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Semi-Discretization for Time-Delay Systems

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To our parents and to our Ágies

Preface

Time delay always arises in engineering models, where the rates of change of state variables depend both on present and on past state variables of the system. Control processes with feedback delay, regenerative machine tool chatter, wheel shimmy models including the elastic contact between the tire and the road, car-following traffic models with the reaction time of the drivers, human motion control with reflex delay, can be mentioned as examples. The analysis of these systems requires the characterization of their local behavior around a desired position or a desired (possibly periodic) path. Such properties can be described by stability charts that present the stability of the linearized system in the plane of the system parameters. These stability charts provide a useful tool for engineers, since they present an overview on the effects of system parameters on the local dynamics of the system.

The main differences between systems with and without time delay is that time delay produces an infinite-dimensional dynamics as opposed to the finite-dimensional dynamics of delay-free systems. For simple time-delay systems, stability charts can be derived analytically. However for complex systems, for instance, when the time-delay effect is coupled with parametric excitation, only numerical techniques can be used.

The scope of this book is to present a numerical technique, called the semi-discretization method, for the stability analysis of linear time-periodic time-delay systems, which is also an essential tool in the study of periodic motions of nonlinear time-delay systems. Semi-discretization is a well-known technique used, for example, in the finite element analysis of solid bodies, or in computational fluid mechanics, where the corresponding partial differential equations are discretized along the spatial coordinates only, while the time coordinates are unchanged. In case of time-delay systems, semi-discretization results in the discretization of delayed terms only, while the actual-time-domain terms are not discretized. In this way, the infinite-dimensional system is approximated by a finite-dimensional one.

The structure of the book is as follows. Chapter 1 gives some introduction to linear time-delay systems. Chapter 2 deals with the construction of the stability charts for some fundamental delay-differential equations. The semi-discretization method is presented in Chapter 3 including higher-order methods, rate convergence estimates, and numerical issues. The semi-discretization method is applied in Chapter 4 to some Newtonian examples with different delay types, such as single point delay, multiple delays, distributed delay, and time-periodic delay. Finally, Chapter 5 presents real-world mechanical engineering applications. Turning and milling processes are considered with varying spindle speed, resulting in time-periodic time delays. Then, the so-called act-and-wait control concept is introduced, and it is analyzed through applications to the stick-balancing problem and to a force-control process with feedback delay. It is shown that the inclusion of waiting periods in the control rule may have a stabilizing effect. This provides the surprising conclusion that doing nothing and rather waiting for the response of a previous action might be a superior control strategy for systems with feedback delay. Finally, the stick-balancing model with reflex delay is investigated in the case of parametric forcing at the stick's base. The book concludes with an appendix that contains Matlab codes for the semi-discretization of the examples presented in Chapter 4.

The book is designed for graduate and PhD students as well as for researchers working in the fields of mechanical, electrical, and chemical engineering, control theory, biomechanics, population dynamics, neurophysiology, even climate research in which time-delay models occur.

The book is based on the authors' research work over the last 10 years, but many colleagues have contributed to different parts. Hereby, the authors thank and acknowledge the useful discussions with and comments of Mikel Zatarain, Jokin Muñoa, Grégoire Peigné, and Sébastien Seguy regarding the computational efficiency of the semi-discretization for different milling applications. The helpful consultations and joint works related to the mathematical issues of the method with Janos Turi, Ferenc Hartung, and Barnabás Garay are greatly appreciated. The comments and novel ideas provided by our young colleagues Zoltán Dombóvári and Dániel Bachrathy are gratefully recognized. Finally, the inspiring long-term cooperation and mutual comparative studies for machining operations with Philip V. Bayly, Brian P. Mann, and Firas A. Khasawneh are gratefully acknowledged.

