Research in Mathematics Education
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Kelly S. Mix · Michael T. Battista *Editors*

Visualizing Mathematics

The Role of Spatial Reasoning in Mathematical Thought

Springer

Research in Mathematics Education

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Foreword

In 2016, we were approached by series editor, Dr. Jinfa Cai, with a novel idea—invite authors from the fields of developmental psychology and mathematics education to write about their work on spatial visualization and mathematics, and then ask them to write commentaries on one another's chapters. The goal was to provide a unique view of research on this topic that encompassed both disciplines, as well as foster cross-field communication and intellectual synergy. We eagerly took up the challenge and invited scholars whose work we knew to be at the forefront of our respective fields. The chapters and commentaries contained in this volume are the products of this esteemed group. They reflect the state of the art in research on spatial visualization and mathematics from at least two perspectives. They highlight important new contributions, but they also reveal the fault lines between our respective disciplines. The commentaries insightfully point out some of these fault lines, as well as the immense common ground and the potential for deeper collaboration in the future.

The basic question of how spatial skill relates to mathematics has received steady attention over the years. In psychology, most of this work has focused on long-term outcomes in STEM fields for individuals with more advanced spatial skill (e.g., Wai, Lubinski, & Benbow, 2009), the possibility that spatial deficits contribute to poor mathematics outcomes in children (e.g., Geary, Hoard, Byrd-Craven, Nugent & Numtee, 2007), and the use of materials that physically embody (via spatial relations) abstract mathematics concepts (see Mix, 2010, for a review). Running through these disparate research programs is the shared notion that spatial thinking plays a major role in understanding mathematics, but it has not been addressed head on in psychology until recent years.

In mathematics education, Clements and Battista, in their 1992 research review, address just this issue. They wrote that both Hadamard and Einstein (renown mathematicians) claimed that much of the thinking required in higher mathematics is spatial, and they cited positive correlations between spatial ability and mathematics achievement at all grade levels. However, even in that time period, the relations between spatial thinking and learning nongeometric concepts did not seem straightforward, and there were conflicting findings. For some tasks, having high-spatial skill seemed to improve performance, whereas in other tasks, processing mathematical information using verbal-logical reasoning enhanced performance compared to students who processed the information visually. Other mathematics education researchers countered that the understanding of some low-spatial students who did well in mathematics was instrumental, whereas high-spatial students' understanding was more relational, a difference often not captured by classroom or standardized assessments. Clements and Battista concluded that even though there was reason to believe that spatial reasoning is important in students' learning and use of mathematical concepts—including nongeometric concepts—the role that such reasoning plays in this learning remained elusive.

Possibly because of this elusiveness, interest in the topic waned in mathematics education. However, currently there is intense interest in this general topic in both psychology and mathematics education due to its potential educational benefits (Newcombe, 2010) and the insights into the relations found through extensive and detailed student interviews (Bruce et al., 2017; Davis et al., 2015). The chapters contributed to this volume represent various approaches to advancing this work in education or moving the work in both fields toward educational application.

The developmental psychology chapters tended to focus on the underlying mental representations used to understand mathematics, and the extent to which these representations already involve, or could be improved by spatial processing. Cipora, Schroeder, Soltanlou, and Nuerk provide a detailed analysis of the link between spatial and numerical processing purportedly demonstrated by spatial-numerical association (SNA) or mental number line effects. They conclude that spatial skills provide a crucial tool for understanding mathematics, but this relation may not be realized in the form of a fixed mental number line. Congdon, Vasileyva, Mix, and Levine examine a deep psychological structure that may underlie a range of mathematics topics—namely, the structure involved in identifying and enumerating spatial units of measurement. They argue that mastery of this structure has the potential to support mathematics learning throughout the elementary grades and perhaps head off misconceptions related to fractions, proportions, and conventional later on. Similarly, Jirout and Newcombe focus on another spatial relation with strong ties to mathematics—namely, relative magnitude—outline its potential role in improving instruction on whole number ordering, fractions, and proportions. Casey and Fell discuss the difference between general spatial skill and spatial skill instantiated in specific mathematics problems, concluding that the most effective way to leverage spatial training to improve mathematics outcomes is likely the latter. They highlight a number of instructional techniques from existing curricula that successfully use spatial representations. Finally, Young, Levine, and Mix considered the multidimensional nature of spatial processing and mathematics processing and the inherent complexity involved in identifying possible instructional levers. Following a critique of the existing literature, including recent factor analytic approaches, they conclude with a set of recommendations for improving these approaches and applying what is already known in educational settings.

