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J. J. Duistermaat
Mathematisch Instituut
Universiteit Utrecht
3508 TA Utrecht
The Netherlands

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Preface

When visiting M.I.T. for two weeks in October 1994, Victor Guillemin made me enthusiastic about a problem in symplectic geometry which involved the use of the so-called spin-c Dirac operator. Back in Berkeley, where I had spent a sabbatical semester[1], I tried to understand the basic facts about this operator: its definition, the main theorems about it, and their proofs. This book is an outgrowth of the notes in which I worked this out. For me this was a great learning experience because of the many beautiful mathematical structures which are involved.

I thank the Editorial Board of Birkhäuser, especially Haim Brézis, for suggesting the publication of these notes as a book. I am also very grateful for the suggestions by the referees, which have led to substantial improvements in the presentation. Finally I would like to express special thanks to Ann Kostant for her help and her prodding me, in her charming way, into the right direction.

J.J. Duistermaat

Utrecht, October 16, 1995.

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Chapter 1

Introduction

1.1 The Holomorphic Lefschetz Fixed Point Formula

Let M be an almost complex manifold of real dimension $2n$, provided with a Hermitian structure. Furthermore, let L be a complex vector bundle over M, provided with a Hermitian connection. We also assume that K^*, the dual bundle of the so-called canonical line bundle K of M, is provided with a Hermitian connection. We write E for the direct sum over q of the bundles of $(0,\, q)$-forms; in it we have the subbundle E^+ and E^-, where the sum is over the even q and odd q, respectively. Write Γ and $\Gamma^\pm$ for the space of smooth sections of $E \otimes L$ and $E^\pm \otimes L$, respectively. From these data, one can construct a first order partial differential operator D, the spin-c Dirac operator mentioned in the title of this book, which acts on Γ. The restriction D^+ of D to Γ^+ maps into Γ^-, and the restriction D^- of D to Γ^- maps into Γ^+. If M is compact, then the fact that D is elliptic implies that the kernel $N^\pm$ of $D^\pm$ is finite-dimensional, and the difference $\dim N^+ - \dim N^-$ is equal to the *index* of D^+.

The Atiyah-Singer index theorem applied to this case [7, Theorem (4.3)]

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expresses this index as the integral over M of a characteristic class in the De Rham cohomology of M, equal to the product of the Todd class of the tangent bundle of M, viewed as a complex vector bundle over M, and the Chern character of L. These characteristic classes are given by polynomial expressions in the curvature forms of the given bundles. If M is a complex analytic manifold, then the index of D^+ is equal to the Riemann-Roch number of M, and the integral formula generalizes the one which Hirzebruch [39] obtained for complex projective algebraic varieties.

If γ is a bundle automorphism of L which leaves all the given structures invariant, then it induces an operator in Γ which commutes with D, and one can form the *virtual character*

$$\chi(\gamma) = \mathrm{trace}_{\mathbf{C}}\, \gamma|_{N^+} - \mathrm{trace}_{\mathbf{C}}\, \gamma|_{N^-}. \tag{1.1}$$

Under the assumption that the fixed point set M^γ of γ in M locally is a smooth almost complex submanifold and that the action of γ in the normal bundle is nondegenerate, the equivariant index theorem of Atiyah-Segal and Atiyah-Singer expresses the virtual character as the sum over the connected components F of M^γ, of similar characteristic classes of the F's. In the complex analytic case, this is called the *holomorphic Lefschetz fixed point formula*, cf. Atiyah and Singer [7, Theorem (4.6)]. In the case of isolated fixed points, it is due to Atiyah and Bott [5, Theorem 4.12].

1.2 The Heat Kernel

The operator $Q^+ = D^- \circ D^+$ maps Γ^+ to Γ^+, and $Q^- = D^+ \circ D^-$ maps Γ^- to Γ^-. Each of the operators Q^+ and Q^- is equal to a Laplace operator, plus a zero order part which involves curvature terms. The corresponding heat diffusion operators $\mathrm{e}^{-t\,Q^\pm}$ are integral operators with a smooth integral kernel $K^\pm(t,\, x,\, y)$, $t > 0$, $x,\, y \in M$. Along the diagonal $x = y$, and for

$t \downarrow 0$, these kernels have an asymptotic expansion of the form

$$K^{\pm}(t,\, x,\, x) \sim t^{-n} \sum_{k=0}^{\infty} t^k\, K_k^{\pm}(x). \tag{1.2}$$

In this asymptotic expansion, each of the coefficients $K_k(x)^{\pm}$ is given by a universal polynomial expression in a finite part of the Taylor expansion of the geometric data at the point x.

