_p]) - \frac{1}{2} N(\gamma[f_p])\big), \tag{1}$$

where, in a natural way, if $f(x,t) = \sum_l u_l(x)\varphi_l(t) \in \mathcal{D}(a) \otimes L^2(dt)$

$$a[f] = \sum_l a[u_l]\varphi_l \text{ and } \gamma[f] = \sum_l \gamma[u_l]\varphi_l.$$

In order to show that A_0 is uniquely defined and is the generator of a Dirichlet form satisfying the required properties, starting from a gradient of the bottom structure we construct a gradient for the upper structure defined first on the test functions. Then we show that this gradient does not depend on the form of the test function and this allows to extend the operators thanks to Friedrichs' property yielding the closedness of the upper structure.

3.2. The bottom gradient

We suppose the space $\mathbf{d}$ separable, then there exists a gradient for the bottom space, i.e., there is a separable Hilbert space and a linear map D from $\mathbf{d}$ into $L^2(X, \nu; H)$ such that $\forall u \in \mathbf{d}$, $\|D[u]\|_H^2 = \gamma[u]$, then necessarily

- If $F : \mathbb{R} \to \mathbb{R}$ is Lipschitz then $\forall u \in \mathbf{d}$, $D[F \circ u] = (F' \circ u)Du$,
- If F is $\mathcal{C}^1$ and Lipschitz from $\mathbb{R}^d$ into $\mathbb{R}$ then $D[F \circ u] = \sum_{i=1}^d (F_i' \circ u) D[u_i]$ $\forall u = (u_1, \ldots, u_d) \in \mathbf{d}^d$.

We take for H a space $L^2(R, \mathcal{R}, \rho)$ where $(R, \mathcal{R}, \rho)$ is a probability space s.t. $L^2(R, \mathcal{R}, \rho)$ is infinite dimensional. The gradient D is denoted by $\flat$:

$$\forall u \in \mathbf{d},\ Du = u^\flat \in L^2(X \times R, \mathcal{X} \otimes \mathcal{R}, \nu \times \rho).$$

Without loss of generality, we assume moreover that the operator $\flat$ takes its values in the orthogonal space of 1 in $L^2(R,\mathcal{R},\rho)$. So that we have

$$\forall u\in \mathbf{d},\ \int u^\flat d\rho = 0\quad \nu\text{-}a.e.$$

3.3. Candidate gradient for the upper space

Now, we introduce the creation operator (resp. annihilation operator) which consists in adding (resp. removing if necessary) a jump at time t with size u:

$$\varepsilon^+_{(t,u)}(w_1) = w_1\mathbf{1}_{\{(t,u)\in \text{supp } w_1\}} + (w_1+\varepsilon_{(t,u)}\})\mathbf{1}_{\{(t,u)\notin \text{supp } w_1\}}$$
$$\varepsilon^-_{(t,u)}(w_1) = w_1\mathbf{1}_{\{(t,u)\notin \text{supp } w_1\}} + (w_1-\varepsilon_{(t,u)}\})\mathbf{1}_{\{(t,u)\in \text{supp } w_1\}}.$$

In a natural way, we extend these operators to the functionals by

$$\varepsilon^+H(w_1,t,u) = H(\varepsilon^+_{(t,u)}w_1,t,u)\quad \varepsilon^-H(w_1,t,u) = H(\varepsilon^-_{(t,u)}w_1,t,u).$$

Definition. *For $F\in\mathcal{D}_0$, we define the pre-gradient*

$$F^\sharp = \int_0^{+\infty}\int_{X\times R}\varepsilon^-((\varepsilon^+F)^\flat)\,dN\odot\rho,$$

where $N\odot\rho$ is the point process N "marked" by ρ, i.e., if N is the family of marked points (T_i,X_i), $N\odot\rho$ is the family (T_i,X_i,r_i) where the r_i are new independent random variables mutually independent and identically distributed with law ρ, defined on an auxiliary probability space $(\hat\Omega,\hat{\mathcal{A}},\hat{\mathbb{P}})$. So $N\odot\rho$ is a Poisson random measure on $[0,+\infty[\times X\times R$.