The mathematics education chapters discuss the spatial processes involved in specific topics in mathematics. Sinclair, Moss, Hawes, and Stephenson examine how children can learn "*through* and *from* drawing," focusing on spatial processes and concepts in primary school geometry. They argue that drawing is not innate but can be improved, and they illustrate through fine-grained analysis how the potential benefits of geometric drawing can be realized in classrooms. Gutiérrez, Ramírez, Benedicto, Beltrán-Meneu, and Jaime analyze the spatial reasoning of mathematically gifted secondary school students as they worked on a collaborative, communication-intensive, task in which they were shown orthogonal projections of cube buildings along with related verbal information. The authors related the objectives of students' actions and their visualization processes and students' solution strategies and cognitive demand. Herbst and Boileau argue that high school geometry instruction can do more than provide names for 3D shapes and formulas for finding surface area and volume. They illustrate, and invite reflection on their design of, a 3D geometry modeling activity in which students write and interpret instructions for how to move pieces of furniture up an L staircase. Lowrie and Logan discuss how the frequency of encountering, and interacting with, information in visual/graphic format, including on the web, has increased our need for research on the role of spatial reasoning in students' encoding and decoding of information in mathematics. To this end, they analyze the representational reasoning of students engaged in tasks that permit different types of representations, from diagrams to equations. Battista, Frazee, and Winer describe the spatial processes involved in reasoning about the geometric topics of measurement, shapes, and isometries. They introduce, and use in their analysis, the construct of *spatial-numerical linked structuring* as the coordinated process in which numerical operations on measurement numbers are linked to spatial structuring of, and operation on, the measured objects in a way that is consistent with properties of numbers and measurement.

As the chapters and commentaries illustrate, there are still fundamental differences between how researchers in psychology and mathematics education view and investigate the fundamental relations between spatial and mathematical reasoning. However, these differences provide fertile ground for exciting new investigations as each field respectively has advanced knowledge in some areas while leaving gaps in others. The commentaries are a starting point for identifying these points of contact and complementarity. We encourage readers to reflect on how the research in the two fields might be further integrated and how to build productive collaborations between the two sets of researchers. We thank all of our authors for taking a first step in this direction.

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Part I
Psychological Perspectives

Chapter 1
How Much as Compared to What: Relative Magnitude as a Key Idea in Mathematics Cognition

Jamie Jirout and Nora S. Newcombe

Abstract Most topics beyond basic arithmetic require relative magnitude reasoning. This chapter describes the link between relative magnitude reasoning and spatial scaling, a specific type of spatial thinking. We discuss use of the number line, proportional reasoning, and fractions. Consideration of the relational reasoning involved in mathematics can advance our understanding of its relation to spatial skills, and has implications for mathematics instruction, such as using spatial reasoning interventions in developing effective methods for supporting relative magnitude understanding. We review evidence that interventions can be successful in promoting better relative magnitude understanding and associated spatial-relational reasoning, and suggest that education considers ways of including relative magnitude learning, along with more traditional whole-number operations, in early educational efforts.

Keywords Spatial scaling · Spatial learning · Spatial development · Spatial visualization · Scale · Spatial representations · Representations · Diagrams · Spatial-relational · Relative magnitude · Absolute magnitude · Magnitude reasoning · Number line estimation · Proportional reasoning · Fractions · Symbolic understanding · Benchmark strategy · Manipulatives · Interventions · Spatial play

Is "½" big or small? Reasoning about this question demonstrates two types of numerical reasoning: if you answered that 1/2 is small, because it is less than one, you are reasoning about the number as an absolute value. If your answer is to ask, ½ of what, you are reasoning about relative magnitude. In this chapter, we suggest that

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number is often interpreted in an absolute sense in mathematics education, though we describe how most topics beyond basic arithmetic require relative magnitude reasoning. We discuss how relative magnitude reasoning might involve a specific type of spatial thinking: spatial scaling. Specifically, use of spatial scaling can contribute to precision in relative magnitude reasoning, perhaps by tapping a more generalized magnitude representation. Focusing on relative magnitude reasoning is an important consideration when determining how spatial thinking relates to mathematics learning, and may have implications for approaches to mathematics instruction, such as using spatial reasoning research and interventions in developing effective methods for supporting relative magnitude understanding. More broadly, consideration of relational reasoning involved in mathematics might advance our understanding of its relation to spatial skills, and even support the inclusion of emphasizing spatial learning as a way to prepare for and support mathematics learning.

The National Research Council has outlined goals for mathematics education in which they suggest that numeracy should be the topic most emphasized early on (NRC, 2009). The content standards of the National Council of Teachers of Mathematics concur (NCTM, 2010). But it can be hard to follow these guidelines because it is not always clear what numeracy or "number" means. In early elementary school, number often means integers, using count words to enumerate sets of discrete objects, and to add and subtract from those sets. In a kindergarten mathematics lesson, for example, students might assemble sets of objects to make a specific number or, later in the year, use sets of objects to do addition and subtraction problems. The kindergarten children are encouraged to think about numbers as referring to these collections, such as telling "how many." When children think about this kind of number—say, they imagine "5"—they need to recognize the Arabic symbol and recall the word "five," remember that five comes after four and before six, and, crucially, imagine a set of five objects. They might also know that five can be used to refer to a time or date, or to a five-dollar-bill or a 5-year-old child. But in most cases, they are thinking about the value of a positive integer referring to a collection of discrete objects.