It was observed by McKean and Singer [57, p. 61] that

$$\text{index}\, D^+ = \int_M \text{trace}_{\mathbf{C}}\, K_n^+(x) - \text{trace}_{\mathbf{C}}\, K_n^-(x)\, \mathrm{d}\, x, \tag{1.3}$$

and they asked the question if not, by some fantastic cancellation, the higher order derivatives in the expression for $K_n^{\pm}(x)$ cancel, to give that the integrand in (1.3) is equal to a characteristic differential form whose cohomology class is equal to the one of the index theorem. This would give a direct analytic proof of the index theorem, with the advantage of having a local interpretation of the integrand. Actually, in [57] the question is asked for the Euler characteristic of M, but it obviously can be generalized to arbitrary index problems.

1.3 The Results

It turned out that, also in the presence of an automorphism γ, the fantastic cancellation indeed takes place. See Theorem 11.1 and Theorem 12.1. In the complex analytic case, the result is referred to as a *local holomorphic Lefschetz fixed point formula*. It is the purpose of this book, to explain both all the ingredients in the formula, and how the answer comes about. In it, we will apply the methods of Berline, Getzler and Vergne [9, Ch. 1-6], and show how these work in the case of the spin-c Dirac operator. (For the comparison: our L is their $\mathcal{W}$, the letter W is the classical notation of Hirzebruch [39]. We have chosen the letter L, because of the connotation of a “linear system”.)

For the index, the result is due to Patodi [64] in the Kähler case, with another proof by Gilkey [28], who in [31] extended the result to almost complex manifolds. In the presence of an automorphism γ, the local formula had been obtained by Patodi [65] under the assumption that the connected components of the fixed point set M^γ of γ in M are Kähler manifolds. A proof for general almost complex manifolds has been indicated by Kawasaki [45, pp. 156-158]. One can also obtain the result in this general setting as a consequence of the local Lefschetz formula for the spinor Dirac operator of Berline and Vergne [11], cf. [9, Theorem 6.11]. That is, by using the comparison (6.20) between the bundle E of $(0, q)$-forms and the spinor bundle S, and observing that it suffices to work locally, where spin structures always exist.

The local formula is particularly suited for the generalization of the Lefschetz formula to compact orbifolds, which we will explain in Chapter 14. I learned this from the proof of Kawasaki [45] for the Riemann-Roch number. For arbitrary elliptic operators on compact orbifolds, the Lefschetz formula has been obtained by Vergne [73]. She used the theory of transversally elliptic operators of Atiyah [2], as Kawasaki [46] did in his proof of the index formula for orbifolds. The use of the local formula avoids the use of the commutative algebra of [2], which may make it more accessible to analysts.

Strictly speaking, this work contains no new results. However, the spin-c Dirac operator is a very important special case among the general Dirac-type operators. As described above, it came originally from the study of complex analytic manifolds. On the other hand, every symplectic manifold (phase space in classical mechanics) also carries an almost complex structure and hence a corresponding spin-c Dirac operator. We will discuss the application of the theory to this case in Chapter 15. As a third application, we mention that recently the Seiberg-Witten theory, an S^1 gauge theory which uses the spin-c Dirac operator, has led to striking progress in the differential topology of four-dimensional compact oriented manifolds. Here one works with spin-c Dirac operators which are defined in terms of spin-c structures which do not necessarily come from an almost complex structure. See the Remark in front of Lemma 5.5. For an exposition of Seiberg-Witten theory see for instance Eichhorn and Friedrich [26] or Morgan [61]. The importance of the spin-c Dirac operator makes it worthwhile to work out the beautiful constructions of [9] for this special kind of Dirac operator.

A large part of the exposition has a wider scope than just the spin-c Dirac operator. For instance, Chapter 8 is an exposition of the asymptotic expansion of heat kernels for generalized Laplace operators, following [9, Ch. 2]. Chapters 9 and 10, on the Berline-Vergne theory of heat kernels on principal bundles, are also written for more general operators than only the spin-c Dirac operator. The point of this theory is, that it gives an explanation for the similarity between the factor $\det \frac{1-e^{-R}}{R}$, which appears in the index formula, and the Jacobian of the exponential mapping from a Lie algebra to the Lie group. (Here R denotes curvature.) Lemma 9.5 and Lemma 9.6 form the starting point of this explanation. Although in general we tried to keep our notations close to our main reference [9], we apologize that at some points we ended up with a different choice.

