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Vicente Cortés
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Geometric Flows and the Geometry of Space-time

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Preface

A major goal in mathematics as well as in physics has been and still is to understand the geometry of space and time. Developments in both subjects have fruitfully influenced each other over the history of science. The formulation of general relativity by Einstein would not have been possible without the concepts of (semi-) Riemannian geometry that had emerged with the visionary ideas of Riemann in the previous century. Conversely, ideas from general relativity influenced mathematical research and the study of Einstein's equation is one of today's major topics in geometric analysis.

Similarly, the development of more recent areas of theoretical physics, such as string theory, is deeply connected to the study of geometric problems in mathematics, such as the study of metrics of special holonomy. It turned out that geometric flows are also of great importance in the interplay between mathematics and physics; e.g., the Riemann Penrose inequality has been shown by Huisken and Ilmanen using the inverse mean curvature flow.

This volume is based on a summer school and workshop entitled "Geometric flows and the geometry of space-time" held at the University of Hamburg in September 2016. The aim of this event was to provide a forum where physicists and mathematicians can exchange ideas and where graduate students and young researchers get the opportunity to learn about recent developments at the intersection of mathematics and physics.

It brought together around 60 participants with mathematical and physical backgrounds. The speakers were Lars Andersson, Helga Baum, Spiros Cotsakis, Pau Figueras, Gary Gibbons, Mark Haskins, Jason Lotay, Thomas Leistner, Jan Metzger, and Oliver C. Schnürer.

Out of these 10 speakers, 7 gave two talks where the first one was more of an introductory nature and the other one was more focused on actual research. These talks covered a broad variety of topics, ranging from special holonomy metrics to various concepts of mass in general relativity and the numerical and analytic study of black hole space-times.

Moreover, three of the speakers gave minicourses where each of them had a total length of 180 min. One minicourse was more of a physical nature and was held by Gary Gibbons about the theory of black holes. The other two lecture courses were more of a mathematical nature. One course was held by Oliver C. Schnürer about geometric flows and focused in particular on mean curvature flow. The other course held by Helga Baum was about special holonomy and parallel spinors in Lorentzian geometry. In addition, we had two related talks about Cauchy problems for Lorentzian manifolds of special holonomy by Thomas Leistner.

This volume consists of two articles. The first is based on the mathematical lecture course by Oliver C. Schnürer and the second on the mathematical lecture course by Helga Baum extended by results presented in the lectures by Thomas Leistner.

Another volume based on the third lecture course about the theory of black holes is planned. The papers are written for graduate students and researchers with a general background in geometry and in the theory of partial differential equations, who want to get acquainted with these central subjects of modern geometry. We hope this volume will be helpful and inspiring.

Hamburg, Germany
July 2018

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Lorentzian Geometry: Holonomy, Spinors, and Cauchy Problems

Helga Baum and Thomas Leistner

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Abstract This review is based on lectures given by the authors during the Summer School *Geometric Flows and the Geometry of Space-Time* at the University of Hamburg, September 19–23, 2016. In the first part we describe the algebraic classification of connected Lorentzian holonomy groups. In particular, we specify the holonomy groups of locally indecomposable Lorentzian spin manifolds with a parallel spinor field. In the second part we explain new methods for the construction of globally hyperbolic Lorentzian manifolds with special holonomy based on the solution of certain Cauchy problems for PDEs that are imposed by the existence of a parallel lightlike vector field or a parallel lightlike spinor field with initial conditions on a spacelike hypersurface. Thereby, we derive a second order evolution equation of Cauchy-Kowalevski type that can be solved in the analytic setting as well as an appropriate first order quasilinear hyperbolic system that yields a solution in the smooth case.

1 Introduction

This review is based on lectures given by the authors during the Summer School *Geometric Flows and the Geometry of Space-Time* at the University of Hamburg, September 19–23, 2016. In these lectures we described at one hand the algebraic classification of connected Lorentzian holonomy groups and explained at the other hand new methods for the construction of Lorentzian manifolds with special holonomy based on the solution of appropriate Cauchy problems with initial conditions on a spacelike hypersurface.

The holonomy group of a semi-Riemannian manifold (M, g) is the group of parallel transports along all curves that are closed at a fixed point $x \in M$. It is a Lie subgroup of the group of all orthogonal transformations of $(T_x M, g_x)$, its connected component is isomorphic to the holonomy group of the universal semi-Riemannian covering of (M, g).

