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Geometry of Moduli

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Foreword

The Norwegian government established the Abel Prize in mathematics in 2002, and the first prize was awarded in 2003. In addition to honoring the great Norwegian mathematician Niels Henrik Abel by awarding an international prize for outstanding scientific work in the field of mathematics, the prize shall contribute toward raising the status of mathematics in society and stimulate the interest for science among school children and students. In keeping with this objective, the Niels Henrik Abel Board has decided to finance annual Abel Symposia. The topic of the symposia may be selected broadly in the area of pure and applied mathematics. The symposia should be at the highest international level and serve to build bridges between the national and international research communities. The Norwegian Mathematical Society is responsible for the events. It has also been decided that the contributions from these symposia should be presented in a series of proceedings, and Springer Verlag has enthusiastically agreed to publish the series. The Niels Henrik Abel Board is confident that the series will be a valuable contribution to the mathematical literature.

Chair of the Niels Henrik Abel Board Kristian Ranestad

Preface

The title of the Abel symposium 2017 was *Geometry of Moduli* and our goal was to highlight important recent developments in algebraic geometry regarding the theory of moduli. This included the geometry of moduli spaces, geometric invariant theory, birational geometry, enumerative geometry, hyper-Kähler geometry, and stability conditions. Moduli theory is ubiquitous in algebraic geometry, as can be seen in the list of moduli spaces treated in the lectures: sheaves on varieties, symmetric tensors, abelian differentials, (log) Calabi–Yau varieties, points on schemes, rational varieties, curves, abelian varieties, and hyper-Kähler manifolds. We believe the proceedings from the conference, which contain both original research and surveys of recent developments, reflect the breadth of and important recent advances in the field.

The speakers and the titles of their lectures at the symposium were:

- Arend Bayer: *Bridgeland stability on Kuznetsov components in families*
- Jim Bryan: *Donaldson-Thomas invariants of the banana manifold and elliptic genera*
- Ana-Maria Castravet: *Derived categories of moduli spaces of stable rational curves*
- Dawei Chen: *Geometry of moduli of abelian differentials*
- Izzet Coskun: *The cohomology and birational geometry of moduli spaces of sheaves on surfaces*
- Barbara Fantechi: *Infinitesimal deformations of log Calabi Yau varieties and orbifolds*
- Maksym Fedorchuk: *Stability of Hilbert points and applications*
- Brendan Hassett: *Rationality in families*
- Klaus Hulek: *Degenerations of Hilbert schemes of degree 0 cycles on surfaces*
- Michael Kemeny: *On the possible Betti tables of a canonical curve*
- Frances Kirwan: *Applications of non-reductive geometric invariant theory*
- Emanuele Macri: *Bridgeland stability and the genus of space curves*
- Kieran O'Grady: *Abelian varieties associated to hyper-Kählers of Kummer type*
- Andrei Okounkov: *Monodromy and derived equivalences*

- Aaron Pixton: *Polynomiality of the double ramification cycle*
- Claire Voisin: *Cubic fourfolds, hyper-Kähler manifolds and their degenerations*

The symposium took place from August 7 to 11, 2017, at Svinøya Rorbuer, Svolvær in Lofoten. The program and organizing committee consisted of Jan Arthur Christophersen (Oslo), John Christian Ottem (Oslo), Ragni Piene (Oslo), Kristian Ranestad (Oslo), Sofia Tirabassi (Bergen), Rahul Pandharipande (ETH Zurich), and Gavril Farkas (Humboldt, Berlin).

We would like to express our gratitude to the Norwegian Mathematical Society for giving us the opportunity to host the Abel Symposium. We would also like to thank the administration of the Department of Mathematics, University of Oslo, for their assistance and Ruth Allewelt at Springer Verlag for her valued support in preparing these proceedings.

Oslo, Norway — Jan Arthur Christophersen
Oslo, Norway — Kristian Ranestad
May 8, 2018

Contents

Stratifying Quotient Stacks and Moduli Stacks

Gergely Bérczi, Victoria Hoskins, and Frances Kirwan

Abstract Recent results in geometric invariant theory (GIT) for non-reductive linear algebraic group actions allow us to stratify quotient stacks of the form $[X/H]$, where X is a projective scheme and H is a linear algebraic group with internally graded unipotent radical acting linearly on X, in such a way that each stratum $[S/H]$ has a geometric quotient S/H. This leads to stratifications of moduli stacks (for example, sheaves over a projective scheme) such that each stratum has a coarse moduli space.

1 Introduction

Let $H = U \rtimes R$ be a linear algebraic group over an algebraically closed field of characteristic 0 with internally graded unipotent radical U; that is, the Levi subgroup R of H has a central one-parameter subgroup (1-PS) $\lambda : \mathbb{G}_m \to R$ which acts on $\operatorname{Lie} U$ with all weights strictly positive. Of course any reductive group G has this form with both U and the central one-parameter subgroup λ being trivial; parabolic subgroups of reductive groups also have internally graded unipotent radicals in this sense, as do automorphism groups of complete toric varieties [2]. Suppose that H acts linearly on an irreducible projective scheme X with respect to an ample line bundle L over X. The aim of this paper is to describe a stratification of the quotient stack $[X/H]$ such that each stratum $[S/H]$ (where S is an H-invariant quasi-projective subscheme of X) has a geometric quotient S/H. When $H = R$

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is reductive this stratification refines the Hesselink–Kempf–Kirwan–Ness (HKKN) stratification associated to the linear action on X (cf. [14, 18, 19, 23]). Potential applications of this construction include moduli stacks which can be filtered by quotient stacks with compatible linearisations; for example, it can be applied to moduli of sheaves of fixed Harder–Narasimhan type over a projective scheme [7], and moduli of unstable projective curves [17].

When H is reductive Mumford's geometric invariant theory (GIT) [22] allows us to find open subschemes $X^s \subseteq X^{ss}$ of X, the stable and semistable loci, such that X^s has a geometric quotient X^s/H and X^{ss} has a good quotient

$$\phi : X^{ss} \to X/\!/H = \operatorname{Proj}\left(\bigoplus_{m\geqslant 0} H^0(X, L^{\otimes m})^H\right).$$

Here $X^s = X^{s,H} = X^{s,H,\mathcal{L}}$ and $X^{ss} = X^{ss,H} = X^{ss,H,\mathcal{L}}$ depend on the choice of linearisation $\mathcal{L}$ (that is, the ample line bundle L and the lift of the action of H to an action on L) and the GIT quotient $X/\!/H = X/\!/_{\mathcal{L}}H$ is a projective scheme with X^s/H as an open subscheme. Moreover when $x, y \in X^{ss}$ then $\phi(x) = \phi(y)$ if and only if the closures of the H-orbits of x and y meet in X^{ss}. The Hilbert–Mumford criteria allow us to determine the stable and semistable loci in a simple way without needing to understand the algebra of invariants $\bigoplus_{m\geqslant 0} H^0(X, L^{\otimes m})^H$.