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Budapest,
April 2011

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Contents

Chapter 1
Introducing Delay in Linear Time-Periodic Systems

Dynamical systems have been described with differential equations since the appearance of the differential calculus; Newton's second law could be considered one of the first examples. A differential equation can serve as a model for how the rate of change of state depends on the present state of a system. However, the rate of change of state may depend on past states, too. It has been known for a long time that several problems can be described by models including past effects. One of the classical examples is the predator–prey model of Volterra [288], where the growth rate of predators depends not only on the present quality of food (say, prey), but also on past quantities (in the period of gestation, say). The first delay models in engineering appeared for wheel shimmy [230] and for ship stabilization [194] in the early 1940s. There are several other engineering applications in which time delay plays a crucial role. As recognized in the late 1940s with the development of control theory, time delay typically arises in feedback control systems due to the finite speed of information transmission and data processing [284, 252]. Another typical application is the stability of machining processes, where time delay appears due to the surface regeneration by the cutting edge [280, 281, 256, 5]. Similar equations describe the car-following traffic models involving the reaction time of drivers [213, 214, 215]. Reflex delay is also a relevant issue to human motion control [24, 259, 192, 13]. Time delay also plays important role in population dynamics [160, 251], in neural networks [49, 216], and in epidemiology models [226, 2].

Systems whose rate of change of state depends on states at deviating arguments are generally described by functional differential equations (FDEs). According to Myshkis [203], FDEs are equations involving the function $x(t)$ of one scalar argument t (called time) and its derivatives for several values of argument t. FDEs can be categorized into retarded, neutral, and advanced types (see, e.g., [74, 152]). If the rate of change of state depends on past states of the system, then the corresponding mathematical model is a retarded functional differential equation (RFDE). If the rate of change of state depends on its own past values as well, then the corresponding equation is called a neutral functional differential equation (NFDE). If the rate of change of state depends on past values of higher derivatives of the state, then the system is described by an advanced functional differential equation (AFDE). These

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equations are also referred to as FDEs of retarded, neutral, or advanced type. While RFDEs and NFDEs have many practical applications, AFDEs are rarely used in engineering modeling due to their inverted causality. Note, however, that there are some special problems even in Newtonian mechanics where the governing equations are related to AFDEs [136, 137].

The literature on FDEs is quite extensive. Several books have appeared summarizing the most important theorems; see, for instance, the books by Myshkis [203], Bellman and Cooke [27], Èl'sgol'c [74], Halanay [98], Hale [99], Driver [72], Kolmanovskii and Nosov [153], Hale and Lunel [100], Kolmanovskii and Myshkis [152], Diekmann et al. [64], just to mention a few. There are also several books dealing with different applications and numerical techniques; see for instance, Stepan [255], Kuang [160], Kuang and Cong [159], Niculescu [207], Hu and Wang [113], Bellen and Zennaro [26], Gu et al. [91], Zhong [306], Michiels and Niculescu [187], Kushner [163], Erneux [79], Balachandran et al. [20], Lakshmanan and Senthilkumar [165], Smith [251], and Yi et al. [299]. It is known that discretization techniques preserve asymptotic stability for RFDEs (see, e.g., [95] or [85]); however, this is not true for NFDEs and AFDEs in general (see, e.g., [81] and [136], respectively).

RFDE is a mathematical terminology. In the engineering literature, RFDEs are referred to as delay-differential equations (DDEs), or simply delay equations. In this monograph, we follow the latter terminology, and use the term DDE rather than RFDE.

This monograph deals with the stability analysis of *linear time-periodic DDEs* using the *semi-discretization method*. These equations often arise during the analysis of delayed systems, since the stability properties of the periodic orbits of nonlinear DDEs are described by linear time-periodic DDEs [143, 158]. This introductory chapter gives a brief overview on some special cases of linear DDEs. The corresponding basic theory is essential for constructing the analytical examples of Chapter 2, which then serve as references for the tests of the numerical method introduced in Chapter 3. The last two chapters investigate pure Newtonian examples of delayed oscillators and the dynamics of real-world engineering problems modeled by time-periodic DDEs.