In later grades, lessons would be different. Third graders might be learning how to move a decimal point to convert percentages to proportions. Fifth graders might be learning about remainders in long division problems. These lessons involve a fundamentally different kind of "number" than integers. Older children need to think about numbers as a number system, which can quantify many kinds of referents, including referring to continuous magnitudes that need not, and often do not, denote collections of discrete objects. When comparison of fractions is required, when calculating a proportion or percentage, or when dividing one number into another but with some quantity remaining, relative magnitude is key. How is this shift, from number as discrete objects to the meaning of number, dependent on the specific problem?

To determine how to support children's underlying representation of number as it becomes more complex, it is important to first make the different representations explicit. Recently, Newcombe and colleagues defined the different categories of quantification types across two dimensions (Newcombe, Frick, & Möhring, 2018).

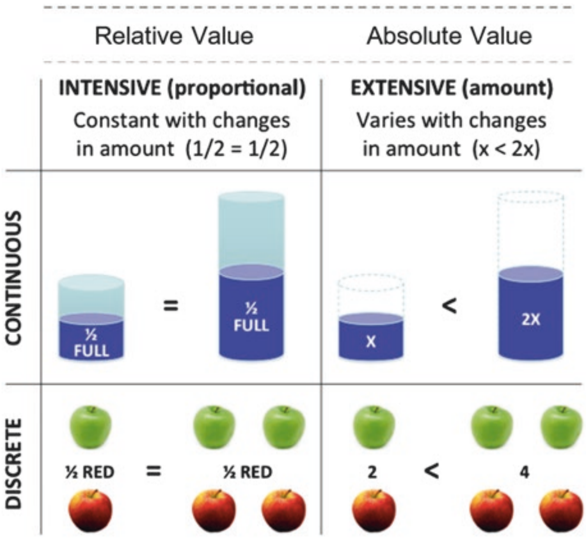

Fig. 1.1 Newcombe et al.'s (2018) categorization of systems of quantification, modified

A first distinction is that number may refer to collections of discrete objects, as shown in the bottom row of Fig. 1.1. However, the number system can also be used to quantify continuous magnitudes, as shown in the top row of Fig. 1.1. A second distinction is that numbers may refer to relative (or intensive or proportional) quantities, as shown in the left column of Fig. 1.1, or to absolute (or extensive) quantities, as shown in the right column of Fig. 1.1. Jointly, these two dimensions generate a four-cell classification system. Kindergarten mathematics generally focuses on the bottom right, but eventually, children must learn about the whole system (Common Core State Standards Initiative, 2010). Thus, thinking about numbers in terms of these varied uses and meanings is important in understanding the development of mathematical cognition, and in identifying effective ways of supporting students' representation of number as they continue in their education.

The distinctions of quantification categories are also important in understanding the link between mathematical and spatial thinking (Newcombe et al., 2018; Newcombe, Levine, & Mix, 2015). Spatial reasoning has multiple aspects, including a distinction between the spatial characteristics of an object itself (intensive) and the spatial position of an object in relation to other objects and its surroundings (extensive) (Newcombe & Shipley, 2015). In this chapter, we focus on the overlap between the relational reasoning processes involved in mathematics and spatial reasoning tasks. We focus on relative magnitude, a concept that is fundamental both to spatial scaling and to proportional reasoning of the kind shown in the left column of Fig. 1.1. Spatial relational reasoning skills are seen in young children and even infants, and could help support similar relational reasoning processes in mathematic tasks involving relative magnitude.

This chapter will begin with examples of relative magnitude in mathematics education, examining number line understanding, proportional reasoning, and fraction learning. We then provide a review of research on relative-magnitude reasoning in spatial processes, especially spatial scaling, and how these spatial processes relate to mathematic skills and are utilized when using external representations. We return to number line, proportional reasoning, and fraction tasks to discuss how spatial representations, and thus spatial-relational reasoning, are used in learning. Finally, we conclude with a discussion of interventions shown to improve children's relative magnitude understanding, and possible connections between spatial learning and developing mathematic skills. We end with a discussion of potential implications of the research on spatial thinking and relative magnitude for mathematics education.

Relative Magnitude in Mathematics Learning

The importance of relative magnitude is evidenced by its necessity for many mathematics tasks both in education and in everyday life. Simple questions like whether a number is "big" or "small" must be evaluated relative to some comparison or scale, for example knowing where to put the number nine on a number line depends on the range of the line (i.e., toward the end on a 0–10 line, but toward the beginning on a 0–100 line). Relative magnitude is important in thinking about proportions to determine what things are being compared and how, for example knowing how much sugar to add for a cup of lemonade when using one lemon, if you know you use a whole cup of sugar for a pitcher using four lemons. Early fraction tasks such as dividing something among friends require understanding the meaning of a part-whole relation, for example, that as the number of friends sharing a cake (the denominator) becomes larger, the portion of the cake that each receives becomes smaller. These different tasks share the common cognitive process of relative magnitude reasoning. We explain this idea further for each of three mathematical tasks now, and return to these tasks later in the chapter to discuss relations across relative magnitude tasks and how spatial representations influence relative magnitude reasoning in the tasks.