The best situation occurs when $X^{ss} = X^s \neq \emptyset$; then $X/\!/H = X^s/H$ is both a projective scheme and a geometric quotient of the open subscheme X^s of X. More generally if $X^s \neq \emptyset$ then the projective completion $X/\!/H$ of the geometric quotient X^s/H has a 'partial desingularisation' $\widetilde{X}/\!/H = \widetilde{X}^{ss}/H$ where $\psi : \widetilde{X}^{ss} \to X^{ss}$ is obtained as follows [20]. If $X^{ss} \neq X^s$ then there exists $x \in X^{ss}$ whose stabiliser in H is reductive of dimension strictly bigger than 0. To construct $\widetilde{X}^{ss}$ we first blow up X^{ss} along its closed subscheme where the stabiliser has maximal dimension in X^{ss} (such stabilisers are always reductive) or equivalently blow up X along the closure of this subscheme in X. We then remove the complement of the semistable locus for a small ample perturbation of the pullback linearisation. The maximal dimension of a stabiliser in this new semistable locus is strictly less than in X^{ss}. When $X^s \neq \emptyset$, repeating this process finitely many times leads to $\widetilde{X}^{ss} = \widetilde{X}^s \neq \emptyset$ and the partial desingularisation $\widetilde{X}/\!/H = \widetilde{X}^s/H$.

When H is non-reductive, then the graded algebra $\bigoplus_{m\geqslant 0} H^0(X, L^{\otimes m})^H$ is not necessarily finitely generated and in general the attractive properties of Mumford's GIT fail [3]. However when the unipotent radical U of $H = U \rtimes R$ is graded in the sense described above by a central 1-PS $\lambda : \mathbb{G}_m \to R$ of the Levi subgroup R, then after twisting the linearisation by an appropriate rational character, so that it becomes 'graded' itself in the sense of [5], some of the desirable properties of classical GIT still hold [4, 6]. More precisely, we first quotient by the linear action of the graded unipotent group $\widehat{U} := U \rtimes \lambda(\mathbb{G}_m)$ using the results of [4, 6] described in Sect. 2.3, then we quotient by the residual action of the reductive group $H/\widehat{U} \cong R/\lambda(\mathbb{G}_m)$. In the best case, when the U-action is free on a certain open subscheme

$X^0_{\min}$ of X (cf. the condition ($*$) in Definition 5 and Theorem 2.7), one can construct a geometric quotient of an open subscheme of 'stable points' for the $\widehat{U}$-action such that the quotient is projective and this stable set has a Hilbert–Mumford type description. If the U-action has positive dimensional stabilisers generically, one can conclude the same results if we assume a weaker condition ($***$) (cf. Theorem 2.11). Even when this weaker condition fails, one can perform an iterated sequence of blow-ups of X along H-invariant subschemes to obtain $\psi : \widetilde{X} \to X$ such that $\widetilde{X}$ has an induced linear H-action satisfying ($***$). Hence, there is a projective and geometric $\widehat{U}$-quotient of an open subscheme of $\widetilde{X}$ that contains as an open subscheme a geometric $\widehat{U}$-quotient of an open subscheme of X, as ψ is an isomorphism away from the exceptional divisor.

Now suppose that G is a reductive group acting linearly on a projective scheme X with respect to an ample line bundle L. Associated to this linear G-action and an invariant inner product on Lie G, there is a stratification (the 'HKKN stratification' cf. [14, 18, 19, 23])

$$X = \bigsqcup_{\beta \in \mathscr{B}} S_\beta$$

of X by locally closed subschemes S_β, indexed by a partially ordered finite set $\mathscr{B}$, such that

1. if $X^{ss} \neq \emptyset$, then $\mathscr{B}$ has a minimal element 0 such that $S_0 = X^{ss}$,
2. for $\beta \in \mathscr{B}$, the closure of S_β is contained in $\bigcup_{\beta' \geqslant \beta} S_{\beta'}$, and
3. for $\beta \in \mathscr{B}$, we have $S_\beta \cong G \times^{P_\beta} Y_\beta^{ss}$, where $G \times^{P_\beta} Y_\beta^{ss}$ is the quotient of $G \times Y_\beta^{ss}$ by the diagonal action of a parabolic subgroup P_β of G acting on the right on G and on the left on a P_β-invariant locally closed subscheme Y_β^{ss} of X.

In fact, Y_β^{ss} is an open subscheme of a projective subscheme $\overline{Y}_\beta$ of X that is determined by the action of a Levi subgroup L_β of P_β with respect to the restriction of the G-linearisation $L \to X$ to the P_β-action on $\overline{Y}_\beta$ twisted by a rational character χ_β of P_β. The index β determines a central (rational) 1-PS $\lambda_\beta : \mathbb{G}_m \to L_\beta$ and χ_β is the corresponding rational character, where the choice of invariant inner product allows us to identify characters and co-characters of a fixed maximal torus (cf. Remark 2.2). Furthermore P_β is the parabolic subgroup $P(\lambda_\beta)$ determined by the 1-PS λ_β, which grades the unipotent radical U_β of P_β. Thus by Property (3) above, to construct a G-quotient of (an open subset of) an unstable stratum S_β, we can study the linear P_β-action on $\overline{Y}_\beta$ and apply the results described above for the action of $\widehat{U} := U_\beta \rtimes \lambda_\beta(\mathbb{G}_m)$.

The G-action on the stratum S_β has a categorical quotient $Z_\beta /\!/ L_\beta$ induced by the morphism

$$p_\beta : Y_\beta^{ss} \to Z_\beta^{ss} = Z_\beta^{ss, L_\beta/\lambda_\beta(\mathbb{G}_m)} \qquad y \mapsto p_\beta(y) := \lim_{t \to 0} \lambda_\beta(t) y,$$

where Z_β is the union of those connected components of the fixed point set for the action of $\lambda_\beta(\mathbb{G}_m)$ on X over which λ_β acts on the fibres of L with weight given by the restriction of β. However this categorical quotient is in general far from being a geometric quotient; it identifies y with $p_\beta(y)$, which lies in the orbit closure but typically not the orbit of y.

In this article we will show that, applying the blow-up sequence needed to construct a quotient by an action of a linear algebraic group with internally graded unipotent radical to the P_β-action on the projective subscheme $\overline{Y}_\beta$ of X, we can refine the stratification $\{S_\beta | \beta \in \mathscr{B}\}$ to obtain a stratification of X such that each stratum is a G-invariant quasi-projective subscheme of X with a geometric quotient by the action of G. This refined stratification is a further refinement of the construction described in [21]. The quotient stack $[X/G]$ has an induced stratification $\{\Sigma_\gamma | \gamma \in \Gamma\}$ such that each stratum Σ_γ has the form

$$\Sigma_\gamma \cong [W_\gamma / H_\gamma]$$

where W_γ is a quasi-projective subscheme of X acted on by a linear algebraic subgroup H_γ of G with internally graded unipotent radical, and this action has a geometric quotient W_γ / H_γ. Moreover under appropriate hypotheses (involving condition ($***$)) for the actions of the subgroups H_γ, the geometric quotients W_γ / H_γ are themselves projective.

This will follow from the following theorem, which is proved in Sect. 3.