1.1 Linear Autonomous ODEs

Linear autonomous ordinary differential equations (ODEs) have the general form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) , \tag{1.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{A}$ is an $n \times n$ matrix, and

$$\dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathrm{col}\left(\frac{\mathrm{d}x_1}{\mathrm{d}t} \ \frac{\mathrm{d}x_2}{\mathrm{d}t} \ \cdots \ \frac{\mathrm{d}x_n}{\mathrm{d}t} \right)$$

with $x_1, x_2, \ldots, x_n$ being the elements of vector $\mathbf{x}$. For a given initial value $\mathbf{x}(0)$, the solution of (1.1) can be written in the form

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{x}(0) , \tag{1.2}$$

where $\mathrm{e}^{\mathbf{A}t}$ is the exponential of matrix $\mathbf{A}t$, defined by the Taylor series of the exponential function (see Appendix A.1). For a general overview on matrix exponentials, see the book of Hirsch and Smale [108], or the book of Perko [219].

The stability of the trivial solution $\mathbf{x}(t) \equiv \mathbf{0}$ is determined by the eigenvalues λ_j, $j = 1, 2, \ldots, n$, of the coefficient matrix $\mathbf{A}$. These eigenvalues are the *characteristic exponents* of (1.1), but they are often called *characteristic roots* or *poles*, too. If each λ_j is unique in the minimal polynomial of $\mathbf{A}$, then each solution of (1.1) can be written in the form

$$\mathbf{x}(t) = \sum_{j=1}^{n} \mathbf{C}_j \mathrm{e}^{\lambda_j t} , \tag{1.3}$$

with $\mathbf{C}_j \in \mathbb{C}^n$ being appropriate vectors depending on the initial condition. If the characteristic exponents have negative real parts, i.e., $\operatorname{Re} \lambda_j < 0$ for all $j = 1, 2, \ldots, n$, then the trivial solution of (1.1) is asymptotically stable. In the general case, the characteristic exponents can be determined by solving the characteristic equation

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0 , \tag{1.4}$$

where $\mathbf{I}$ stands for the $n \times n$ identity matrix. Development of (1.4) results in an nth-degree polynomial of λ, whose roots (i.e., the characteristic exponents) can be determined by a number of numerical methods. Stability analysis, however, does not require the exact calculation of the characteristic exponents; only the sign of the real part of the critical (i.e., rightmost) exponent must be determined. This analysis can be performed by the celebrated Routh–Hurwitz criterion [227, 114], which gives a necessary and sufficient condition for stability based on the coefficients of the characteristic polynomial (for details, see Appendix A.2).

Depending on the location of the critical characteristic exponents, there are two typical mechanisms for loss of stability of linear autonomous systems [92]:

1. The critical characteristic exponents form a complex conjugate pair moving from the left-hand side of the complex plane to the right-hand side; they cross the imaginary axis, as shown by case (a) in Figure 1.1. This case is an essential necessary condition for the so-called *Hopf* (or *Andronov–Hopf* or *Poincaré–Andronov–Hopf*) bifurcation of the corresponding nonlinear system, for which the equation under analysis is the variational system. The systematic study of the conditions and a proof of the corresponding bifurcation theorem have been done by Andronov and Leontovich [10] for the two-dimensional case, and by Hopf [109] for the n-dimensional case. According to the theory of nonlinear systems, either stable or unstable periodic motion may exist around the equilibrium of the corresponding nonlinear system, called supercritical and subcritical bifurcation, respectively.

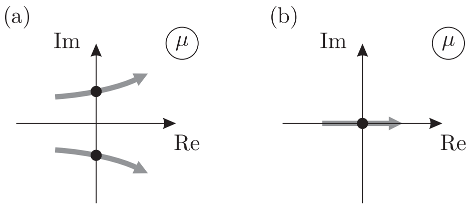

Fig. 1.1 Critical characteristic exponents for linear autonomous ODEs: (a) Hopf bifurcation, (b) and saddle-node bifurcation.