Number Lines as Relative Magnitude

Although some mathematic tasks in research measure absolute value knowledge, others, including the widely used number line estimation, involve relative magnitude reasoning. In early development, understanding of number is often assessed with tasks asking children to give a specified number of objects, or the "give-N" task (e.g., Wynn, 1990). But many studies of mathematics cognition or number understanding use measures of relative rather than absolute magnitude reasoning (e.g., using estimates of *more* or *less* rather than absolute value). Rather than asking how many, these tasks rely on children considering *how much/many compared to what*. For example, on the widely used number-line estimation task, children are asked to show "how much" of the given line is equivalent to a specific value, but often they

Fig. 1.2 Example of the number line estimation task. In one version, the participant is asked to mark a line to show where a given Arabic numeral would go, relative to the scale provided. In another version, the participant is asked to give the value of the mark provided on the number line, relative to the scale provided. Here, the mark shows a value of 50 relative to the 0–100 scale. If the scale changed to 0–1000, the mark would show 500

are provided beginning and endpoint values, including a scale for comparison to the value (Siegler & Booth, 2004). The child's specific task is to place a mark on the line to represent where a specific number would fall, or to provide an estimate of the number represented by a mark shown on the line (see Fig. 1.2). As opposed to the give N tasks, successful performance on the number line task requires relative magnitude understanding. The magnitude of the space between the provided endpoints is relative to the scale (i.e., 1 in. on a 10-in. number line represents a single unit for a scale of 0–10, but represents 10 units if the scale is 0–100). This requires children to shift from their early mathematics experience in thinking about number as absolute values of discrete objects, to thinking about number as a value relative to the scale. Younger children sometimes use a more familiar discrete counting strategy, ignoring the provided endpoint value. This can result in overestimating lower numbers across the line, and then squeezing the larger numbers toward the end of the scale, resulting in a logarithmic representation of the number scale. As they get older, children begin to use more proportional strategies and thus their representations become more linear, though a shift back to the earlier strategy is observed as the scale increases (e.g., 0–1000 to 0–10,000). That is, children progress from basic concepts such as knowing that the word two means two objects, to having linear representations of numbers on scales of 1–10 (age 3–5), 1–100 (age 5–7), 1–1000 (age 7–11), and eventually understanding fractions in a similar way, beginning around age eight, advancing through adulthood (Siegler & Braithwaite, 2017).

In mathematics education, much emphasis is placed on understanding the absolute magnitude of numbers in early elementary school. Yet, even as early as first grade when addition and subtraction are emphasized, current standards explicitly mention the importance of understanding relative magnitude of numbers as well (Common Core State Standards Initiative, 2010). As children begin to learn more complex mathematics, reasoning about relative magnitude becomes much more central, as the content begins to rely much more on relational reasoning than absolute values of number when fractions, proportions, functions, probabilities, etc. are introduced (Common Core State Standards Initiative, 2010). Beginning in third grade, children whose education thus far focused on the absolute value of numbers are expected to shift to reasoning about relative magnitude, where paying attention to whole numbers would lead to less accurate performance (DeWolf & Vosniadou, 2011). It is important, then, to consider how this shift can be supported in educational practice.

Proportional Reasoning

Proportional reasoning is typically used to solve a problem in order to reach a specific value of interest, yet, like number line estimation, it requires relative magnitude reasoning. Reasoning about proportion is observed in everyday problems, such as when comparing costs of two items that are different prices and amounts or to estimate total cost with sales tax. In mathematics instruction, proportional reasoning is necessary when calculating concentrations of a solution, or in traditional multiplication problems such as using a given proportion—say, the speed a car travels—to determine how long it will take for the car to reach a destination of a specific distance. This task is dependent on the ability to reason about one quantity relative to another, determining a ratio, and often to apply this relational information to another context. Though the algorithmic use of formulas often studied to solve this type of problem does not seem to involve relative magnitude (i.e., plugging numbers in), many direct measures of proportional reasoning require children to use relative magnitude reasoning. For instance, proportional reasoning tasks in research typically use spatial displays of concentrations, such as proportions of water to juice, with children choosing a matching concentration that would taste the same (i.e., have the same concentration) or showing the concentration using a scale from very weak to very strong (Boyer, Levine, & Huttenlocher, 2008; Möhring, Newcombe, Levine, & Frick, 2016a).

Fraction Learning

Like proportions, fractions are part-whole relations in which their meaning is derived from relative magnitude. But unlike proportions, fractions are considered numbers themselves that can be represented on a number line, and they can be greater than one. In fact, many researchers and educators argue that thinking about fractions as numbers by representing them on a number line (relative to other fractions and whole numbers) improves learning. Specifically, the widely used Common Core standards suggest that students learn to place fractions on number lines, implicitly considering them to be absolute magnitudes (Common Core State Standards Initiative, 2010). Yet when fractions are included in mathematics problems, understanding them as relative magnitudes helps conceptual understanding. For example, it is fairly easy to place the fraction one half between zero and one on a number line, but if you are multiplying a number by one half, it is helpful to think of it as signifying one of two equal parts of a quantity, which may of course be much greater than one. The relation between the numerator and denominator is what makes a fraction meaningful, and knowing that this relation can be applied to any quantity just as proportions can be. Understanding fractions is important for success in algebra (NMAP, 2008), with algebra considered to be "the gatekeeper to higher learning in mathematics and science" (Booth & Newton, 2012, p. 247), and it lays the foundation to more advanced mathematics. It has been suggested that the relation between fraction knowledge and later success in algebra may be due to more general underlying knowledge of number systems and magnitude, supporting more abstract mathematical reasoning (Ketterlin-Geller, Gifford, & Perry, 2015).