Theorem 1.1 *Let $H = U \rtimes R$ be a linear algebraic group with internally graded unipotent radical U acting on a projective scheme X over an algebraically closed field $\Bbbk$ of characteristic 0 with respect to an ample linearisation and fix an invariant inner product on* Lie R. *Then X has a stratification $\{\mathscr{S}_\gamma | \gamma \in \Gamma\}$ induced by the linearisation $\mathscr{L}$ and grading $\lambda : \mathbb{G}_m \to R$ for the action of H on X, such that the following properties hold.*

(i) *The index set Γ is finite and partially ordered such that for all $\gamma \in \Gamma$, we have*

$$\overline{\mathscr{S}_\gamma} \subseteq \mathscr{S}_\gamma \cup \bigcup_{\delta \in \Gamma, \delta > \gamma} \mathscr{S}_\delta.$$

(ii) *Each $\mathscr{S}_\gamma$ is an H-invariant quasi-projective subscheme of X with a geometric quotient $\mathscr{S}_\gamma / H$.*

(iii) *If Y is an H-invariant projective subscheme of X then the stratification $\{\mathscr{S}^Y_\gamma | \gamma \in \Gamma^Y\}$ of Y induced by the restriction $\mathscr{L}|_Y$ of the linearisation $\mathscr{L}$ to Y is (up to taking connected components) the restriction to Y of the stratification $\{\mathscr{S}_\gamma | \gamma \in \Gamma\}$ of X, so that there is a map of indexing sets $\phi_Y : \Gamma^Y \to \Gamma$ such that if $\gamma \in \Gamma^Y$ then $\mathscr{S}^Y_\gamma$ is a connected component of $\mathscr{S}_{\phi_Y(\gamma)} \cap Y$;*

Moreover, if $H = G$ is reductive, then this stratification satisfies the following additional properties.

(iv) *The stratification* $\{\mathscr{S}_\gamma | \gamma \in \Gamma\}$ *is a refinement of the HKKN stratification* $\{S_\beta | \beta \in \mathscr{B}\}$ *for the linearisation* $\mathscr{L}$ *(cf. Sect. 2.1.1).*

(v) *If* $\beta \in \mathscr{B}$ *(which we recall determines a 1-PS* $\lambda_\beta : \mathbb{G}_m \to P_\beta \leq G$*) satisfies*

$$x \in Z_\beta^{ss} \quad \Rightarrow \quad \dim(\mathrm{Stab}_G(x)/\lambda_\beta(\mathbb{G}_m)) = 0, \tag{†}$$

then the connected components of GZ_β^{ss} *and* $S_\beta \setminus GZ_\beta^{ss}$ *(if these are nonempty) are strata in the refined stratification* $\{\mathscr{S}_\gamma | \gamma \in \Gamma\}$.

As a consequence, we obtain a stratification of the quotient stack $[X/H]$ by locally closed substacks, each of which admits a coarse moduli space (cf. Corollary 1). Inspired by the reductive GIT notion of a good quotient, Alper introduces a notion of a good moduli space for a stack [1]. However, in general the strata appearing in this stratification of $[X/H]$ will not admit good moduli spaces, because a necessary condition for a stack to admit a good moduli space is that its closed points have reductive stabiliser groups (cf. [1, Proposition 12.14]). In general (even when $H = G$ is reductive) the points in the strata of $[X/H]$ will have non-reductive stabiliser groups.

If $H = G$ is reductive, then a stacky version of the HKKN stratification has been studied by Halpern-Leister, and by abstracting this concept he obtains a notion of a Θ-stratification [12]. Indeed, the linearisation of the G-action on X and the choice of invariant inner product is precisely the data required to construct a Θ-stratification of $[X/G]$, and this Θ-stratification is the stratification $\{[\mathscr{S}_\beta/G] : \beta \in \mathscr{B}\}$ obtained from the HKKN stratification of X. The stratification described above thus refines this Θ-stratification without depending on any additional data.

Since the construction of the refined stratification involves studying the blow-up procedures used in partial desingularisations of reductive GIT quotients [20] and for constructing geometric quotients by linear algebraic groups with internally graded unipotent radical [6], one can ask how this compares with the stack-theoretic blow-up constructions. The ideas in [20] have been generalised to stacks by Edidin and Rydh [11] to show that for a smooth Artin stack $\mathfrak{X}$ admitting a stable good moduli space, there is a sequence of birational morphisms of smooth Artin stacks $\mathfrak{X}_n \to \cdots \mathfrak{X}_1 \to \mathfrak{X}$ such that the good moduli space of $\mathfrak{X}_n$ is an algebraic space with only tame quotient singularities and is a partial desingularisation of the good moduli space of $\mathfrak{X}$. However, for H non-reductive, it is often the case that $[X/H]$ will not have a good moduli space, and so one cannot apply this result.

The picture provided by Theorem 1.1 has potential applications to moduli stacks which are filtered by quotient stacks, and to the construction of moduli spaces of ‘unstable’ objects (for example, moduli of sheaves over a projective scheme [7] or moduli of projective curves [17]). Suppose that $\mathscr{M}$ is a moduli stack which can be expressed as an increasing union

$$\mathscr{M} = \bigcup_{n \geqslant 0} \mathscr{U}_n$$

of open substacks of the form

$$\mathscr{U}_n \cong [V_n/G_n]$$

where $[V_n/G_n]$ is the quotient stack associated to a linear action on a quasi-projective scheme V_n by a group G_n which is reductive (or more generally has internally graded unipotent radical). We can look for suitable 'stability conditions' on $\mathscr{M}$: linearisations $(\mathscr{L}_n)_{n \geqslant 0}$ for the actions of G_n on projective completions $\overline{V_n}$ of V_n and invariant inner products on Lie G_n which are compatible in the sense that the stratification induced by $\mathscr{L}_n$ on $[V_n/G_n]$ restricts to the stratification induced by $\mathscr{L}_m$ on $[V_m/G_m]$ when $n > m$.

This situation arises for sheaves over a projective scheme [7, 16], for example, and also for projective curves [17], and we obtain a stratification $\{\Sigma_\gamma | \gamma \in \Gamma\}$ of the stack $\mathscr{M}$ such that each stratum Σ_γ is isomorphic to a quotient stack $[W_\gamma/H_\gamma]$, where W_γ is quasi-projective acted on by a linear algebraic group H_γ with internally graded unipotent radical, and there is a geometric quotient W_γ/H_γ which is a coarse moduli space for Σ_γ. The geometric quotient W_γ/H_γ will be projective if semistability coincides with stability in an appropriate sense for the action of H_γ on a suitable projective completion of W_γ with respect to an induced linearisation.

The layout of this article is as follows. In Sect. 2, we will review classical and non-reductive GIT, describing how to construct quotients by actions of linear algebraic groups with internally graded unipotent radical. The heart of the paper is Sect. 3, in which we describe how to stratify a quotient stack $[X/H]$ into strata $\Sigma_\gamma = [W_\gamma/H_\gamma]$ where the action of H_γ on W_γ has a geometric quotient W_γ/H_γ. The argument is an inductive one, so the assumption on X and H is that H is a linear algebraic group with internally graded unipotent radical, and X is a projective scheme which has an amply linearised action of H. In Sect. 4, this construction is applied to stacks which are suitably filtered by quotient stacks, and Sect. 5 contains a brief discussion of examples including moduli of unstable curves and moduli of sheaves of given Harder–Narasimhan type over a fixed projective scheme.

We would like to thank Brent Doran, Daniel Halpern-Leistner, Eloise Hamilton and Joshua Jackson for helpful discussions about this material.