3.4. Main result

The above candidate may be shown to extend in a true gradient for the upper structure. The argument is based on the extension of the pregenerator A_0 thanks to Friedrichs' property (cf. for instance [7] p. 4): A_0 is shown to be well defined on $\mathcal{D}_0$ which is dense, A_0 is non-positive and symmetric and therefore possesses a selfadjoint extension. Before stating the main theorem, let us introduce some notation. We denote by $\underline{\mathbb{D}}$ the completion of $\mathcal{D}_0\otimes L^2([0,+\infty[,dt)\otimes\mathbf{d}$ with respect to the norm

$$\begin{aligned}\|H\|_{\underline{\mathbb{D}}} &= \left(\mathbb{E}\int_0^{+\infty}\int_X \varepsilon^-(\gamma[H])(w,t,u)N(dt,du)\right)^{\frac12}\\ &\quad+\mathbb{E}\int_0^{+\infty}\int_X(\varepsilon^-|H|)(w,t,u)\eta(t,u)N(dt,du)\\ &= \left(\mathbb{E}\int_0^{+\infty}\int_X\gamma[H](w,t,u)\nu(du)dt\right)^{\frac12}\\ &\quad+\mathbb{E}\int_0^{+\infty}\int_X|H|(w,t,u)\eta(t,u)\nu(du)dt,\end{aligned}$$

where η is a fixed positive function in $L^2(\mathbb{R}^+\times X, dt\times d\nu)$.

As we shall see below, a peculiarity of the method comes from the fact that it involves, in the computation, successively mutually singular measures, such as measures $\mathbb{P}_N = \mathbb{P}_1(d\omega)N(\omega, dt, dx)$ and $\mathbb{P}_1 \times dt \times \nu$. This imposes some care in the applications.

Main theorem. *The formula*

$$\forall F \in \mathbb{D},\ F^\sharp = \int_0^{+\infty} \int_{X\times R} \varepsilon^-((\varepsilon^+ F)^\flat)\, dN \odot \rho,$$

extends from $\mathcal{D}_0$ to $\mathbb{D}$, it is justified by the following decomposition:

$$F \in \mathbb{D} \overset{\varepsilon^+ - I}{\mapsto} \varepsilon^+ F - F \in \underline{\mathbb{D}} \overset{\varepsilon^-((.)^\flat)}{\mapsto} \varepsilon^-((\varepsilon^+ F)^\flat) \in L^2_0(\mathbb{P}_N \times \rho) \overset{d(N\odot\rho)}{\mapsto} F^\sharp \in L^2(\mathbb{P}_1 \times \hat{\mathbb{P}})$$

where each operator is continuous on the range of the preceding one and where $L^2_0(\mathbb{P}_N \times \rho)$ is the closed set of elements G in $L^2(\mathbb{P}_N \times \rho)$ such that $\int_R G d\rho = 0$ $\mathbb{P}_N$-a.s.

Furthermore for all $F \in \mathbb{D}$

$$\Gamma[F] = \hat{\mathbb{E}}(F^\sharp)^2 = \int_0^{+\infty} \int_X \varepsilon^- \gamma[\varepsilon^+ F]\, dN. \tag{2}$$

Let us explain the steps of a typical calculation applying this theorem.

Let $H = \Phi(F_1, \dots, F_n)$ with $\Phi \in \mathcal{C}^1 \cap \mathrm{Lip}(\mathbb{R}^n)$ and $F = (F_1, \dots, F_n)$ with $F_i \in \mathbb{D}$, we have:

a) $\gamma[\varepsilon^+ H] = \sum_{ij} \Phi'_i(\varepsilon^+ F)\Phi'_j(\varepsilon^+ F)\gamma[\varepsilon^+ F_i, \varepsilon^+ F_j]$ $\mathbb{P} \times \nu$-a.e.

b) $\varepsilon^-\gamma[\varepsilon^+ H] = \sum_{ij} \Phi'_i(F)\Phi'_j(F)\varepsilon^-\gamma[\varepsilon^+ F_i, \varepsilon^+ F_j]$ $\mathbb{P}_N$-a.e.

c) $\Gamma[H] = \int \varepsilon^-\gamma[\varepsilon^+ H]dN = \sum_{ij} \Phi'_i(F)\Phi'_j(F) \int \varepsilon^-\gamma[\varepsilon^+ F_i, \varepsilon^+ F_j]dN$ $\mathbb{P}$-a.e.