Relative Magnitude and Spatial Thinking

Similar to the varied conceptualizations of number, spatial thinking is a broad label for many different types of reasoning. For example, planning how to navigate from one point to another requires spatial thinking, but so does determining which size pot will hold the amount of food you plan to cook; children's spatial thinking in play can involve matching up pieces of a puzzle or building with Lego diagrams, or comparing lengths of sticks to choose which should be the *daddy* vs. *baby*. Spatial thinking is also multidimensional, similar to the conceptualizations of number.

Relational thinking can be observed in intensive spatial tasks similar to number tasks. For example, relational position of discrete objects in a model or diagram (i.e., attending to the relative position within an array) is used to solve spatial analogies. In arrays of toys where array 1 is "pig," "dog," "chick" and array 2 is "horse," "mouse," "dog," the relational match to "dog" from array 1 is "mouse" in array 2. This relational matching of relative position can also be done with continuous spaces, such as finding a location in real space on a map. When using a map, the task draws on relative magnitude when determining the position of a target in the continuous space, creating a representation or estimation of relative position that can then be applied to another space. When this relational matching is done across spaces and/or representations of different size, the process of spatial scaling is used. Spatial scaling is one area of focus in research investigating the relation between spatial thinking and mathematics tasks requiring relative magnitude.

Spatial Scaling

Spatial scaling is the ability to reason about spatial relations in one context and to apply this relational information to a different sized spatial area. Just as proportions and fractions often involve identifying relational information and applying it to solve a problem, spatial scaling involves two steps: first, recognizing the relational correspondence between the two areas; second, mentally transforming the spatial-relational information from one space to the other (Möhring, Newcombe, & Frick, 2014). An example of spatial scaling is the process used when looking at the distance between two points on a map and then determining that distance in real space. Just as mathematics-relational reasoning can be assessed with number-line estimation, spatial-relational reasoning can be assessed using a spatial scaling measure (see Fig. 1.3). Instead of a symbolic number, children are shown a scaled map

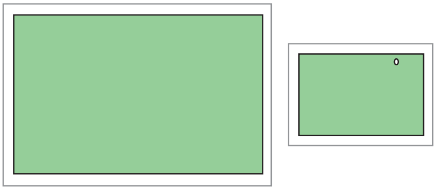

Fig. 1.3 Example of a spatial scaling task with a map (right) and referent space (left)

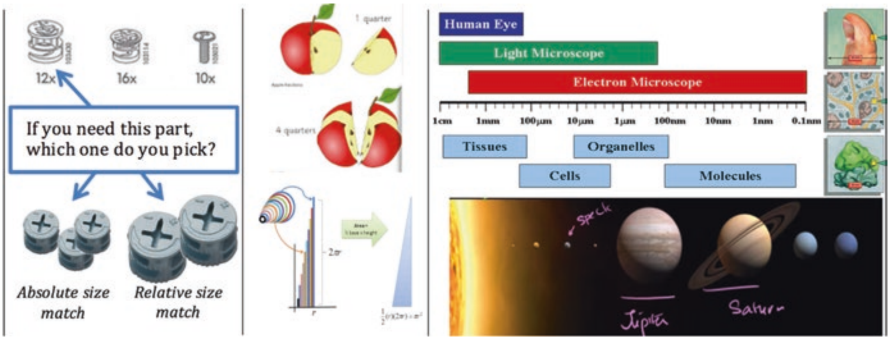

Fig. 1.4 Examples of diagrams requiring relational-reasoning, shown for every-day tasks (left), mathematics learning (middle), and science learning with microscopic (right-top) and telescopic (right-bottom) scales

marking a location, and are asked to match that location on a larger space (Frick & Newcombe, 2012).

By definition, spatial scaling requires a representation and a referent. A key function of representations is to convey various types of relational information about the referent. Examples of everyday representation use include navigating a space (referent) that is larger than one can see from a single viewpoint, such as with scaled maps (representation), or showing things too large or small to be seen with the human eye (referent), like in scaled photographs (representation). Other examples include configuring complex materials into a desired state, like when using building instructions, or for displaying information to show relations visually as charts and graphs do. In all of these examples, relational reasoning is critical, and scaling is often used—both in understanding the scale of the representation and, in several cases, extrapolating the information to apply it to another scale (e.g., navigating through a park, or determining which correct size part to use for a step when building furniture, see Fig. 1.4).

Focus on superficial or perceptual information can lead to ineffective use of the representation, for example not going far enough when driving because the distance on a map looked small, while the scale actually indicated it was quite far. Similarly, focus on perceptual rather than relational information in mathematics learning can cause difficulty. For example, in the original conservation of number tasks, Piaget found that children focused on the perceptually salient spatial information of how far a sequence of objects stretched as indicating that it had more, even after just observing that the sequence was equivalent to another before being rearranged (Piaget, 1952). In more advanced magnitude comparison, children often struggle with ordering fractions by magnitude because although magnitude decreases as the denominator increases, they mistakenly associate this increase with an increase in magnitude (e.g., saying one fourth is larger than one half, because four is greater than two).