2 Classical and Non-reductive GIT

2.1 Classical GIT for Reductive Groups

In Mumford's GIT [22], a linearisation of an action of a reductive group G on a projective scheme X over an algebraically closed field $\Bbbk$ of characteristic 0 is given by a line bundle L (which we will always assume to be ample) on X and a lift of the action to L. Since G is reductive, the algebra of G-invariant sections $\widehat{\mathscr{O}}_L(X)^G$ is finitely generated as a graded algebra with associated projective scheme

$X//G = \mathrm{Proj}(\widehat{\mathcal{O}}_L(X)^G)$.

$$\begin{array}{ccccl} (X,L) & \rightsquigarrow & \widehat{\mathcal{O}}_L(X) & := & \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}) \\ \Big\downarrow & & \cup\!\mid & & \\ X//G & \leftsquigarrow & \widehat{\mathcal{O}}_L(X)^G & & \text{algebra of invariants.} \end{array}$$

The inclusion of $\widehat{\mathcal{O}}_L(X)^G$ in $\widehat{\mathcal{O}}_L(X)$ determines a rational map $X - - \to X//G$ which fits into a diagram

$$\begin{array}{rcccl} & X & -- \to & X//G & \text{projective} \\ & \bigcup & & \| & \\ \text{semistable} & X^{ss} & \twoheadrightarrow & X//G & \\ & \bigcup & & \bigcup & \text{open} \\ \text{stable} & X^s & \twoheadrightarrow & X^s/G & \end{array}$$

where X^s (the stable locus) and X^{ss} (the semistable locus) are open subschemes of X, there is a geometric quotient X^s/G for the action of G on X^s, the GIT quotient $X//G$ is a good quotient for the action of G on X^{ss} via the G-invariant surjective morphism $\phi : X^{ss} \to X//G$, and

$$\phi(x) = \phi(y) \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset.$$

The semistable and stable loci X^{ss} and X^s of X are characterised by the following properties ([22, Chapter 2], [24]).

Proposition 2.1 (Hilbert–Mumford Criteria for Reductive Groups) *Let T be a maximal torus of G.*

1. *A point $x \in X$ is semistable (respectively stable) for the G-action on X if and only if for every $g \in G$ the point gx is semistable (respectively stable) for the T-action.*
2. *A point $x \in X \subset \mathbb{P}^n$ with homogeneous coordinates $[x_0 : \ldots : x_n]$ is semistable (respectively stable) for a diagonal T-action on $\mathbb{P}^n$ with weights $\alpha_0, \ldots, \alpha_n$ if and only if*

$$0 \in \mathrm{Conv}\{\alpha_i : x_i \neq 0\}$$

(respectively 0 is contained in the interior of this convex hull).

2.1.1 The HKKN Stratification

Associated to the linear action of G on X and an invariant inner product on the Lie algebra of G, there is a stratification (the ‘HKKN stratification’, which in the case

$\Bbbk = \mathbb{C}$ is the Morse stratification for the norm-square of an associated moment map [14, 18, 19, 23]).

Remark 2.2 Let us clarify what is meant by this invariant inner product, whose associated norm we denote by $|| - ||$. If $\Bbbk = \mathbb{C}$, then G is the complexification of its maximal compact group K; then the Lie algebra of K is a real vector space, and we choose an inner product on this Lie algebra that is invariant under the adjoint action of K. In fact, we will also assume that we fix a maximal compact torus $T_c \subset K$ such that the inner product is integral on the co-character lattice $X_*(T_c) \subset \mathrm{Lie}\, K$. For an arbitrary algebraically closed field $\Bbbk$ of characteristic zero, one can fix a maximal torus T of G and choose an inner product on the co-character space $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ that is invariant for the Weyl group of T and is integral on the co-character lattice (for example, see [15, §2]). Then this inner product gives an identification between characters and co-characters (i.e. 1-PSs) of T.

The HKKN stratification associated to the action of G on X with respect to L and the norm $|| - ||$ is a stratification

$$X = \bigsqcup_{\beta \in \mathscr{B}} S_\beta$$

of X by locally closed subschemes S_β, indexed by a partially ordered finite subset $\mathscr{B}$ of rational elements in a positive Weyl chamber for the reductive group G, with the following properties.

1. If $0 \in \mathscr{B}$, then this is the minimal element and $S_0 = X^{ss}$.

Moreover, for each $\beta \in \mathscr{B}$, we additionally have the following properties:

2. the closure of S_β is contained in $\bigcup_{\beta' \geqslant \beta} S_{\beta'}$ where $\gamma \geqslant \beta$ if and only if $\gamma = \beta$ or $\|\gamma\| > \|\beta\|$;
3. $S_\beta \cong G \times^{P_\beta} Y_\beta^{ss} := (G \times Y_\beta^{ss})/P_\beta$ where this quotient is of the diagonal action of P_β on the right on G and on the left on Y_β^{ss}.

Here P_β is a parabolic subgroup of G which acts on a locally closed subscheme Y_β^{ss} of X.

More precisely, $\beta \in \mathscr{B}$ determines a (rational) 1-PS $\lambda_\beta : \mathbb{G}_m \to G$ and an associated parabolic subgroup $P_\beta = P(\lambda_\beta) = U_\beta \rtimes L_\beta$ with Levi subgroup $L_\beta = \mathrm{Stab}_G(\beta)$ such that the conjugation action of $\lambda_\beta(\mathbb{G}_m)$ on $\mathrm{Lie}\, U_\beta$ has strictly positive weights; thus P_β has internally graded unipotent radical. Let Z_β be the union of components in the fixed locus $X^{\lambda_\beta(\mathbb{G}_m)}$ on which this 1-PS acts on the fibres of L with weight given by the restriction of β, and let $Y_\beta \subset X$ be the subscheme of points $x \in X$ such that $\lim_{t \to 0} \lambda_\beta(t) y \in Z_\beta$; thus there is a retraction

$$p_\beta : Y_\beta \to Z_\beta \qquad y \mapsto p_\beta(y) := \lim_{t \to 0} \lambda_\beta(t) y$$

which is equivariant with respect to the quotient homomorphism $q_\beta : P_\beta \to L_\beta$ obtained by identifying L_β with P_β/U_β. Let $\mathscr{L}_\beta$ denote the restriction of the G-linearisation $\mathscr{L}$ on X to the P_β-action on $\overline{Y}_\beta$ twisted by the (rational) character χ_β of P_β corresponding to the 1-PS λ_β (via the norm $||-||$). We also let $\mathscr{L}_\beta$ denote the restriction of this linearisation to the L_β-action on Z_β. Then Y_β^{ss} (respectively Z_β^{ss}) is the semistable locus for the P_β-action on $\overline{Y}_\beta$ (respectively the $L_\beta/\lambda_\beta(\mathbb{G}_m)$-action on Z_β) linearised by the twisted linearisation $\mathscr{L}_\beta$; furthermore, $Y_\beta^{ss} = p_\beta^{-1}(Z_\beta^{ss})$.

Finally, we make the following observation about quotienting the unstable strata (cf. [16]).

Remark 2.3 The G-action on S_β has a categorical quotient $Z_\beta/\!/L_\beta$ induced by the map $p_\beta : Y_\beta^{ss} \to Z_\beta^{ss}$. In general, this quotient is far from being a geometric quotient (even after restriction to the pre-image of any nonempty open subscheme of $Z_\beta/\!/L_\beta$), as $y \in Y_\beta^{ss}$ is identified with $p_\beta(y) \in \overline{Gy}$.

By (3), constructing a quotient of the G-action on a G-invariant open subset of S_β is equivalent to constructing a P_β-quotient of a P_β-invariant open subset of Y_β^{ss} (or its closure); the latter perspective will lead to a geometric quotient by using GIT for the non-reductive group P_β, whose unipotent radical U_β is internally graded by λ_β (cf. Sect. 2.3).