Finally, in Chapter 13 the formulas of Theorem 11.1 and Theorem 12.1 are translated into the language of characteristic classes, in which the formulas of Hirzebruch and Atiyah-Singer originally were phrased. We use the occasion to explain, in Chapter 16, the Weil homomorphism in its natural setting of equivariant differential forms in the presence of an action of a Lie group, and under the assumption that the action admits a connection form.

I am very grateful to Victor Guillemin for arousing my interest in the subject, in connection with the question how the Riemann-Roch number of a reduced phase space for a torus action is related to multiplicities of intermediate phase spaces. And I apologize for spending so much time on writing up this text, instead of "adorning the dendrites". Finally I would like to thank the Department of Mathematics of UC Berkeley, for providing me with an ideal environment to work on this.

Chapter 2

The Dolbeault-Dirac Operator

In this chapter we set the stage, by introducing complex and almost structures, the Dolbeault complex and Hermitian structures. The holomorphic Lefschetz number, defined as the alternating sum of the trace of the automorphism acting on the cohomology of the sheaf of holomorphic sections, will be expressed in terms of a selfadjoint operator, which is built out of the Dolbeault operator and its adjoint; the Dolbeault-Dirac operator in the title of this chapter. This material is very well-known but, also in order to fix the notations, we have taken our time for the description of these structures. Just for convenience, we will assume that all objects are smooth (infinitely differentiable).

2.1 The Dolbeault Complex

Let M be a manifold of even dimension $2n$, provided with an almost complex structure J. That is, for each $x \in M$, J_x is a real linear transformation in $\mathrm{T}_x\, M$ such that ${J_x}^2 = -1$. A real linear mapping A from $\mathrm{T}_x\, M$ to a complex vector space V is called complex linear and complex antilinear with respect

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to the complex structure J_x in $\mathrm{T}_x\, M$, if

$$A\,(J_x(v)) = i\,A(v),\ v \in \mathrm{T}_x\, M \tag{2.1}$$

and

$$A\,(J_x(v)) = -i\,A(v),\ v \in \mathrm{T}_x\, M, \tag{2.2}$$

respectively.

The space of complex linear and complex antilinear forms ($V = \mathbf{C}$) on $\mathrm{T}_x\, M$ is denoted by $\mathrm{T}_x^*\, M^{(1,\,0)}$ and $\mathrm{T}_x^*\, M^{(0,\,1)}$, respectively. With this notation, the space of complex linear and antilinear mappings from $\mathrm{T}_x\, M$ to V becomes equal to $\mathrm{T}_x^*\, M^{(1,\,0)} \otimes V$ and $\mathrm{T}_x^*\, M^{(0,\,1)} \otimes V$, respectively. One has the complementary projections

$$\pi^{(1,\,0)} : \xi \mapsto \xi^{(1,\,0)} := \tfrac{1}{2}\,(\xi - i\,\xi \circ J_x)\,, \tag{2.3}$$

and

$$\pi^{(0,\,1)} : \xi \mapsto \xi^{(0,\,1)} := \tfrac{1}{2}\,(\xi + i\,\xi \circ J_x)\,, \tag{2.4}$$

from $\mathrm{T}_x^*\, M \otimes \mathbf{C}$ onto $\mathrm{T}_x^*\, M^{(1,\,0)}$ along $\mathrm{T}_x^*\, M^{(0,\,1)}$, and from $\mathrm{T}_x^*\, M \otimes \mathbf{C}$ onto $\mathrm{T}_x^*\, M^{(0,\,1)}$ along $\mathrm{T}_x^*\, M^{(1,\,0)}$, respectively.

A complex-valued function f on M is called complex-differentiable or complex-analytic, or holomorphic, if, for every $x \in M$, $\mathrm{d}f_x$ is complex linear. If $\bar{\partial} = \pi^{(0,\,1)} \circ \mathrm{d}$ denotes the operator d followed by the projection (2.4), then this condition is equivalent to the differential equation $\bar{\partial} f = 0$. One also writes $\partial = \pi^{(1,\,0)} \circ \mathrm{d}$, so that $\mathrm{d} = \partial + \bar{\partial}$ on functions, and $\partial f = \mathrm{d}f$ if and only if f is holomorphic.