Let us eventually remark that the lent particle formula (2) has been encountered previously by some authors for test functions (see, e.g., [17] before Prop. 8). Here, it is proved on the whole domain $\mathbb{D}$, this is essential to apply the method to SDE's and to exploit the full strength of the functional calculus of Dirichlet forms.

4. Applications

4.1. Sup of a stochastic process on $[0, t]$

The fact that the operation of taking the maximum is typically a Lipschitz operation makes it easy to apply the method.

Let Y be a centered Lévy process as in §2.2. Let K be a càdlàg process independent of Y. We put

$$H_s = Y_s + K_s.$$

Proposition. *If $\nu(\mathbb{R}\backslash\{0\}) = +\infty$ and if $\mathbb{P}_1[\sup_{s \leqslant t} H_s = H_0] = 0$, the random variable $\sup_{s \leqslant t} H_s$ has a density.*

As a consequence, *any Lévy process starting from zero and immediately entering* $\mathbb{R}_+^*$, *whose Lévy measure dominates a measure* ν *satisfying Hamza condition and infinite, is such that* $\sup_{s\leqslant t} X_s$ *has a density.*

Let us recall that the Hamza condition (cf. Fukushima *et al.* [10] Chapter 3) gives a necessary and sufficient condition of existence of a Dirichlet structure on $L^2(\nu)$. Such a necessary and sufficient condition is only known in dimension one.

4.2. Regularity without Hörmander

Consider the following SDE driven by a two-dimensional Brownian motion

$$\begin{cases} X_t^1 = z_1 + \int_0^t dB_s^1 \\ X_t^2 = z_2 + \int_0^t 2X_s^1 dB_s^1 + \int_0^t dB_s^2 \\ X_t^3 = z_3 + \int_0^t X_s^1 dB_s^1 + 2\int_0^t dB_s^2. \end{cases} \tag{3}$$

This diffusion is degenerate and the Hörmander conditions are not fulfilled. The generator is $A = \frac{1}{2}(U_1^2 + U_2^2) + V$ and its adjoint $A^* = \frac{1}{2}(U_1^2 + U_2^2) - V$ with $U_1 = \frac{\partial}{\partial x_1} + 2x_1\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}$, $U_2 = \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3}$ and $V = -\frac{\partial}{\partial z_2} - \frac{1}{2}\frac{\partial}{\partial z_3}$. The Lie brackets of these vectors vanish and the Lie algebra is of dimension 2: the diffusion remains on the quadric of equation $\frac{3}{4}x_1^2 - x_2 + \frac{1}{2}x_3 - \frac{3}{4}t = C$.

Consider now the same equation driven by a Lévy process:

$$\begin{cases} Z_t^1 = z_1 + \int_0^t dY_s^1 \\ Z_t^2 = z_2 + \int_0^t 2Z_{s_-}^1 dY_s^1 + \int_0^t dY_s^2 \\ Z_t^3 = z_3 + \int_0^t Z_{s_-}^1 dY_s^1 + 2\int_0^t dY_s^2 \end{cases}$$

under hypotheses on the Lévy measure such that the bottom space may be equipped with the carré du champ operator $\gamma[f] = y_1^2 f_1'^2 + y_2^2 f_2'^2$ satisfying the hypotheses yielding (EID). Let us apply in full details the lent particle method.