2.1.2 Partial Desingularisations of Reductive GIT Quotients

The geometric quotient X^s/G has at most orbifold singularities when X is nonsingular, since the stabiliser subgroups of stable points are finite subgroups of G. If $X^{ss} \neq X^s \neq \emptyset$, the singularities of $X/\!/G$ are typically more severe even when X is itself nonsingular, but $X/\!/G$ has a 'partial desingularisation' $\widetilde{X}/\!/G$ which is also a projective completion of X^s/G and is itself a geometric quotient

$$\widetilde{X}/\!/G = \widetilde{X}^{ss}/G$$

by G of an open subscheme $\widetilde{X}^{ss} = \widetilde{X}^s$ of a G-equivariant blow-up $\widetilde{X}$ of X [20]. Here $\widetilde{X}^{ss}$ is obtained from X^{ss} by successively blowing up along the subschemes of semistable points stabilised by reductive subgroups of G of maximal dimension and removing the complement of the semistable locus from the resulting blow-up.

For the construction of the partial desingularisation $\widetilde{X}/\!/G$ in [20], it is assumed that $X^s \neq \emptyset$. There exist semistable points of X which are not stable if and only if there exists a non-trivial connected reductive subgroup of G fixing a semistable point. Let $r > 0$ be the maximal dimension of a reductive subgroup of G fixing a point of X^{ss} and let $\mathscr{R}(r)$ be a set of representatives of conjugacy classes of all connected reductive subgroups R of dimension r in G such that

$$Z_R^{ss} := \{x \in X^{ss} : Rx = x\}$$

is non-empty. Then

$$Z^{ss}_{\mathscr{R}(r)} := \bigcup_{R \in \mathscr{R}(r)} GZ^{ss}_R$$

is a disjoint union of closed G-invariant subschemes of X^{ss}. The action of G on X^{ss} lifts to an action on the blow-up $X_{(1)}$ of X^{ss} along $Z^{ss}_{\mathscr{R}(r)}$, and this action can be linearised so that the complement of $X^{ss}_{(1)}$ in $X_{(1)}$ is the proper transform of the closed subscheme $\pi^{-1}(\pi(GZ^{ss}_R))$ of X^{ss} where $\pi : X^{ss} \to X//G$ is the quotient map (see [20, 7.17]). The G-linearisation on $X_{(1)}$ used here is (a tensor power of) the pullback of the ample line bundle L on X along $\psi_{(1)} : X_{(1)} \to X$ perturbed by a sufficiently small multiple of the exceptional divisor $E_{(1)}$; then, if the perturbation is sufficiently small, we have

$$\psi^{-1}_{(1)}(X^s) \subseteq X^s_{(1)} \subseteq X^{ss}_{(1)} \subseteq \psi^{-1}_{(1)}(X^{ss}) = X_{(1)},$$

and the stable and semistable loci $X^s_{(1)}$ and $X^{ss}_{(1)}$ will be independent of the choice of perturbation. Moreover, no point $x \in X^{ss}_{(1)}$ is fixed by a reductive subgroup of G of dimension at least r, and $x \in X^{ss}_{(1)}$ is fixed by a reductive subgroup R of dimension less than r in G if and only if it belongs to the proper transform of the closed subscheme Z^{ss}_R of X^{ss}.

Remark 2.4 In [20], X itself is blown up along the closure $\overline{Z^{ss}_{\mathscr{R}(r)}}$ of $Z^{ss}_{\mathscr{R}(r)}$ in X (or in a projective completion of X^{ss} with a G-equivariant morphism to X which is an isomorphism over X^{ss}). This gives a projective scheme $\overline{X}_{(1)}$ and blow-down map $\overline{\psi_{(1)}} : \overline{X}_{(1)} \to X$ restricting to $\psi_{(1)} : X_{(1)} \to X$ where $(\overline{\psi_{(1)}})^{-1}(X^{ss}) = X_{(1)}$. For a sufficiently small perturbation of the pullback to $\overline{X}_{(1)}$ of the linearisation on X, we have $(\overline{\psi_{(1)}})^{-1}(X^s) \subseteq \overline{X}^s_{(1)} \subseteq \overline{X}^{ss}_{(1)} \subseteq (\overline{\psi_{(1)}})^{-1}(X^{ss}) = X_{(1)}$, and moreover the restriction of the linearisation to $X_{(1)}$ is obtained from the pullback of L by perturbing by a sufficiently small multiple of the exceptional divisor $E_{(1)}$.

If $r > 1$, we can apply the same procedure to $X^{ss}_{(1)}$ to obtain $X^{ss}_{(2)}$ such that no point of $X^{ss}_{(2)}$ is fixed by a reductive subgroup of G of dimension at least $r - 1$. If $X^s \neq \emptyset$ then repeating this process at most r times gives us $\psi : \widetilde{X}^{ss} \to X^{ss}$ such that ψ is an isomorphism over X^s and no positive-dimensional reductive subgroup of G fixes a point of $\widetilde{X}^{ss}$. The partial desingularisation $\widetilde{X}//G = \widetilde{X}^{ss}/G$ can be obtained by blowing up $X//G$ along the proper transforms of $\pi(GZ^{ss}_R) \subset X//G$ in decreasing order of the dimension of R.

Remark 2.5 Suppose for simplicity that X is irreducible.

a) If $X^s \neq \emptyset$, then this is the situation considered in [20] and the partial desingularisation construction is described above. If $X^{ss} = X^s$, then $\widetilde{X} = X$.

b) If $X^{ss} = \emptyset$, then there is an unstable stratum S_β with $\beta \neq 0$ in the HKKN stratification (cf. Sect. 2.1.1) which is a non-empty open subscheme of X, and thus when X is irreducible $X = \overline{S_\beta}$. Then constructing a quotient of a non-empty open subscheme of X reduces to non-reductive GIT for the action of the parabolic subgroup P_β on $\overline{Y_\beta}$ as described in Sect. 2.3 below, where a blow-up sequence may also need to be performed.

c) If $X^s = \emptyset \neq X^{ss}$ then the partial desingularisation construction can be applied to X^{ss}, and there are different ways in which it can terminate.

(i) If $X^{ss} = GZ_R^{ss} \cong G \times^{N_R} Z_R^{ss}$ for a positive-dimensional connected reductive subgroup R of G with normaliser N_R in G, then N_R and N_R/R are also reductive, and

$$X//G \cong Z_R//N_R \cong Z_R//(N_R/R)$$

where Z_R is the closed subscheme of X which is the fixed point set for the action of R. Then we can apply induction on the dimension of G to study this case.

(ii) If $GZ_R^{ss} \neq X^{ss}$ for each positive-dimensional connected reductive subgroup R of G, then we can perform the first blow-up in the partial desingularisation construction to obtain $\psi_{(1)} : X_{(1)} \to X^{ss}$ such that $X_{(1)}^{ss} \subseteq X_{(1)}$ and $X_{(1)}^s = \emptyset$ as above (as $X_{(1)}^s$ is open and $X_{(1)}^s \setminus E_{(1)} \cong X^s = \emptyset$, where $E_{(1)}$ is the exceptional divisor). If $X_{(1)}^{ss} = \emptyset$, then $X_{(1)}$ has a dense open stratum $S_{(1),\beta}$ for $\beta \neq 0$ as in Case *b)*. If we have $X_{(1)}^{ss} = GZ_{(1),R}^{ss}$ for a positive-dimensional connected reductive subgroup R of G, where $Z_{(1),R}^{ss} = \{x \in X_{(1)}^{ss} : Rx = x\}$, then we proceed as in Case *(i)* above. Otherwise we can repeat the process, until it terminates in one of these two ways.

2.2 GIT for Non-reductive Groups

Now suppose that X is a projective scheme over an algebraically closed field $\Bbbk$ of characteristic 0 and let H be a linear algebraic group, with unipotent radical U, acting on X with respect to an ample linearisation L.