Research on performance in proportional reasoning and fractions shows a correlation between these processes and spatial scaling. Despite these tasks being different, this relation supports the common reliance on relative magnitude for successful performance (Möhring et al., 2016a; Möhring, Newcombe, Levine, & Frick, 2016b). More broadly, spatial magnitude information is used when interpreting numerical magnitude, for better or for worse (Newcombe et al., 2015); spatial thinking is necessary when completing tasks like placing numbers on a number line, but can also cause bias or error such as Piaget's number conservation mistakes or when estimations are pulled toward discrete markers or benchmarks. Further, children often learn about proportional reasoning and fractions using diagrams and representations, thus it is important for them to understand how to use them correctly, which often requires spatial scaling.

In fact, the importance of understanding scale of diagrams and using scaled diagrams, as well as an emphasis on scaling skills more generally, is made explicit by its inclusion throughout the common core mathematics standards (Common Core State Standards Initiative, 2010). Although early developmental theory suggested that processes like proportional reasoning are well beyond what young children are capable of (e.g., Piaget, 1952; Piaget & Inhelder, 1956), measures of relational reasoning, especially spatial-relational reasoning and magnitude understanding, suggest that even infants show early aptitude and children become progressively more skilled in early years, with both improved scaling accuracy and more advanced proportional reasoning with age (Lourenco & Longo, 2011).

Spatial Representations

Visual representations like diagrams are common in and important for mathematics education (National Council of Teachers of Mathematics, 2010). These representations rely on spatial thinking, but also have the potential to help develop relational-reasoning skills (Davies & Uttal, 2007; Gentner, 1988) and promote broader learning in mathematics (Woodward et al., 2012). Spatial-relational reasoning might be specifically important for fully understanding and using representations. Representations are not always helpful, for example, if they are perceived as pictures or objects of their own rather than as representing information or a referent (Garcia Garcia & Cox, 2010; Uttal & O'Doherty, 2008). Several varying factors of implementing representations might influence their effectiveness for learning: the structure of representation implementation, type of contact with the representations, the relation of the representation to familiar symbols, and the amount of exposure to the representations (Mix, 2010). The type of representation and the learning context can vary further in the mechanisms of support representations may offer, from reducing fine motor and cognitive demands and to providing metaphors or embodied experiences (Mix, 2010). The implementation methods, support sought and available from representations and needs of different learners, and the specific learning goals more broadly are all important considerations in using representations for mathematics instruction. Importantly, though, children must have the necessary skills for understanding representations in the first place, such as their representational nature (Uttal & O'Doherty, 2008).

As discussed above, interpreting information from spatial representations often requires spatial scaling. Although children rarely receive explicit instruction on scaling, research shows they are capable of spatial scaling from very young ages. Success in using scaled representations relies on matching the *relational structure* of the representation, which they learn to do quite early (Gentner, 1988). Failure in this step, for example, would be in choosing the absolute rather than relative size match in Fig. 1.4 (left). Children develop more accurate scaling ability in the early elementary school years, though even adults perform poorly when reasoning about very small or very large scales, thus there is room for improvement. One potential mechanism for developing spatial scaling is improved relational reasoning, such as learning the related processes of proportions or fractions taught in late elementary and middle school grades. Alternatively, early practice with scaled spatial representations could help prepare children for later mathematics learning by strengthening their relational reasoning.

To understand how spatial representations might help support learning of relative magnitude mathematics tasks, an important question is, how do children learn relational reasoning with spatial representations? Though instances of children's relational thinking can be observed in very young children, and even infants, research shows improvement in representation use, and changes in strategies or processes involved, with age. For example, children demonstrate the ability to understand symbolic correspondence, or that a model represents something else, around the age of 3 years old (DeLoache, 1987). However, understanding spatial representations typically also requires geometric correspondence (Newcombe & Huttenlocher, 2000). Children must be able to encode length or distance within representations and then map that to the referent space, while conducting any necessary scaling if the representation is a different size (Möhring et al., 2014). Young children can successfully use models, typically by using landmark mapping and perceptual coding of the space, and there is some evidence they use categorical or even distance coding (Huttenlocher, Newcombe, & Vasilyeva, 1999). However, coding distances and scaling are not simple tasks. Difficulty can easily be increased by requiring children to code distances along both horizontal and vertical axes or by removing landmark cues. Even adults completing a simple scaling task (i.e., using a map to mark a target on a two-dimensional space) show variation in accuracy as a function of the scale of the representation to the referent space (Möhring et al., 2014).