For $\alpha \leqslant t$

$$\varepsilon^+_{(\alpha,y_1,y_2)} Z_t = Z_t + \begin{pmatrix} y_1 \\ 2Y_{\alpha-}^1 y_1 + 2\int_{]\alpha}^t y_1 dY_s^1 + y_2 \\ Y_{\alpha-}^1 y_1 + \int_{]\alpha}^t y_1 dY_s^1 + 2y_2 \end{pmatrix} = Z_t + \begin{pmatrix} y_1 \\ 2y_1 Y_t^1 + y_2 \\ y_1 Y_t^1 + 2y_2 \end{pmatrix},$$

where we have used $Y_{\alpha-}^1 = Y_\alpha^1$ because ε^+ send into $\mathbb{P}_1 \times dt \times \nu$ classes. That gives

$$\gamma[\varepsilon^+ Z_t] = \begin{pmatrix} y_1^2 & y_1^2 2Y_t^1 & y_1^2 Y_t^1 \\ id & y_1^2 4(Y_t^1)^2 + y_2^2 & y_1^2 2(Y_t^1)^2 + 2y_2^2 \\ id & id & y_1^2 (Y_t^1)^2 + 4y_2^2 \end{pmatrix}$$

and

$$\varepsilon^-\gamma[\varepsilon^+ Z_t] = \begin{pmatrix} y_1^2 & y_1^2 2(Y_t^1 - \Delta Y_\alpha^1) & y_1^2 (Y_t^1 - \Delta Y_\alpha^1) \\ id & y_1^2 4(Y_t^1 - \Delta Y_\alpha^1)^2 + y_2^2 & y_1^2 2(Y_t^1 - \Delta Y_\alpha^1)^2 + 2y_2^2 \\ id & id & y_1^2 (Y_t^1 - \Delta Y_\alpha^1)^2 + 4y_2^2 \end{pmatrix},$$

where *id* denotes the symmetry of the matrices. Hence

$$\Gamma[Z_t] = \sum_{\alpha \leqslant t} (\Delta Y^1_\alpha)^2 \begin{pmatrix} 1 & 2(Y^1_t - \Delta Y^1_\alpha) & (Y^1_t - \Delta Y^1_\alpha) \\ id & 4(Y^1_t - \Delta Y^1_\alpha)^2 & 2(Y^1_t - \Delta Y^1_\alpha)^2 \\ id & id & (Y^1_t - \Delta Y^1_\alpha)^2 \end{pmatrix} + (\Delta Y^2_\alpha)^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}.$$

With this formula we can reason, trying to find conditions for the determinant of $\Gamma[Z]$ to be positive. For instance if the Lévy measures of Y^1 and Y^2 are infinite, it follows that Z_t has a density as soon as

$$\dim \mathcal{L} \left\{ \begin{pmatrix} 1 \\ 2(Y^1_t - \Delta Y^1_\alpha) \\ (Y^1_t - \Delta Y^1_\alpha) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \alpha \in JT \right\} = 3.$$

But Y^1 possesses necessarily jumps of different sizes, hence Z_t has a density on $\mathbb{R}^3$.

It follows that the integro-differential operator

$$\tilde{A} f(z) = \int \left[f(z) - f\begin{pmatrix} z_1 + y_1 \\ z_2 + 2z_1 y_1 + y_2 \\ z_3 + z_1 y_1 + 2y_2 \end{pmatrix} - (f_1'(z)\ f_2'(z)\ f_3'(z)) \begin{pmatrix} y_1 \\ 2z_1 y_1 + y_2 \\ z_1 y_1 + 2y_2 \end{pmatrix} \right] \sigma(dy_1 dy_2)$$

is hypoelliptic at order zero, in the sense that its semigroup P_t has a density. No minoration is supposed on the growth of the Lévy measure near 0 as assumed by some authors.

This result implies that for any Lévy process Y satisfying the above hypotheses, even a subordinated one in the sense of Bochner, the process Z is never subordinated of the Markov process X solution of equation (3) (otherwise it would live on the same manifold as the initial diffusion).