Let $p,\ q,\ r \in \mathbf{Z}_{\geq 0}$, with $p + q = r$. A complex-valued antisymmetric r-linear form on $\mathrm{T}_x\, M$ is called of type $(p,\ q)$, if it is equal to a finite sum of forms $\alpha \wedge \beta$, where $\alpha \in \Lambda^p\, \mathrm{T}_x^*\, M^{(1,\,0)}$ and $\beta \in \Lambda^q\, \mathrm{T}_x^*\, M^{(0,\,1)}$. The space of forms of type $(p,\ q)$ is denoted by $\mathrm{T}_x^*\, M^{(p,\,q)}$. The point is that

$$\Lambda^r\, \mathrm{T}_x^*\, M \otimes \mathbf{C} = \bigoplus_{p,\,q,\,p+q=r} \mathrm{T}_x^*\, M^{(p,\,q)}, \tag{2.5}$$

so we have the projection $\pi_{p,q}$ from $\Lambda^r \mathrm{T}_x^* M \otimes \mathbf{C}$ onto $\mathrm{T}_x^* M^{(p,q)}$ along the sum of the other components. An L-valued version is obtained by tensoring $\mathrm{T}_x^* M^{(p,q)}$ with L_x. A (p, q)-form ω_x on $\mathrm{T}_x M$, which depends smoothly on $x \in M$, is called a (p, q)-form on M. The space of (p, q)-forms on M is denoted by $\Omega^{(p,q)}(M)$.

In particular we will be interested in the case $p = 0$, for which we will use the following abbreviation throughout:

$$E_x^q := \mathrm{T}_x^* M^{(0,q)} = \Lambda^q \mathrm{T}_x^* M^{(0,1)}, \quad E_x^0 = \mathbf{C}. \tag{2.6}$$

Note that $E_x^q = 0$ if $q > n$, and $\dim_{\mathbf{C}} E_x^q = \binom{n}{q}$ if $0 \leq q \leq n$. We will write

$$E_x := \bigoplus_{q=0}^{n} E_x^q, \tag{2.7}$$

$$E_x^+ = E_x^{\text{even}} = \bigoplus_{\text{even } q} E_x^q, \tag{2.8}$$

$$E_x^- = E_x^{\text{odd}} = \bigoplus_{\text{odd } q} E_x^q. \tag{2.9}$$

With the exterior product of forms and the splitting in E_x^+ and E_x^-, E_x is a supercommutative superalgebra over $\mathbf{C}$. (See [9, Section 1.3] for the definition of such algebras.)

The $E_x, x \in M$, form a complex vector bundle E over M with subbundles $E^+ = \bigcup_{x \in M} E_x^+$ and $E^- = \bigcup_{x \in M} E_x^-$. The space of sections of E, E^+ and E^- is equal to the direct sum of the spaces $\Omega^{(0,q)}$, where q runs over all the even and the odd integers $0 \leq q \leq n$, respectively.

In Chapter 5 we will introduce the spin-c Dirac operator, which will be used in the general case of an almost complex structure. In order to motivate its definition and to understand its relation to complex analysis, we assume in the remainder of this chapter that M is a complex analytic manifold. This means that around every $x \in M$ there is a system of local coordinates in which J is equal to the standard complex structure of $\mathbf{C}^n$. This is equivalent to the condition that, at every $x \in M$, there exist n holomorphic functions z_j in a neighborhood of x in M, such that the $\mathrm{d}z_j$ at x are linearly independent over $\mathbf{C}$.

In such coordinates z_j, each $(p,\, q)$-form is of the form

$$\omega = \sum_{J,\, K} \omega_{J,\, K}\, \mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K, \tag{2.10}$$

where J and K runs over the set of strictly increasing sequences $J = (j_i)_{i=1}^{p}$ and $K = (k_i)_{i=1}^{q}$, respectively, each $\omega_{J,\,K}$ is a complex-valued function, and

$$\mathrm{d}z_J = \mathrm{d}z_{j_1} \wedge \mathrm{d}z_{j_2} \wedge \ldots \wedge \mathrm{d}z_{j_p}, \tag{2.11}$$

$$\mathrm{d}\bar{z}_K = \mathrm{d}\bar{z}_{k_1} \wedge \mathrm{d}\bar{z}_{k_2} \wedge \cdots \wedge \mathrm{d}\bar{z}_{k_q}. \tag{2.12}$$

From this we see that $\mathrm{d}\omega$ is the sum of a $(p+1,\, q)$-form and a $(p,\, q+1)$-form. Or, one again has $\mathrm{d} = \partial + \bar{\partial}$, if one writes

$$\partial = \pi^{(p+1,\, q)} \circ \mathrm{d} \tag{2.13}$$

and

$$\bar{\partial} = \pi^{(p,\, q+1)} \circ \mathrm{d} \tag{2.14}$$

on $(p,\, q)$-forms.