Definition 1 (Semistability for the Unipotent Group cf. [10, §4] and [10, 5.3.7]) For an invariant section $f \in I = \bigcup_{m>0} H^0(X, L^{\otimes m})^U$, let X_f be the U-invariant affine open subset of X on which f does not vanish, and let $\mathcal{O}(X_f)$ denote its coordinate ring.

1. The *semistable locus* for the U-action on X linearised by L is $X^{ss,U} = \bigcup_{f \in I^{\mathrm{fg}}} X_f$ where

$$I^{\mathrm{fg}} = \{f \in I \mid \mathcal{O}(X_f)^U \text{ is finitely generated}\}.$$

2. The *(locally trivial) stable locus* for the linearised U-action on X is $X^{\mathrm{lts},U} = \bigcup_{f \in I^{\mathrm{lts}}} X_f$ where

$$I^{\mathrm{lts}} := \{f \in I^{\mathrm{fg}} \mid q_U : X_f \to \mathrm{Spec}(\mathcal{O}(X_f)^U) \text{ is a locally trivial geometric quotient}\}.$$

3. The *enveloped quotient* of $X^{ss,U}$ by the linear U-action is $q_U : X^{ss,U} \to q_U(X^{ss,U})$, where $q_U : X^{ss,U} \to \mathrm{Proj}(\widehat{\mathcal{O}}_L(X)^U)$ is the natural morphism of schemes and $q_U(X^{ss,U})$ is a dense constructible subset of the *enveloping quotient*

$$X \wr\wr U = \bigcup_{f \in I^{fg}} \mathrm{Spec}(\mathcal{O}(X_f)^U).$$

Remark 2.6

1. Even when $\widehat{\mathcal{O}}_L(X)^U$ is finitely generated, so that $X \wr\wr U = \mathrm{Proj}(\widehat{\mathcal{O}}_L(X)^U)$, the enveloped quotient $q_U(X^{ss,U})$ is not necessarily a subscheme of $X \wr\wr U$ (for example, see [10, §6]).
2. The enveloping quotient $X \wr\wr U$ has quasi-projective open subschemes ('inner enveloping quotients' $X /\!\!/_{\circ} U$) that contain the enveloped quotient $q_U(X^{ss})$ and have ample line bundles which under the natural map $q_U : X^{ss} \to X \wr\wr U$ pull back to positive tensor powers of L (see [3] for details).

The H-semistable locus $X^{ss} = X^{ss,H}$ and enveloped and (inner) enveloping quotients

$$q_H : X^{ss} \to q_H(X^{ss}) \subseteq X /\!\!/_{\circ} H \subseteq X \wr\wr H$$

for the linear action of H are defined as for the unipotent case in Definition 1 and Remark 2.6 (cf. [3]), but the definition of the stable locus $X^{\mathrm{lts}} = X^{\mathrm{lts},H}$ for the linear action of H combines (and extends) the definitions for unipotent and reductive groups.

Definition 2 (Stability for Linear Algebraic Groups) Let H be a linear algebraic group acting linearly on X with respect to an ample line bundle L; then the *(locally trivial) stable locus* is the open subscheme $X^{\mathrm{lts}} = \bigcup_{f \in I^{\mathrm{lts}}} X_f$ of X^{ss}, where $I^{\mathrm{lts}} \subseteq \bigcup_{r>0} H^0(X, L^{\otimes r})^H$ is the subset of H-invariant sections f satisfying the following conditions:

1. the H-invariant open subscheme X_f is affine;
2. the H-action on X_f is closed with finite stabiliser groups; and
3. the restriction of the U-enveloping quotient map

$$q_U : X_f \to \mathrm{Spec}((\widehat{\mathcal{O}}_L(X)^U)_{(f)})$$

is a locally trivial geometric quotient for the U-action on X_f.

2.3 GIT for Linear Algebraic Groups with Internally Graded Unipotent Radicals

Now suppose that $H = U \rtimes R$ is a linear algebraic group with internally graded unipotent radical U in the following sense.

Definition 3 We say $H = U \rtimes R$ has *internally graded unipotent radical* U if there is a central 1-PS $\lambda : \mathbb{G}_m \to R$ whose conjugation action on the Lie algebra of U has strictly positive weights. We write $\widehat{U} = U \rtimes \lambda(\mathbb{G}_m) \leqslant H$ for the associated semi-direct product.

Suppose also that H acts linearly on X with respect to an ample line bundle L. It is shown in [4, 6] that the algebra of H-invariant sections is finitely generated provided:

(a) L is replaced with a suitable tensor power $L^{\otimes m}$, with $m \geq 1$ sufficiently divisible, and the linearisation of the action of H is twisted by a suitable (rational) character, and
(b) condition (∗) described below (also known as 'semistability coincides with stability for the unipotent radical U') holds.

Moreover, in this situation the natural quotient morphism q_H from the semistable locus $X^{ss,H}$ to the enveloping quotient $X \wr\wr H$ is surjective, and expresses the projective scheme $X//H = X/\!\!/_{\circ} H = X \wr\wr H$ as a good quotient of $X^{ss,H}$. Furthermore this locus $X^{ss,H}$ can be described using Hilbert–Mumford criteria. It is also shown in [6] that when condition (∗) is not satisfied, but is replaced with a slightly weaker condition, such as (∗∗) below, then there is a sequence of blow-ups of X along H-invariant subschemes (similar to that of [20] when H is reductive) resulting in a projective scheme $\widehat{X}$ with an induced linear action of H satisfying condition (∗). In fact, these results can be generalised to allow actions where the U-action has positive dimensional stabilisers (cf. Theorems 2.11 and 2.12 below). Before giving a precise description of the condition (∗) and its variants, we define the notion of an adapted linearisation, which is also needed for the statement of the main results of [4, 6].

Let $\chi : H \to \mathbb{G}_m$ be a character of H; the restriction to $\widehat{U}$ of χ contains U in its kernel and can be identified naturally with an integer so that the integer 1 corresponds to the character of $\widehat{U}$ which fits into the exact sequence $U \hookrightarrow \widehat{U} \twoheadrightarrow \lambda(\mathbb{G}_m)$. By replacing L with a sufficiently high power, we can without loss of generality assume that L is very ample. Let $\omega_{\min} := \omega_0 < \omega_1 < \cdots < \omega_{\max}$ be the weights with which the 1-PS $\lambda : \mathbb{G}_m \to \widehat{U} \leq H$ acts on the fibres of the tautological line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,L)^*)}(-1)$ over points of the fixed locus $\mathbb{P}((H^0(X, L)^*)^{\lambda(\mathbb{G}_m)}$. Without loss of generality we may assume that there exist at least two distinct such weights, as otherwise the grading hypothesis implies that the U-action on X is trivial, in which case we can take a quotient by the action of the reductive group $R = H/U$.

Definition 4 For a character χ of H as above and a positive integer c, we say the rational character χ/c is *adapted* to the linear action of $\widehat{U}$ if

$$\omega_{\min} := \omega_0 < \frac{\chi}{c} < \omega_1. \tag{1}$$

Furthermore, we say L is adapted to the $\widehat{U}$-action if $\omega_{\min} := \omega_0 < 0 < \omega_1$.