5. Application to SDE's driven by a Poisson measure

5.1. The equation we study

We consider another probability space $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ on which an $\mathbb{R}^n$-valued semimartingale $Z = (Z^1, \ldots, Z^n)$ is defined, $n \in \mathbb{N}^*$. We adopt the following assumption on the bracket of Z and on the total variation of its finite variation part. It is satisfied if both are dominated by the Lebesgue measure uniformly:

Assumption on Z: There exists a positive constant C such that for any square integrable $\mathbb{R}^n$-valued predictable process h:

$$\forall t \geqslant 0,\ \mathbb{E}[(\int_0^t h_s dZ_s)^2] \leqslant C^2 \mathbb{E}[\int_0^t |h_s|^2 ds]. \tag{4}$$

We shall work on the product probability space: $(\Omega, \mathcal{A}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \times \mathbb{P}_2)$.
For simplicity, we fix a finite terminal time $T > 0$.
Let $d \in \mathbb{N}^*$, we consider the following SDE:

$$X_t = x + \int_0^t \int_X c(s, X_{s^-}, u) \tilde{N}(ds, du) + \int_0^t \sigma(s, X_{s^-}) dZ_s \tag{5}$$

where $x \in \mathbb{R}^d$, $c : \mathbb{R}^+ \times \mathbb{R}^d \times X \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^{d \times n}$ satisfy the set of hypotheses below denoted (R).

Hypotheses (R):

1. There exists $\eta \in L^2(X, \nu)$ such that:
   a) for all $t \in [0, T]$ and $u \in X$, $c(t, \cdot, u)$ is differentiable with continuous derivative and
   $$\forall u \in X, \quad \sup_{t \in [0,T], x \in \mathbb{R}^d} |D_x c(t, x, u)| \leqslant \eta(u),$$
   b) $\forall (t, u) \in [0, T] \times X,\ |c(t, 0, u)| \leqslant \eta(u)$,
   c) for all $t \in [0, T]$ and $x \in \mathbb{R}^d$, $c(t, x, \cdot) \in \mathbf{d}$ and
   $$\sup_{t \in [0,T], x \in \mathbb{R}^d} \gamma[c(t, x, \cdot)](u) \leqslant \eta^2(u),$$
   d) for all $t \in [0, T]$, all $x \in \mathbb{R}^d$ and $u \in X$, the matrix $I + D_x c(t, x, u)$ is invertible and
   $$\sup_{t \in [0,T], x \in \mathbb{R}^d} \left| (I + D_x c(t, x, u))^{-1} \times \mathbf{c(t, x, u)} \right| \leqslant \eta(u).$$
2. For all $t \in [0, T]$, $\sigma(t, \cdot)$ is differentiable with continuous derivative and
   $$\sup_{t \in [0,T], x \in \mathbb{R}^d} |D_x \sigma(t, x)| < +\infty.$$
3. As a consequence of hypotheses 1. and 2. above, it is well known that equation (5) admits a unique solution X such that $\mathbb{E}[\sup_{t \in [0,T]} |X_t|^2] < +\infty$. We suppose that for all $t \in [0, T]$, the matrix $(I + \sum_{j=1}^n D_x \sigma_{\cdot, j}(t, X_{t^-}) \Delta Z_t^j)$ is invertible and its inverse is bounded by a deterministic constant uniformly with respect to $t \in [0, T]$.

Remark. We have defined a Dirichlet structure $(\mathbb{D}, \mathcal{E})$ on $L^2(\Omega_1, \mathbb{P}_1)$. Now, we work on the product space, $\Omega_1 \times \Omega_2$. Using natural notations, we consider from now on that $(\mathbb{D}, \mathcal{E})$ is a Dirichlet structure on $L^2(\Omega, \mathbb{P})$. In fact, it is the product structure of $(\mathbb{D}, \mathcal{E})$ with the trivial one on $L^2(\Omega_2, \mathbb{P}_2)$ (see [7]). Of course, all the properties remain true. In other words, we only differentiate w.r.t. the Poisson noise and not w.r.t. the one introduced by Z.