2. The critical characteristic exponent is a real one moving from the left-hand side of the complex plane to the right-hand side through the origin, as shown by case (b) in Figure 1.1. This case is called *saddle-node* bifurcation of the corresponding nonlinear system.

1.2 Linear Periodic ODEs

The general form of linear periodic ODEs reads

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) , \quad \mathbf{A}(t) = \mathbf{A}(t+T) , \tag{1.5}$$

with $\mathbf{x}(t) \in \mathbb{R}^n$. Here, the $n \times n$ coefficient matrix $\mathbf{A}(t)$ is time-periodic at period T, called the *principal period* in contrast to the constant-coefficient matrix of the autonomous system (1.1). The main theorems on general periodic systems are summarized in the book of Farkas [83].

For periodic ODEs, a stability condition is provided by the Floquet theory [84]. The solution of (1.5) with the initial condition $\mathbf{x}(0)$ is given by $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$, where $\mathbf{\Phi}(t)$ is a fundamental matrix of (1.5). According to the Floquet theory, the fundamental matrix can be written in the form $\mathbf{\Phi}(t) = \mathbf{P}(t)\mathrm{e}^{\mathbf{B}t}$, where $\mathbf{P}(t) = \mathbf{P}(t+T)$ is a periodic matrix with initial value $\mathbf{P}(0) = \mathbf{I}$, and $\mathbf{B}$ is a constant matrix. The matrix $\mathbf{\Phi}(T) = \mathrm{e}^{\mathbf{B}T}$ is called the *monodromy matrix* (or *principal matrix* or *Floquet transition matrix*) of (1.5). This matrix gives the connection between the initial state and the state one principal period later: $\mathbf{x}(T) = \mathbf{\Phi}(T)\mathbf{x}(0)$.

The eigenvalues of $\mathbf{\Phi}(T)$ are the *characteristic multipliers* (μ_j, $j = 1, 2, \ldots, n$) (also called *Floquet multipliers* or the *poles* of $\mathbf{\Phi}(T)$) calculated from

$$\det(\mu\mathbf{I} - \mathbf{\Phi}(T)) = 0 . \tag{1.6}$$

The eigenvalues of matrix $\mathbf{B}$ are the *characteristic exponents* (λ_j, $j = 1, 2, \ldots, n$) given by

$$\det(\lambda\mathbf{I} - \mathbf{B}) = 0 . \tag{1.7}$$

If μ is a characteristic multiplier, then there are characteristic exponents λ such that $\mu = \exp(\lambda T)$, and vice versa. Due to the periodicity of the complex exponential function, each characteristic multiplier is associated with infinitely many character-

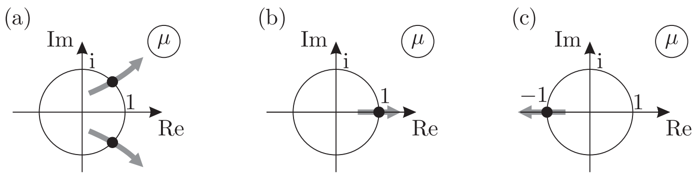

Fig. 1.2 Critical characteristic multipliers for periodic systems: (a) secondary Hopf bifurcation, (b) cyclic-fold bifurcation, and (c) period-doubling bifurcation.

istic exponents of the form $\lambda_k = \gamma + \mathrm{i}(\omega + k2\pi/T)$, where $\gamma, \omega \in \mathbb{R}$, $k \in \mathbb{Z}$, and $T\omega \in (-\pi, \pi]$.

The trivial solution $\mathbf{x}(t) \equiv \mathbf{0}$ of (1.5) is asymptotically stable if and only if all the characteristic multipliers have modulus less than one, that is, all the characteristic exponents have negative real parts.