If the rational character χ/c is adapted to the linear action of $\widehat{U}$, then the H-linearisation $\mathscr{L}_\chi^{\otimes c}$ on X given by twisting the ample line bundle $L^{\otimes c}$ by the character χ (that is, so that the weights ω_j are replaced with $\omega_j c - \chi$) is adapted. Let $X^{s,\mathbb{G}_m}_{\min+} = X^{s,\lambda(\mathbb{G}_m)}_{\min+}$ denote the stable locus in X for the linear action of $\mathbb{G}_m$ via λ with respect to the adapted linearisation $\mathscr{L}_\chi^{\otimes c}$ and, for a maximal torus T of H containing $\lambda(\mathbb{G}_m)$, let $X^{(s)s,T}_{\min+}$ denote the (semi)stable locus in X for the linear action of T with respect to the adapted linearisation $L_\chi^{\otimes c}$. By the theory of variation of (classical) GIT [9, 28], the stable locus $X^{s,\lambda(\mathbb{G}_m)}_{\min+} = X^{ss,\lambda(\mathbb{G}_m)}_{\min+}$ is independent of the choice of adapted rational character χ/c. In fact, by the Hilbert–Mumford criterion, we have $X^{s,\lambda(\mathbb{G}_m)}_{\min+} = X^0_{\min} \setminus Z_{\min}$, where if $V_{\min}$ denotes the weight space of $\omega_{\min}$ in $V = H^0(X, L)^*$, then

$$Z_{\min} := X \cap \mathbb{P}(V_{\min}) = \left\{ x \in X^{\lambda(\mathbb{G}_m)} : \lambda(\mathbb{G}_m) \text{ acts on } L^*|_x \text{ with weight } \omega_{\min} \right\}$$

and

$$X^0_{\min} := \{x \in X \mid \lim_{t \to 0,\ t \in \mathbb{G}_m} \lambda(t) \cdot x \in Z_{\min}\}.$$

Definition 5 (Conditions (∗)–(∗ ∗ ∗) Generalising 'Semistability Equals Stability' cf. [6]) With the above notation, we define the following conditions for the $\widehat{U}$-action on X.

$$\mathrm{Stab}_U(z) = \{e\} \text{ for every } z \in Z_{\min}. \tag{$*$}$$

$$\mathrm{Stab}_U(x) = \{e\} \text{ for generic } x \in X^0_{\min}. \tag{$**$}$$

Moreover, if $U \geqslant U^{(1)} \geqslant \ldots \geqslant U^{(s)} \geqslant \{e\}$ denotes the derived series of U, we define condition

$$\text{for } 1 \le j \le s, \text{ there exists } d_j \in \mathbb{N} \text{ such that } \dim \mathrm{Stab}_{U^{(j)}}(x) = d_j \text{ for all } x \in X^0_{\min}. \tag{$***$}$$

Note that condition (∗) holds if and only if we have $\mathrm{Stab}_U(x) = \{e\}$ for all $x \in X^0_{\min}$. This condition is also referred to in [6] as the condition 'semistability coincides with stability' for the action of $\widehat{U}$ (or, when the 1-PS $\lambda : \mathbb{G}_m \to R$ is fixed, for the linear action of U).

Definition 6 Let $T \le R$ be a maximal torus containing $\lambda(\mathbb{G}_m)$. The *min-stable* (respectively *min-semistable*) locus for a linear H-action satisfying condition (∗ ∗ ∗) for the action of the graded unipotent group $\widehat{U}$ is

$$X^{s,H}_{\min+} := \bigcap_{h \in H} h X^{s,T}_{\min+}$$

(respectively $X^{ss,H}_{\min+} := \bigcap_{h \in H} h X^{ss,T}_{\min+}$).

The min-stable locus for the $\widehat{U}$-action then satisfies

$$X^{s,\widehat{U}}_{\min+} = X^{ss,\widehat{U}}_{\min+} = \bigcap_{u \in U} u X^{s,\lambda(\mathbb{G}_m)}_{\min+} = X^0_{\min} \setminus U Z_{\min}.$$

Theorem 2.7 ($\widehat{U}$-Theorem When Semistability Coincides with Stability for $\widehat{U}$) *[6] Suppose that the linearisation for the action of H on X is adapted as in Definition 4 and that the $\widehat{U}$-action on X satisfies condition ($*$). Then the following statements hold.*

(i) *The open subscheme $X^{s,\widehat{U}}_{\min+}$ of X has a projective geometric quotient $X/\!/\widehat{U} = X^{s,\widehat{U}}_{\min+}/\widehat{U}$ by $\widehat{U}$.*

(ii) *The open subscheme $X^{ss,H}_{\min+}$ of X has a good quotient $X/\!/H = (X/\!/\widehat{U})/\!/(R/\mathbb{G}_m)$ by $H = U \rtimes R$, which is also projective.*

Remark 2.8 In the proof of Theorem 2.7 (and its variants below), one replaces the adapted linearisation by a 'well adapted' linearisation (which can be achieved by twisting by a rational character); this is a slightly stronger notion. This strengthening does not alter $X^{s,\widehat{U}}_{\min+}$ or its quotient $X/\!/\widehat{U}$, but it affects what can be said about induced ample line bundles on $X/\!/\widehat{U}$ and $X/\!/H = (X/\!/\widehat{U})/\!/(R/\lambda(\mathbb{G}_m))$. The proofs in [4, 6] that the algebras of invariants $\oplus_{m\geq 0} H^0(X, L^{\otimes cm}_{m\chi})^{\widehat{U}}$ and

$$\bigoplus_{m\geq 0} H^0(X, L^{\otimes cm}_{m\chi})^H = (\bigoplus_{m\geq 0} H^0(X, L^{\otimes cm}_{m\chi})^{\widehat{U}})^{(R/\lambda(\mathbb{G}_m))}$$

are finitely generated, and that the enveloping quotients $X/\!/\widehat{U} = X^{s,\widehat{U}}_{\min+}/\widehat{U}$ and $X/\!/H$ are the associated projective schemes, require that the linearisation is twisted by a well adapted rational character χ/c. More precisely, it is shown in [4, 6] that, given a linear action of H on X with respect to an ample line bundle L, there exists $\epsilon > 0$ such that if χ/c is a rational character of $\mathbb{G}_m$ (lifting to H) with c sufficiently divisible and $\chi : H \to \mathbb{G}_m$ a character of H such that

$$\omega_{\min} < \frac{\chi}{c} < \omega_{\min} + \epsilon,$$

then the algebras of invariants $\oplus_{m\geq 0} H^0(X, L^{\otimes cm}_{m\chi})^{\widehat{U}}$ and $\oplus_{m\geq 0} H^0(X, L^{\otimes cm}_{m\chi})^H$ are finitely generated, and the associated projective schemes $X/\!/\widehat{U}$ and $X/\!/H$ satisfy the conclusions of Theorem 2.7.

Theorem 2.7 describes the good case when semistability coincides with stability for the linear action of $\widehat{U}$. The following versions proved in [6] apply more generally.

Theorem 2.9 ($\widehat{U}$-Theorem Giving Projective Completions [6]) *Suppose that the linear action of $\widehat{U}$ on X is adapted and satisfies condition ($**$). Then there exists a sequence of blow-ups along H-invariant projective subschemes, resulting in a* *projective scheme $\widehat{X}$ (with blow-down map $\widehat{\psi} : \widehat{X} \to X$) such that the conditions of Theorem 2.7 hold for a suitable ample linearisation.*

If in addition the stable locus $(\widehat{X}//\widehat{U})^{s,R/\lambda(\mathbb{G}_m)}$ for the induced linear action of $R/\lambda(\mathbb{G}_m)$ on $\widehat{X}//\widehat{U}$ is nonempty, then this sequence of blow-ups along H-invariant projective subschemes of X can be extended, resulting in a projective scheme $\widetilde{X}$ (with blow-down map $\widetilde{\psi} : \widetilde{X} \to X$) such that the conditions of Theorem 2.7 still hold for a suitable ample linearisation, and such that the quotient given by that theorem is a geometric quotient of an open subscheme $\widetilde{X}^{s,H}$ of $\widetilde{X}$.