The concept of holonomy was probably first successfully applied in differential geometry by E. Cartan [31–33], who used it to classify symmetric spaces. Since then, it has proved to be a very important concept. In particular, it allows to describe parallel sections in geometric vector bundles over (M, g)—such as tangent, tensor or spinor bundles—as holonomy invariant objects and therefore by purely algebraic tools. Moreover, geometric properties like curvature properties can be read off if the holonomy group is special, i.e., a proper subgroup of $\mathrm{O}(T_x M, g_x)$. One of the important consequences of the holonomy notion is its application to the ‘classification’ of special geometries that are compatible with Riemannian geometry. For each of these geometries an own branch of differential geometry has developed, for example Kähler geometry (holonomy $\mathrm{U}(m)$), geometry of Calabi-Yau manifolds ($\mathrm{SU}(m)$), hyper-Kähler geometry ($\mathrm{Sp}(k)$), quaternionic Kähler geometry ($\mathrm{Sp}(k) \cdot \mathrm{Sp}(1)$), or the exceptional geometry of G_2-manifolds or of $\mathrm{Spin}(7)$-manifolds. In physics there is much interest in semi-Riemannian manifolds with special holonomy, since they often allow to construct spaces with additional supersymmetries (Killing spinors). The development of holonomy theory has a long history. We refer for details to [22, 25, 26, 51].

The *irreducible* holonomy representations of simply connected semi-Riemannian manifolds were classified by M. Berger in the 1950s [19, 20]. Since any holonomy representation of a *Riemannian* manifold splits into irreducible subrepresentations, Berger's results yield the classification of the connected holonomy groups of Riemannian manifolds. The situation in Lorentzian geometry is more difficult. The only connected *irreducible* Lorentzian holonomy group is the group $SO^0(1, n-1)$. Hence, if a connected Lorentzian holonomy group is a proper subgroup of $SO^0(1, n-1)$, then it acts decomposable or it acts indecomposable but non-irreducible, i.e., it admits an invariant degenerate subspace.

The holonomy groups of 4-dimensional Lorentzian manifolds were classified by physicists working in General Relativity [49, 72, 73]. The general dimension was long time ignored. Due to the development of supergravity and string theory in the last decades physicists as well as mathematicians became more interested in higher dimensional Lorentzian geometry. The search for special supersymmetries required the classification of holonomy groups in higher dimension. In the beginning of the 1990s, L. Berard-Bergery and his students began a systematic study of Lorentzian holonomy groups. They discovered many special features of Lorentzian holonomy. Their groundbreaking paper [18] on the algebraic structure of subgroups $H \subset SO^0(1, n-1)$ acting with a degenerate invariant subspace was the starting point for the classification. The second author [60, 61] completed the classification of the connected Lorentzian holonomy groups by the full description of the structure of such $H \subset SO(1, n-1)$ which can appear as holonomy groups. It remained to show that any of the groups in this holonomy list can be realised by a Lorentzian metric. Many realisations were known before but some cases were still open until A. Galaev [44] finally found a realisation for all of the groups.

In the first part of this review we describe these results in more detail. In Sect. 2 we first recall some basic notions of Lorentzian geometry in order to clarify the conventions. For all fundamental differential geometric concepts such as Levi-Civita connection, Lie derivative, etc. we refer to [68]. In Sect. 3 we give a short introduction to holonomy theory of semi-Riemannian manifolds and recall the classification of connected holonomy groups of Riemannian manifolds. Afterwards we explain the classification of connected holonomy groups of Lorentzian manifolds. Special holonomy groups always appear if the manifold is spin and admits a non-trivial parallel spinor field. For this reason we consider in Sect. 4 the relation between holonomy groups and parallel spinor fields. In particular, we discuss the properties of the Ricci curvature of Lorentzian spin manifolds with a parallel spinor field and describe the indecomposable Lorentzian holonomy groups which allow parallel spinors.

In the second part of the review we explain new approaches to construct globally hyperbolic Lorentzian manifolds with special holonomy by solving appropriate Cauchy problems with initial conditions along a spacelike hypersurface based on recent results in [16, 65] and [62], see also [17] for related results. We focus on the case of Lorentzian manifolds which admit non-trivial lightlike parallel vector fields or non-trivial lightlike parallel spinor fields. In both cases the holonomy representation is of special form, it admits an invariant degenerate subspace. At first, in Sect. 5 we derive the necessary constraint conditions, which lightlike parallel vector and spinor fields impose on spacelike hypersurfaces. In the vector field case, the local geometry of Riemannian manifolds satisfying these constraint conditions is completely described. In the spinor field case, the constraint conditions can be expressed as the existence of an so-called imaginary W-Killing spinor of a special algebraic type, where W is the Weingarten operator of the spacelike hypersurface. As an application of the solutions of the Cauchy problem described in Sect. 7 we obtain a local classification of Riemannian manifolds with imaginary W-Killing spinors of this algebraic type (Sect. 8). It is natural to ask whether the constraint conditions for Riemannian manifolds (Σ, h) described in Sect. 5 are not only necessary but also sufficient for (Σ, h) being a Cauchy hypersurface in a Lorentzian manifold with a lightlike parallel vector or spinor field. By studying certain Cauchy problems for PDEs that are induced by the existence of lightlike parallel vector and spinor fields, we show in Sect. 7 that this is indeed the case. Since the methods for the existence of a solution are in part analogous to the approach for the vacuum Einstein equation, we give in Sect. 6 a short review of the approaches for the Einstein equation. After deriving the constraint equations we first describe the vacuum Einstein equation as a second order evolution equation for a family of Riemannian metrics that is of Cauchy-Kowalevski form, that can be solved in the real-analytic setting. Afterwards we explain the method of hyperbolic reduction which allows to consider the vacuum Einstein equation as symmetric hyperbolic system and solve it in the smooth setting. In Sect. 7 we derive in a similar way an evolution equation of Cauchy-Kowalevski type for a parallel lightlike vector field in the analytic setting as well as an appropriate symmetric hyperbolic system which can be solved in the smooth case. Finally we show, that in both cases the solution admits a parallel lightlike spinor field if, in addition, the contraint conditions for parallel spinors on the initial hypersurface are satisfied.