Similarly to autonomous systems, the basic types of loss of stability can be classified according to the location of the critical characteristic multipliers [92]. For periodic systems, there are three typical cases:

1. The critical characteristic multipliers form a complex conjugate pair crossing the unit circle, i.e., $|\mu| = 1$ and $|\overline{\mu}| = 1$, as shown by case (a) in Figure 1.2. This case is topologically equivalent to the Hopf bifurcation of autonomous systems and is called *secondary Hopf* (or *Neimark–Sacker*) bifurcation.
2. The critical characteristic multiplier is real and crosses the unit circle at $+1$, as shown by case (b) in Figure 1.2. The bifurcation that arises is topologically equivalent to the saddle-node bifurcation of autonomous systems and is called *cyclic-fold* (or *period-one*) bifurcation.
3. The critical characteristic multiplier is real and crosses the unit circle at -1, as shown by case (c) in Figure 1.2. There is no topologically equivalent type of bifurcation for autonomous systems. This case is called *period-doubling* (or *period-two* or *flip*) bifurcation.

Generally, the monodromy matrix cannot be determined in closed form, but there exist several numerical and semi-analytical techniques to approximate it, such as Hill's infinite determinant method and its generalizations [107, 266, 32, 205], the method of strained parameters [205], the method of multiple scales [205], and the Chebyshev polynomial approach [247, 246]. A simple numerical method is the piecewise constant approximation of the periodic matrix $\mathbf{A}(t)$ in the form

$$\mathbf{A}(t) \approx \mathbf{A}_i := \int_{(i-1)h}^{ih} \mathbf{A}(s)\,\mathrm{d}s\,, \quad t \in [t_i, t_{i+1})\,, \tag{1.8}$$

where $t_i = ih$ is the discrete time with $i \in \mathbb{Z}$, $h = T/p$ is the length of the discretization step, and p is an integer [111, 83]. The original system can be approximated by

$$\dot{\mathbf{y}}(t) = \mathbf{A}_i \mathbf{y}(t)\,, \quad t \in [t_i, t_{i+1})\,, \tag{1.9}$$

for which the solution over a discretization interval is

$$\mathbf{y}(t_{i+1}) = \mathrm{e}^{\mathbf{A}_i h}\mathbf{y}(t_i) \; . \tag{1.10}$$

Application of (1.10) over p repeated discretization steps with initial state $\mathbf{y}(0)$ results in

$$\mathbf{y}(T) = \tilde{\mathbf{\Phi}}(T)\mathbf{y}(0) \; , \tag{1.11}$$

where

$$\tilde{\mathbf{\Phi}}(T) = \mathrm{e}^{\mathbf{A}_{p-1}h}\,\mathrm{e}^{\mathbf{A}_{p-2}h}\,\cdots\,\mathrm{e}^{\mathbf{A}_0 h} \tag{1.12}$$

is an approximation for the monodromy matrix $\mathbf{\Phi}(T)$. Eigenvalue analysis of $\tilde{\mathbf{\Phi}}(T)$ gives then an approximate description of the stability properties of (1.5). A higher-order generalization of this piecewise constant approximation technique is the method of Magnus expansion, which involves higher-order terms of the so-called Magnus series of the logarithm of the fundamental matrix $\mathbf{\Phi}(h)$ (see, e.g., [175, 138, 139, 46]). Approximation (1.8) corresponds to the first-order Magnus expansion of $\ln(\mathbf{\Phi}(h))$.