This implies that for each $(0,\, 1)$-form ω, $\pi^{(2,\, 0)}\, \mathrm{d}\omega = 0$. For a general almost complex structure J, it need no longer be true that $\mathrm{d} = \partial + \bar{\partial}$. For each $x \in M$, one has the antisymmetric bilinear mapping $[J,\, J]_x$ from $\mathrm{T}_x\, M \times \mathrm{T}_x\, M$ to $\mathrm{T}_x\, M$, which is defined by

$$[J,\, J](v,\, w) = [Jv,\, Jw] - J\,[Jv,\, w] - J\,[v,\, Jw] - [v,\, w], \tag{2.15}$$

for any vector fields v and w in M. Using the formula

$$(\mathrm{d}\omega)(v,\, w) = v\,\omega(w) - w\,\omega(v) - \omega\left([v,\, w]\right), \tag{2.16}$$

one gets for each (0, 1)-form ω that

$$\mathrm{d}\omega(v - i\,Jv,\, w - i\,Jw) = \omega\left([J,\, J](v,\, w)\right).$$

So the condition that $\pi^{(2,0)}\,\mathrm{d}\omega = 0$ for every (0, 1)-form ω is equivalent to the condition that $[J,\, J] = 0$. The *theorem of Newlander and Nirenberg* now says that an almost complex manifold $(M,\, J)$ is complex analytic if and only if $[J,\, J] = 0$. This theorem is already valid if the first order derivatives of J are Hölder-continuous. Cf. Newlander and Nirenberg [62], Hörmander [41], Malgrange [55].

We continue the discussion of complex analytic manifolds. Identifying the types in $0 = \mathrm{d}^2 = \partial^2 + \partial\,\bar{\partial} + \bar{\partial}\,\partial + \bar{\partial}^2$ on $\Omega^{(p,\,q)}$-forms, one sees that $\partial^2 = 0$, $\partial\,\bar{\partial} + \bar{\partial}\,\partial = 0$, and $\bar{\partial}^2 = 0$. In particular, the operator $\bar{\partial}$ defines a complex

$$0 \longrightarrow \Omega^{(p,\,0)} \xrightarrow{\bar{\partial}} \Omega^{(p,\,1)} \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \Omega^{(p,\,n)} \longrightarrow 0, \tag{2.17}$$

called the *Dolbeault complex*. On the sheaves of locally defined forms, this sequence is exact, and one gets the theorem of Dolbeault that

$$\ker \bar{\partial}^{(p,\,q)} / \operatorname{range} \bar{\partial}^{(p,\,q-1)} \simeq \mathrm{H}^q\left(M,\, \mathcal{O}\left(\Omega^{(p,\,0)}\right)\right), \tag{2.18}$$

where the right hand side denotes the q-th cohomology group of the sheaf $\mathcal{O}\left(\Omega^{(p,\,0)}\right)$ of holomorphic $(p,\, 0)$-forms over M. See for instance Griffiths and Harris [33, p. 45]. If M is compact, then the ellipticity of the complex yields that the spaces in the left hand side are finite-dimensional. We will mainly be interested in the case that $p = 0$.

A holomorphic vector bundle L over M is defined as a complex vector bundle over M for which the retrivializations are given by elements of $\mathrm{GL}(l,\, \mathbf{C})$ which depend holomorphically on the base point. (Here $l = \dim_{\mathbf{C}} L_x$.) All the above remains valid for L-valued $(p,\, q)$-forms, that is, the sections of the vector bundle

$$\mathrm{T}_x^*\, M^{(p,\,q)} \otimes L_x, \quad x \in M.$$

In particular, we have the "twisted Dolbeault complex" defined by

$$\bar{\partial}^{(0,q)} : E^q \otimes L \to E^{q+1} \otimes L,$$

and the corresponding cohomology groups

$$\ker \bar{\partial}^{(0,q)} / \operatorname{range} \bar{\partial}^{(0,q-1)} \simeq \mathrm{H}^q\left(M,\, \mathcal{O}(L)\right), \tag{2.19}$$

the q-th cohomology group of the sheaf $\mathcal{O}(L)$ of holomorphic sections of L over M.