5.2. Spaces of processes and functional calculus

We denote by $\mathcal{P}$ the predictable sigma-field on $[0,T]\times\Omega$ and we define the following sets of processes:

- $\mathcal{H}_{\mathbb{D}}$: the set of real-valued processes $(H_t)_{t\in[0,T]}$, which belong to $L^2([0,T];\mathbb{D})$, i.e., such that
$$\|H\|^2_{\mathcal{H}_{\mathbb{D}}} = \mathbb{E}[\int_0^T |H_t|^2 dt] + \int_0^T \mathcal{E}(H_t)dt < +\infty.$$
- $\mathcal{H}_{\mathbb{D},\mathcal{P}}$: the subvector space of predictable processes in $\mathcal{H}_{\mathbb{D}}$.
- $\mathcal{H}_{\mathbb{D}\otimes\mathbf{d},\mathcal{P}}$: the set of real-valued processes H defined on $[0,T]\times\Omega\times X$ which are predictable and belong to $L^2([0,T];\mathbb{D}\otimes\mathbf{d})$, i.e., such that
$$\begin{aligned}\|H\|^2_{\mathcal{H}_{\mathbb{D}\otimes\mathbf{d},\mathcal{P}}} &= \mathbb{E}\left[\int_0^T\int_X |H_t|^2\nu(du)dt\right]\\ &\quad+\int_0^T\int_X \mathcal{E}(H_t(\cdot,u))\nu(du)dt + \mathbb{E}\left[\int_0^T e(H_t)dt\right] < +\infty.\end{aligned}$$

The main idea is to differentiate equation (5). To do that, we need some functional calculus. It is given by the next proposition that we prove by approximation:

Proposition 2. *Let $H\in\mathcal{H}_{\mathbb{D}\otimes\mathbf{d},\mathcal{P}}$ and $G\in\mathcal{H}^n_{\mathbb{D},\mathcal{P}}$, then:*

1) *The process*
$$\forall t\in[0,T],\ X_t = \int_0^t\int_X H(s,w,u)\tilde{N}(ds,du)$$
is a square integrable martingale which belongs to $\mathcal{H}_{\mathbb{D}}$ and such that the process $X^- = (X_{t-})_{t\in[0,T]}$ belongs to $\mathcal{H}_{\mathbb{D},\mathcal{P}}$. The gradient operator satisfies for all $t\in[0,T]$:
$$\begin{aligned}X^\sharp_t(w,\hat{w}) = &\int_0^t\int_X H^\sharp(s,w,u,\hat{w})d\tilde{N}(ds,du)\\ &+\int_0^t\int_{X\times R} H^\flat(s,w,u,r)N\odot\rho(ds,du,dr).\end{aligned}\tag{6}$$

2) *The process*
$$\forall t\in[0,T],\ Y_t = \int_0^t G(s,w)dZ_s$$
is a square integrable semimartingale which belongs to $\mathcal{H}_{\mathbb{D}}$, $Y^- = (Y_{t-})_{t\in[0,T]}$ belongs to $\mathcal{H}_{\mathbb{D},\mathcal{P}}$ and
$$\forall t\in[0,T],\ Y^\sharp_t(w,\hat{w}) = \int_0^t G^\sharp(s,w,\hat{w})dZ_s.\tag{7}$$

5.3. Computation of the Carré du champ matrix of the solution

Applying the standard functional calculus related to Dirichlet forms, the previous proposition and a Picard iteration argument, we obtain:

Proposition 3. *The equation* (5) *admits a unique solution* X *in* $\mathcal{H}^d_{\mathbb{D}}$. *Moreover, the gradient of* X *satisfies:*

$$\begin{aligned} X^\sharp_t = & \int_0^t \int_X D_x c(s, X_{s-}, u) \cdot X^\sharp_{s-} \tilde{N}(ds, du) \\ & + \int_0^t \int_{X \times R} c^\flat(s, X_{s-}, u, r) N \odot \rho(ds, du, dr) \\ & + \int_0^t D_x \sigma(s, X_{s-}) \cdot X^\sharp_{s-} dZ_s. \end{aligned}$$