1.3 Linear Autonomous DDEs

The general form of linear autonomous DDEs is

$$\dot{\mathbf{x}}(t) = \mathbf{L}(\mathbf{x}_t) \; , \tag{1.13}$$

where $\mathbf{L} : C \to \mathbb{R}^n$ is a continuous linear functional (C is the Banach space of continuous functions) and the continuous function $\mathbf{x}_t$ is defined by the shift

$$\mathbf{x}_t(\vartheta) = \mathbf{x}(t+\vartheta) \; , \quad \vartheta \in [-\sigma, 0] \; . \tag{1.14}$$

According to the Riesz representation theorem (see [99]), the linear functional $\mathbf{L}$ can be represented in the matrix form

$$\mathbf{L}(\mathbf{x}_t) = \int_{-\sigma}^{0} \mathrm{d}\boldsymbol{\eta}(\vartheta)\,\mathbf{x}(t+\vartheta) \; , \tag{1.15}$$

where $\boldsymbol{\eta} : [-\sigma, 0] \to \mathbb{R}^{n\times n}$ is a matrix function of bounded variation, and the integral is a Stieltjes one, i.e., (1.15) contains both point delays and distributed delays.

The characteristic equation can be obtained by substituting the nontrivial solution $\mathbf{x}(t) = \mathbf{C}\,\mathrm{e}^{\lambda t}$, $\mathbf{C} \in \mathbb{C}^n$, into (1.13), which gives

$$\underbrace{\det\left(\lambda\mathbf{I} - \int_{-\sigma}^{0} \mathrm{e}^{\lambda\vartheta}\,\mathrm{d}\boldsymbol{\eta}(\vartheta)\right)}_{:= D(\lambda)} = 0 \; . \tag{1.16}$$

The left-hand side of this equation defines the characteristic function $D(\lambda)$ of (1.13). The characteristic exponents are the zeros of the characteristic function. As opposed to the characteristic polynomial of autonomous ODEs, the characteristic function $D(\lambda)$ has, in general, an infinite number of zeros in the complex plane, all of which should be considered during the stability analysis. Stability charts that present the stability properties as a function of the system parameters have therefore a rich and intricate structure even for the simplest DDEs.

DDEs containing only point/discrete delays can be given in the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{j=1}^{g} \mathbf{B}_j\, \mathbf{x}(t - \tau_j)\,, \tag{1.17}$$

where $\mathbf{A}$ and the $\mathbf{B}_j$'s are $n \times n$ matrices, $\tau_j > 0$ for all j, and $g \in \mathbb{Z}^+$. In this case, only discrete values of the past have influence on the present rate of change of state.

An example of a DDE with distributed delay is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \int_{-\sigma_1}^{-\sigma_2} \mathbf{K}(\vartheta)\, \mathbf{x}(t + \vartheta)\, \mathrm{d}\vartheta\,, \tag{1.18}$$

where $\mathbf{K}(\vartheta)$ is an $n \times n$ measurable kernel function, $\sigma_1, \sigma_2 \in \mathbb{R}$, and $\sigma_1 > \sigma_2 \geq 0$. The kernel function $\mathbf{K}(\vartheta)$ describes the weight of the past effects over the interval $[t - \sigma_1, t - \sigma_2]$. If the kernel is a constant matrix multiplied by the shifted Dirac delta distribution, i.e., $\mathbf{K}(\vartheta) = \mathbf{K}_0\, \delta(\vartheta + \tau)$ with $\sigma_1 \leq \tau \leq \sigma_2$, then the integral in (1.18) gives the point delay $\mathbf{K}_0\, \mathbf{x}(t - \tau)$.

Linear autonomous DDEs with distributed delay and with a finite number of point delays can be given in the general form

$$\dot{\mathbf{x}}(t) = \int_{-\sigma}^{0} \mathbf{K}(\vartheta)\mathbf{x}(t + \vartheta)\, \mathrm{d}\vartheta\,, \tag{1.19}$$

where $\mathbf{K}(\vartheta)$ is an $n \times n$ measurable kernel function that may comprise a measurable distribution and finitely many shifted Dirac delta distributions. That is, $\mathbf{K}(\vartheta)$ can also be given in the form