If M is a compact complex analytic manifold, then an important quantity is the *Riemann-Roch number*

$$\mathrm{RR}(M,\, L) := \sum_{q=0}^{n} (-1)^q \dim_{\mathbf{C}} \mathrm{H}^q\left(M,\, \mathcal{O}(L)\right). \tag{2.20}$$

More generally, if γ is a complex analytic automorphism of L, then γ acts on $\mathcal{O}(L)$, and one can define its *holomorphic Lefschetz number*

$$\chi(\gamma) = \chi_{M,\,L}(\gamma) := \sum_{q=0}^{n} (-1)^q \operatorname{trace}_{\mathbf{C}} \gamma|_{\mathrm{H}^q(M,\,\mathcal{O}(L))}. \tag{2.21}$$

This *is* a generalization because $\chi_{M,\,L}(1) = \mathrm{RR}(M,\, L)$.

If L is a holomorphic complex line bundle over M for which $K^* \otimes L$ is positive, then Kodaira's vanishing theorem says that $\mathrm{H}^q\left(M,\, \mathcal{O}(L)\right) = 0$ for every $q > 0$, cf. (6.34). If the latter is the case, the holomomorphic Lefschetz number is equal to the trace of the action of γ on the space $\mathrm{H}^0\left(M,\, \mathcal{O}(L)\right)$ of all holomorphic sections of L over M, and the Riemann-Roch number is equal to the dimension of that space.

2.2 The Dolbeault-Dirac Operator

In order to define adjoints, we now introduce Hermitian structures h and h^L in the tangent bundle $\mathrm{T}\, M$ of M and the fibers of L, respectively. For each $x \in M$, h_x is a complex-valued bilinear form on $\mathrm{T}_x\, M$, such that

$$h_x(v,\, v) > 0 \text{ if } v \in \mathrm{T}_x\, M,\; v \neq 0, \tag{2.22}$$

and

$$h_x\left(J_x(v),\, w\right) = i\, h_x(v,\, w) = -h_x\left(v,\, J_x(w)\right),\ v,\, w \in \mathrm{T}_x\, M. \qquad (2.23)$$

Similarly with h_x and $\mathrm{T}_x\, M$ replaced by h_x^L and L_x, respectively.

It follows that the real part $\beta = \mathrm{Re}\, h$ of h is a Riemannian structure in M, and J_x is antisymmetric with respect to β_x. Furthermore, the imaginary part $\sigma = \mathrm{Im}\, h$ is a nowhere degenerate two-form in M, and J_x is infinitesimally symplectic for σ_x. Finally,

$$\beta(v,\, w) = \sigma\left(Jv,\, w\right) \qquad (2.24)$$

shows that choosing two of the three structures $J,\ \beta,\ \sigma$ determines the third.

In order to get a Hermitian structure in E, we begin by observing that

$$h_x : v \mapsto (w \mapsto h_x(v,\, w)) : \mathrm{T}_x\, M \to \mathrm{T}_x^*\, M^{(0,\,1)} \qquad (2.25)$$

is a complex linear isomorphism from $\mathrm{T}_x\, M$ onto $\mathrm{T}_x^*\, M^{(0,\,1)}$. Using this isomorphism, we transplant the Hermitian structure of $\mathrm{T}\, M$ to a Hermitian structure $h^{(0,\,1)}$ in $\mathrm{T}^*\, M^{(0,\,1)}$. That is, $h^{(0,\,1)}$ is determined by the condition that, if e_j, $1 \le j \le n$, is a unitary local frame in $\mathrm{T}\, M$ for h, then $\epsilon_j := h\, e_j$ forms a unitary local frame in $\mathrm{T}^*\, M^{(0,\,1)}$ for $h^{(0,\,1)}$. It is dual to the frame e_j, in the sense that $\langle e_j,\, \epsilon_k\rangle = \delta_{jk}$.

The Hermitian structure $h^{(0,\,q)}$ on $E^q = \mathrm{T}^*\, M^{(0,\,q)}$ can now be defined by the conditon that the ϵ_K form a unitary local frame in E^q, if for each strictly increasing sequence $K = (k_i)_{i=1}^q$ we write

$$\epsilon_K := \epsilon_{k_1} \wedge \epsilon_{k_2} \wedge \ldots \wedge \epsilon_{k_q}. \qquad (2.26)$$

The Hermitian structure h_E on the direct sum E of the E^q is defined by requiring the summands to be mutually orthogonal. And the Hermitian structure $h_{E\otimes L}$ on $E \otimes L$ by the condition that if e_j and l_k are unitary local frames in E and L, respectively, then the $e_j \otimes l_k$ form a unitary local frame in $E \otimes L$.