Let us define the $\mathbb{R}^{d\times d}$-valued processes U by

$$dU_s = \sum_{j=1}^n D_x \sigma_{.,j}(s, X_{s-}) dZ^j_s,$$

and the derivative of the flow generated by X:

$$K_t = I + \int_0^t \int_X D_x c(s, X_{s-}, u) K_{s-} \tilde{N}(ds, du) + \int_0^t dU_s K_{s-}.$$

Proposition 4. *Under our hypotheses, for all* $t \geqslant 0$, *the matrix* K_t *is invertible and its inverse* $\bar{K}_t = (K_t)^{-1}$ *satisfies:*

$$\begin{aligned} \bar{K}_t = I - & \int_0^t \int_X \bar{K}_{s-} (I + D_x c(s, X_{s-}, u))^{-1} D_x c(s, X_{s-}, u) \tilde{N}(ds, du) \\ & - \int_0^t \bar{K}_{s-} dU_s + \sum_{s \leqslant t} \bar{K}_{s-} (\Delta U_s)^2 (I + \Delta U_s)^{-1} \\ & + \int_0^t \bar{K}_s d < U^c, U^c >_s . \end{aligned}$$

We are now able to calculate the carré du champ matrix. This is done in the next theorem whose proof is sketched to show how simple is the *lent particle method.*

Theorem 5. *For all* $t \in [0, T]$,

$$\Gamma[X_t] = K_t \int_0^t \int_X \bar{K}_s \gamma[c(s, X_{s-}, \cdot)] \bar{K}^*_s \, N(ds, du) K^*_t.$$

Proof. Let $(\alpha, u) \in [0,T] \times X$. We put $X_t^{(\alpha,u)} = \varepsilon^+_{(\alpha,u)} X_t$.

$$\begin{aligned}X_t^{(\alpha,u)} = x &+ \int_0^\alpha \int_X c(s, X_{s^-}^{(\alpha,u)}, u')\tilde{N}(ds, du') \\ &+ \int_0^\alpha \sigma(s, X_{s^-}^{(\alpha,u)}) dZ_s + c(\alpha, X_{\alpha^-}^{(\alpha,u)}, u) \\ &+ \int_{]\alpha,t]} \int_X c(s, X_{s^-}^{(\alpha,u)}, u')\tilde{N}(ds, du') + \int_{]\alpha,t]} \sigma(s, X_{s^-}^{(\alpha,u)}) dZ_s.\end{aligned}$$

Let us remark that $X_t^{(\alpha,u)} = X_t$ if $t < \alpha$ so that, taking the gradient with respect to the variable u, we obtain:

$$\begin{aligned}(X_t^{(\alpha,u)})^\flat = (c(\alpha, X_{\alpha^-}^{(\alpha,u)}, u))^\flat &+ \int_{]\alpha,t]} \int_X D_x c(s, X_{s^-}^{(\alpha,u)}, u') \cdot (X_{s^-}^{(\alpha,u)})^\flat \tilde{N}(ds, du') \\ &+ \int_{]\alpha,t]} D_x \sigma(s, X_{s^-}^{(\alpha,u)}) \cdot (X_{s^-}^{(\alpha,u)})^\flat dZ_s.\end{aligned}$$

Let us now introduce the process $K_t^{(\alpha,u)} = \varepsilon^+_{(\alpha,u)}(K_t)$ which satisfies the following SDE:

$$K_t^{(\alpha,u)} = I + \int_0^t \int_X D_x c(s, X_{s^-}^{(\alpha,u)}, u') K_{s-}^{(\alpha,u)} \tilde{N}(ds, du') + \int_0^t dU_s^{(\alpha,u)} K_{s-}^{(\alpha,u)}$$

and its inverse $\bar{K}_t^{(\alpha,u)} = (K_t^{(\alpha,u)})^{-1}$. Then, using the flow property, we have:

$$\forall t \geqslant 0,\ (X_t^{(\alpha,u)})^\flat = K_t^{(\alpha,u)} \bar{K}_\alpha^{(\alpha,u)} (c(\alpha, X_{\alpha^-}, u))^\flat.$$