$$\mathbf{K}(\vartheta) = \mathbf{W}(\vartheta) + \sum_{j=1}^{g} \mathbf{B}_j \delta(\vartheta + \tau_j)\,, \tag{1.20}$$

where $\mathbf{W}(\vartheta)$ is an $n \times n$ measurable function (a weight function), the $\mathbf{B}_j$'s are $n \times n$ constant matrices, $\delta(\vartheta)$ denotes the Dirac delta distribution, $\tau_j \geq 0$ for all j, and $g \in \mathbb{N}$. Thus, (1.19) can be written as

$$\dot{\mathbf{x}}(t) = \int_{-\sigma}^{0} \mathbf{W}(\vartheta)\mathbf{x}(t + \vartheta)\mathrm{d}\vartheta + \sum_{j=1}^{g} \mathbf{B}_j \mathbf{x}(t - \tau_j)\,. \tag{1.21}$$

A necessary and sufficient condition for the asymptotic stability of DDE (1.13) with (1.15) is that all the infinite number of characteristic exponents have negative real parts and there exist a scalar $\nu > 0$ such that

$$\int_{-\infty}^{0} \mathrm{e}^{-\nu\vartheta} \left|\mathrm{d}\eta_{jk}(\vartheta)\right| < \infty \,, \quad j,k = 1,2,\ldots,n \,, \tag{1.22}$$

where $\eta_{jk}(\vartheta)$ are the elements of $\boldsymbol{\eta}(\vartheta)$. Condition (1.22) means that the past effect decays exponentially in the past. Obviously, this condition holds if σ in the lower limit of the integral in (1.15) is finite.

Although there are infinitely many characteristic exponents, it is not necessary to compute all of them, since stability analysis requires only the sign of the real part of the rightmost one(s). There exist several analytical and semi-analytical methods to derive the stability conditions for the system parameters. The first attempts for determining stability criteria for first- and second-order scalar DDEs were made by Bellmann and Cooke [27] and by Bhatt and Hsu [28]. They used the D-subdivision method of Neimark [206] combined with a theorem of Pontryagin [221]. The book of Kolmanovskii and Nosov [153] summarizes the main theorems on the stability of DDEs, and contains several examples as well. A sophisticated method was developed by Stepan [255] (generalized also by Hassard [103]) that can be applied even for a combination of multiple point delays and for distributed delays. There exist several efficient numerical methods to determine the rightmost exponents for a delayed system; see, for instance, the celebrated DDE-BIFTOOL developed by Engelborghs et al. [76, 77], the pseudospectral differencing method by Breda et al. [34, 35], the cluster treatment method by Olgac and Sipahi [210, 211], the Galerkin projection by Wahi and Chatterjee [289, 290], the mapping algorithm by Vyhlídal and Zítek [287], the harmonic balance by Liu and Kalmár-Nagy [171], or the Lambert W function approach by Ulsoy et al. [14, 298].

The stability properties of DDEs are often represented in the form of stability charts that show the stable and unstable domains, or alternatively, the number of unstable characteristic exponents (also called instability degree) in the space of system parameters. Stability charts for autonomous DDEs can be constructed by the *D-subdivision method*. The curves where changes in the number of unstable exponents happen are given by the so-called *D-curves* (also called exponent-crossing curves or transition curves) given by

$$R(\omega) = 0 \,, \quad S(\omega) = 0 \,, \qquad \omega \in [0,\infty) \,, \tag{1.23}$$

where

$$R(\omega) := \operatorname{Re} D(\mathrm{i}\omega) \,, \quad S(\omega) := \operatorname{Im} D(\mathrm{i}\omega) \,, \tag{1.24}$$

with $D(\lambda)$ being the characteristic function defined in (1.16) and ω the parameter of the curves [256]. Due to the continuity of the characteristic exponents with respect to changes in the system parameters (see, for instance, [187]), the D-curves separate the parameter space into domains where the numbers of unstable characteristic exponents are constant. The determination of these numbers for the individual