2 Basic Notions

Let (M^n, g) be an n-dimensional manifold[1] with a metric g of signature (p, q), where p denotes the number of -1 and q the number of $+1$ in the normal form of the metric g. We call (M, g) *Riemannian manifold* if $p = 0$, *Lorentzian manifold* if $p = 1 < n$ and *pseudo-Riemannian manifold* if $1 \leq p < n$. If we do not want to specify the signature we use the term *semi-Riemannian manifold*.

Contrary to the Riemannian case, not every manifold admits a Lorentzian metric. There is a topological obstruction (see [68, Chapter 5, Proposition 37] for a proof):

[1] We assume all manifolds to be smooth, connected and without boundary.

Theorem 1 *Let M be a manifold of dimension $n \geq 2$. Then there exists a Lorentzian metric on M if and only if M is non-compact or M is compact with vanishing Euler characteristic.*

Now, let (M, g) be a Lorentzian manifold.

Definition 1 A tangent vector $v \in T_x M$ is called

- timelike, if $g_x(v, v) < 0$,
- spacelike, if $g_x(v, v) > 0$ or $v = 0$,
- lightlike, if $g_x(v, v) = 0$ and $v \neq 0$,
- causal, if v is timelike or lightlike.

Correspondingly, a vector field X is called timelike, spacelike, etc., if $X(x)$ is timelike, spacelike, etc., for all $x \in M$. A a smooth curve $\gamma : I \to M$ is called timelike, spacelike, etc., if all its tangent vectors $\gamma'(t)$ are timelike, spacelike, etc., for all $t \in I$.

Definition 2 Let (M, g) be a Lorentzian manifold. A vector field ξ on M is called *time-orientation* if $g(\xi, \xi) = -1$. If there exists a time-orientation ξ on (M, g), (M, g) is called time-orientable.

A time-oriented Lorentzian manifold is also called spacetime. A time-orientation ξ on a Lorentzian manifold (M, g) singles out one of the two time-cones $\tau^{\pm}(x)$ in any point $x \in M$ in a smooth way, where $\tau^{+}(x)$ and $\tau^{-}(x)$ denote the connected components of $\{v \in T_x M \mid g_x(v, v) < 0\}$. A causal vector field X on M is called *future-directed*, if $g(X, \xi) < 0$, i.e. $X(x)$ and $\xi(x)$ belong to the same time-cone.

In the following we will denote by ∇^g the Levi-Civita connection of (M, g), i.e. the unique metric and torsion free covariant derivative on (M, g). Our convention for the curvature tensor $R^g \in \Gamma(\Lambda^2 T^*M \otimes End(TM))$, and $R^g \in \Gamma(\Lambda^2 T^*M \otimes \Lambda^2 T^*M)$ is the following:

$$R^g(X, Y)Z := \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z,$$
$$R^g(X, Y, Z, W) := g(R^g(X, Y)Z, W).$$

The curvature tensor satisfies the first and second Bianchi-identities,

$$R^g(X, Y)Z + R^g(Y, Z)X + R^g(Z, X)Y = 0,$$
$$\nabla^g_X R^g(Y, Z, U, V) + \nabla^g_Y R^g(Z, X, U, V) + \nabla^g_Z R^g(X, Y, U, V) = 0.$$

Then the Ricci tensor Ric^g and the scalar curvature $scal^g$ of (M, g) are given by

$$Ric^g(X, Y) := \mathrm{tr}_g\, R^g(X, \cdot, \cdot, Y), \qquad scal^g := \mathrm{tr}_g Ric^g.$$

The second Bianchi identity for R^g implies

$$d\, scal^g = 2\mathrm{div}^g(Ric^g), \tag{1}$$