The adult strategy for using scaled representations is likely determined by the need for accuracy; while adults can quickly and quite accurately interpret scaled information using perceptual methods (on familiar scales), they are also able to use a slower metric coding strategy, which should provide even more precision. This more sophisticated metric strategy is difficult for children, because it relies on proportional reasoning. Though they do not calculate exact proportions, young children have been found to be capable of using basic spatial categories—for example using a midpoint to create categories within the space (Huttenlocher, Newcombe, & Sandberg, 1994). Just as estimation errors on the number line are seen around benchmarking points, suggesting the use of proportional reasoning strategies, spatial estimation responses also show error patterns that suggest they are being slightly biased by these categorical boundaries, where their responses are pulled toward corners or midpoints (Sandberg, Huttenlocher, & Newcombe, 1996; Slusser, Santiago, & Barth, 2013). This more implicit or categorical way of reasoning about proportions might prepare children for later mathematics learning that requires relative magnitude understanding. Further, spatial representations can be used specifically to help children reason about relative magnitude in mathematics learning.

Spatial Representations in Relative Magnitude Mathematics Tasks

We return now to our discussion of the relative magnitude mathematics tasks discussed above, with a focus on spatial representations and how they might support mathematics learning. Common Core standards include using number lines (similar to the research task discussed earlier) to represent whole numbers and fractions (Common Core State Standards Initiative, 2010). Studies show that number line estimation performance relates to spatial thinking, perhaps by emphasizing the spatial importance in number understanding as a linear representation of numbers (Gunderson, Ramirez, Beilock, & Levine, 2012). The question of what specific processes are involved in number line estimation is the subject of much research. Many studies show that there are spatial patterns in developing number line estimation skills by what is described as a representation shift. Performance on the number line estimation task transitions from estimates that are better fit by a logarithmic function (i.e., lower numbers tend to be, with larger numbers being grouped too closely toward the top of the scale) to a linear fit where numbers are placed relatively equally spaced apart and in order. Children's estimations become more linear with age, and the timing of this transition is positively correlated with scale; the older children get, the higher the scale showing a linear representation (Siegler & Opfer, 2003). This body of research emphasizes this representation shift as an important developmental milestone of number understanding, for good reason, but it should also be noted that the shift is referring to children's *spatial* representations.

One reason linear estimates on the number line are important is that they show understanding of equal spacing between whole numbers, and correct knowledge of order in which the numbers follow; this tends to occur when scales are familiar to the participant (Ebersbach, Luwel, Frick, Onghena, & Verschaffel, 2008). Yet even when one knows the order of numbers, the one-to-one correspondence on a number line is not as easy as counting objects—the space provided must be divided into the equal units. Studies investigating *how* people use the given scale to determine where a number goes provide evidence that relational reasoning is involved. Specifically, studies using eye-tracking and differences in error patterns show that both children and adults use proportional reasoning strategies when making estimations (Barth & Paladino, 2011; Schneider et al., 2008; Slusser et al., 2013). These studies indicate the use of landmarks—the endpoints as well as fractional landmarks (e.g., ½, ¼). The level of proportional benchmarking relates to improved performance; the more precise proportions used (i.e., more benchmarks), the more accurate estimations were (Peeters, Verschaffel, & Luwel, 2017). Peeters et al. (2017) found that providing explicit proportional markers can induce finer grained proportional reasoning with landmarks, but the use of implicit landmarks (i.e., the proportional ones) also appears to increase with age (Slusser et al., 2013). Thus, not only is children's developing number line estimation measured as a spatial representation, it also draws on mathematical reasoning related to relative magnitude understanding, and can be improved using spatial-relational interventions with representations.

The use of proportional landmarks is observed in other mathematics tasks involving representations as well. Children at the earliest ages of formal schooling show the ability to use ½ as a benchmark when comparing proportions (Singer-Freeman & Goswamani, 2001; Spinillo & Bryant, 1991). On basic part-whole fraction tasks, emerging understanding appears by 4 years of age (Mix, Levine, & Huttenlocher, 1999; Singer-Freeman & Goswamani, 2001). Some research on early proportional reasoning suggests that the specific features of diagrams or materials used to represent proportions might impact children's attention to the relative vs. absolute magnitude information. Boyer and Levine (2012) asked children to match proportions displayed using a "juice task," where different concentrations of juice to water were displayed using red and blue segments (see Fig. 1.5). Sometimes the diagram included a single red and a single blue bar, differing in size, to show continuous proportion of whole (e.g., 50% each; Fig. 1.5, bottom). Other diagrams showed the