Now, we calculate the carré du champ and then we take back the particle:

$$\forall t \geqslant 0,\ \varepsilon^-_{(\alpha,u)} \gamma[(X_t^{(\alpha,u)})] = K_t \bar{K}_\alpha \gamma[c(\alpha, X_{\alpha^-}, \cdot)] \bar{K}_\alpha^* K_t^*.$$

Finally integrating with respect to N we get

$$\forall t \geqslant 0,\ \Gamma[X_t] = K_t \int_0^t \int_X \bar{K}_s \gamma[c(s, X_{s^-}, \cdot)](u) \bar{K}_s^* N(ds, du) K_t^*. \qquad \square$$

5.4. First application: the regular case

An immediate consequence of the previous theorem is:

Proposition 6. *Assume that X is a topological space, that the intensity measure $ds \times \nu$ of N is such that ν has an infinite mass near some point u_0 in X. If the matrix $(s, y, u) \to \gamma[c(s, y, \cdot)](u)$ is continuous on a neighborhood of $(0, x, u_0)$ and invertible at $(0, x, u_0)$, then the solution X_t of (5) has a density for all $t \in]0, T]$.*

5.5. Application to SDE's driven by a Lévy process

Let Y be a Lévy process with values in $\mathbb{R}^d$, independent of another variable X_0.

We consider the following equation

$$X_t = X_0 + \int_0^t a(X_{s-}, s)\, dY_s, \quad t \geqslant 0$$

where $a : \mathbb{R}^k \times \mathbb{R}^+ \to \mathbb{R}^{k\times d}$ is a given map.

Proposition 7. *We assume that:*

1) *The Lévy measure, ν, of Y satisfies hypotheses of the example given in Section 2.4 with $\nu(O) = +\infty$ and $\xi_{i,j}(x) = x_i\delta_{i,j}$. Then we may choose the operator γ to be*

$$\gamma[f] = \frac{\psi(x)}{k(x)} \sum_{i=1}^d x_i^2 \sum_{i=1}^d (\partial_i f)^2 \quad \text{for } f \in \mathcal{C}_0^\infty(\mathbb{R}^d).$$

2) *a is $\mathcal{C}^1 \cap \mathrm{Lip}$ with respect to the first variable uniformly in s and*

$$\sup_{t,x} |(I + D_x a \cdot u)^{-1}(x,t)| \leqslant \eta(u),$$

where $\eta \in L^2(\nu)$.

3) *a is continuous with respect to the second variable at 0, and such that the matrix $aa^*(X_0, 0)$ is invertible;*

then for all $t > 0$ the law of X_t is absolutely continuous w.r.t. the Lebesgue measure.

Proof. We just give an idea of the proof in the case $d = 1$:

Let us recall that $\gamma[f](u) = \dfrac{\psi(u)}{k(u)} u^2 f'^2(u)$.

We have the representation: $Y_t = \int_0^t \int_{\mathbb{R}} u\tilde{N}(ds, du)$, so that

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} a(s, X_{s-})u\, \tilde{N}(ds, du).$$

The lent particle method yields:

$$\Gamma[X_t] = K_t^2 \int_0^t \int_X \bar{K}_s^2 a^2(s, X_{s-})\gamma[j](u) N(ds, du)$$

where j is the identity application: $\gamma[j](u) = \dfrac{\psi(u)}{k(u)} u^2$.

So

$$\begin{aligned}\Gamma[X_t] &= K_t^2 \int_0^t \int_X \bar{K}_s^2 a^2(s, X_{s-}) \frac{\psi(u)}{k(u)} u^2 N(ds, du)\\ &= K_t^2 \sum_{\alpha<t} \bar{K}_s^2 a^2(s, X_{s-}) \frac{\psi(\Delta Y_s)}{k(\Delta Y_s)} \Delta Y_s^2,\end{aligned}$$

and it is easy to conclude. □

Remarks

(i) We refer to [5] for other examples and applications.
(ii) Let us finally remark that, as easily seen, the gradient we have introduced may be naturally iterated. This yields a criteria of regularity for the density of Poisson functionals such as solutions of SDE's, this is the object of a forthcoming paper.

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