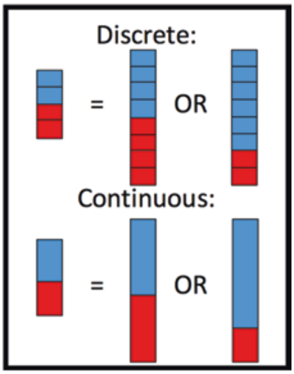

Fig. 1.5 Discrete vs. continuous proportional representations

same information, but the blue and red bars were divided using discrete units (e.g., two red, two blue; Fig. 1.5, top). Children were then asked to choose which of two options was a match to the concentration (i.e., "Which of these two would taste like Wally Bear's Juice?"). Children were better at matching the continuous bars, though their performance decreased as the comparison became less similar by becoming increasingly different in size (for example, choosing 2/8 instead of 4/8 as a match for 2/4). This effect of scale difference is similar to that observed in tasks of spatial scaling, where performance decreases as the scale factor increases (Möhring et al., 2014). When the visualizations included the discrete units, children often defaulted to an inaccurate counting strategy of the units shown rather than using proportional reasoning; when they were asked to produce a matching proportion, the lines of the discrete units actually amplified errors across scale factors, pulling responses further from the correct location compared to the continuous displays (Boyer & Levine, 2012). This difficulty in applying proportional information from one display to another with discrete units seems unfortunate, since real-world use of proportional reasoning often involves discrete units. For example, if you figure out that pancakes come out best when you combined 2-tbs water to 3-tbs boxed mix and want to now make more than a single pancake, you can determine the proportional composition of 2:3 or 40% water to 60% boxed mix, and then apply this proportion to a unit of measurement (rather than estimating), for example 1-cup water to 1.5-cups mix. The key is to translate the initial discrete amounts into a proportion, rather than trying to apply the raw values from the first amounts. It may not be that children are unable to apply proportional information to quantities of discrete units, but that they are being distracted from determining the proportional information in the first place.

Boyer and Levine (2015) tested this by having children select a proportional match with the comparison proportion presented in spatial representations using either discrete or continuous amounts. As expected, when children are presented first with proportions shown as continuous quantities, they then do better on proportions shown as discrete quantities, though only the older children benefited from the different displays (Boyer & Levine, 2015). The authors suggest that providing the continuous representation first encourages more spatial proportional reasoning strategies rather than a "count-and-match" method. More generally, these finding suggest that spatial visualization using representations might help direct attention to relative magnitude (i.e., the continuous presentation) rather than whole number counting (i.e., the discrete presentation).

Spatial representations of proportions like those in Fig. 1.5 can also be used to show fractions concepts, e.g., 1:2 or 50% is the same as ½, and fractions can be similarly displayed using part-whole diagrams. However, fractions can be larger than one to include any rational numbers, sometimes making diagrams of part-whole relations confusing. Reasoning about fractions can also become difficult when the fraction is to be applied to another number, such as multiplying or dividing by a fraction. Even a simple magnitude comparison task can be challenging if the numbers in the larger fraction have smaller components, (e.g., 3/4 vs. 8/20; Lortie-Forgues, Tian, & Siegler, 2015). Siegler and Lortie-Forgues (2017) outline the reasons why fractions and rational numbers more generally are difficult to understand, some of which relate to the differences between proportions and fractions. First, the same value can be symbolized in more than one way, such as 1/2 and 2/4 or 1½, 3/2 and 9/6. Second, the presentation of fractions as a numerator over denominator can cause difficulty in correctly understanding the fraction based on inaccurate application of whole-number knowledge. If much of your experience is with whole numbers, it is hard to switch to thinking of the numerical symbols not as individual whole numbers—one half is not "one" and "two." In this case, spatial representations of fractions could be helpful in demonstrating the meaning of the fraction. Even more complicated is that when performing fraction problems sometimes you *do* treat numerators as whole numbers—but not always. Consider subtraction of fractions that have the same vs. different denominators (e.g., 6/8–4/8 is simply solved as 6–4, to give the answer 2/8, vs. 6/8–1/2 in which 6–1 is incorrect; 1/2 must first be converted to 4/8). And, of course, when multiplying fractions, you treat the numerators and denominators as whole numbers to multiply across whether or not the denominators are the same, however with division of fractions you also multiply across, but using the reciprocal of the devisor. This formulaic thinking involves considering fractions independently as absolute values rather than using relative magnitude reasoning, which perhaps contributes to the difficulty in fraction learning.

Often these "rules" or procedures are memorized by students, but without the conceptual understanding of why they work, it can be difficult to have a sense of the general ballpark the answer should fall, resulting in higher risk of error and mistakes. Using conceptual understanding of relative magnitude could help students catch themselves when they make these mistakes. For example, if they are multiplying by a fraction that is less than one, or finding a proportion of some amount, conceptual understanding would help them catch a mistake where the answer should not be larger than the original value. Similarly, when comparing magnitudes, having an idea of where a fraction would fall on a number line can help students to more quickly compare without having to do calculations. For example, when comparing 112/250 vs. 167/310: realizing that one number is less than half and one is more can allow you to quickly respond with the larger fraction. Understanding why these different procedures work can also help you remember them accurately in the first place, and using relative magnitude is necessary for this understanding.

Some studies have shown that using spatial representations of fractions, such as manipulatives, can help to build stronger foundational knowledge about fractions and support later problem solving (Carbonneau, Marley, & Selig, 2013), though Mix (2010) suggests that there are important considerations that may influence learning. Concrete manipulatives can be used to direct attention from absolute to relative magnitude. These materials are similar to the 2-dimensional diagrams in their representational nature, but they also typically allow physical manipulation, and can provide scaffolding for moving from concrete objects to abstract meaning. Spatial features of the manipulatives like size can be used to help children observe the relational information when making comparisons. For example, in Montessori classrooms children can manipulate bars or beads made up of 10 connected units, comparing its relation to the 100-board made up 10 of the same bars, and then relat-