Remark 2.10 For the first stage of this blow-up sequence the centres are determined by considering the stabiliser subgroups for $\widehat{U}$ to obtain $\widehat{X}$, while for the second stage one blows up by considering the stabiliser subgroups for the reductive group R to obtain $\widetilde{X}$.

In the first step, one performs a blow-up sequence to obtain $\widehat{\psi} : \widehat{X} \to X$ such that the $\widehat{U}$-action on $\widehat{X}$ satisfies condition (∗) with respect to a linearisation $\widehat{\mathscr{L}}$, which is an arbitrarily small perturbation of $\widehat{\psi}^*(\mathscr{L})$. The centres of the blow-ups used to obtain $\widehat{X}$ from X are determined by the dimensions of the stabilisers in $\widehat{U}$ of $x \in X^0_{\min}$ (for details, see [6]). Then one can construct a projective and geometric quotient of the $\widehat{U}$-action on $\widehat{X}^{s,\widehat{U}}_{\min+}$ by Theorem 2.7 and, as $\widehat{\psi}$ is an isomorphism away from the exceptional divisor, one obtains a geometric quotient of a $\widehat{U}$-invariant open subset $X^{\hat{s},\widehat{U}}_{\min+}$ of X as an open subscheme of $\widehat{X}//\widehat{U}$, where $X^{\hat{s},\widehat{U}}_{\min+}$ is the image under $\widehat{\psi}$ of the intersection of $\widehat{X}^{s,\widehat{U}}_{\min+}$ with the complement of the exceptional divisor in $\widehat{X}$. Another characterisation of $X^{Ms,\widehat{U}}$, when $\mathrm{Stab}_U(z) = \{e\}$ for generic $z \in Z_{\min}$, is as

$$\{x \in \psi(\widehat{X}^{s,\widehat{U}}_{\min+}) \mid \dim \mathrm{Stab}_U(\lim_{t\to 0} \lambda(t) \cdot x) = 0\};$$

if (∗) holds then $\widehat{X} = X$ and $X^{\hat{s},\widehat{U}}_{\min+} = X^{s,\widehat{U}}_{\min+}$. If one is only interested in obtaining a good quotient for the H-action, then the second stage of the blow-up procedure is not needed: one can then take a reductive GIT quotient of the residual action on $\widehat{X}//_{\widehat{\mathscr{L}}}\widehat{U}$ of $H/\widehat{U} = R/\lambda(\mathbb{G}_m)$, and thus one obtains a good quotient of the H-action on an open subset of X as an open subscheme of $(\widehat{X}//\widehat{U})//(R/\lambda(\mathbb{G}_m))$. Moreover, this good quotient restricts to a geometric quotient on an open subscheme of stable points.

To go from $\widehat{X}$ to the blow-up $\widetilde{X}$ in Theorem 2.9, one performs an additional blow up sequence by considering the stabiliser groups for the action of the reductive group $R/\lambda(\mathbb{G}_m)$ as in the partial desingularisation procedure described in Sect. 2.1.2. This gives us an H-invariant open subscheme of X with a geometric quotient by H which is isomorphic to an open subscheme of the projective scheme $\widetilde{X}//_{\widetilde{\mathscr{L}}}H$ (and also of $\widehat{X}//_{\widehat{\mathscr{L}}}H$).

Theorem 2.7 can be generalised by weakening the condition (∗) further to (∗ ∗ ∗) to allow for actions with positive dimensional stabiliser groups generically.

Theorem 2.11 ($\widehat{U}$-Theorem with Positive-Dimensional Stabilisers in U) *[6] Suppose that condition $(***)$ holds for an adapted linear $\widehat{U}$-action on X. Then the conclusions of Theorem 2.7 hold.*

In fact, this theorem still holds if we replace the derived series in condition $(***)$ with any series $U > U^{(1)} > \ldots > U^{(s)} > \{e\}$ which is normalised by H and whose successive quotients $U^{(j)}/U^{(j+1)}$ are abelian, provided that $(***)$ holds for this series.

Finally, there is a version of the theorem without requiring any hypothesis related to semistability coinciding with stability.

Theorem 2.12 ($\widehat{U}$-Theorem with Positive-Dimensional Stabilisers in U, Giving Projective Completions) *[6] For a linear H-action on X with respect to an adapted ample linearisation $\mathscr{L}$, there is a sequence of blow-ups along H-invariant projective subschemes, resulting in a projective scheme $\widehat{X}$ (with blow-down map $\widehat{\psi} : \widehat{X} \to X$) with a linear H-action such that condition $(***)$ is satisfied, and so the conclusions of Theorem 2.7 hold.*

If in addition the stable locus $(\widehat{X}/\!/\widehat{U})^{s,R/\lambda(\mathbb{G}_m)}$ for the induced linear action of the reductive group $R/\lambda(\mathbb{G}_m)$ on $\widehat{X}/\!/\widehat{U}$ is nonempty, then this sequence of blow-ups along H-invariant projective subschemes of X can be extended, resulting in another projective scheme $\widetilde{X}$ such that condition $(***)$ holds, and such that the resulting H-quotient $\widetilde{X}/\!/H = (\widetilde{X}/\!/\widehat{U})/\!/(R/\lambda(\mathbb{G}_m))$ is a geometric quotient of an open subscheme of $\widetilde{X}$.

This theorem provides a non-reductive analogue of the partial desingularisation construction for reductive GIT described at the end of Sect. 2.1.

Definition 7 For a linear $\widehat{U}$-action on X with respect to an adapted ample linearisation $\mathscr{L}$, we define $X^{\hat{s},\widehat{U}}_{\min+}$ to be the image of the intersection of $\widehat{X}^{s,\widehat{U}}_{\min+}$ with the complement of the exceptional divisor in $\widehat{X}$ under the blow-down map $\widehat{\psi} : \widehat{X} \to X$ given by Theorem 2.12.

Since the projective scheme $\widehat{X}/\!/\widehat{U}$ is a geometric quotient of the $\widehat{U}$-action on $\widehat{X}^{s,\widehat{U}}_{\min+}$, the $\widehat{U}$-action on $X^{\hat{s},\widehat{U}}_{\min+}$ has a geometric quotient $X^{\hat{s},\widehat{U}}_{\min+}/\widehat{U} \subset \widehat{X}/\!/\widehat{U}$, which is invariant under the induced action of $R/\lambda(\mathbb{G}_m)$.

Note that if the $\widehat{U}$-action already satisfies condition $(***)$, then $\widehat{X} = X$ and thus the locus $X^{\hat{s},\widehat{U}}_{\min+}$ coincides with the min-stable locus $X^{s,\widehat{U}}_{\min+}$ given by Definition 6.

Remark 2.13 As in Theorem 2.9 (cf. Remark 2.10), one first constructs a blow-up $\widehat{X} \to X$ by considering stabiliser subgroups for the action of $\widehat{U}$, and then if $(\widehat{X}/\!/\widehat{U})^{s,R/\lambda(\mathbb{G}_m)} \neq \emptyset$ one constructs a further blow-up sequence $\widetilde{X} \to \widehat{X}$ by considering the stabiliser subgroups for the residual action of the reductive subgroup $R/\lambda(\mathbb{G}_m)$. The slight difference is that in the first step, we look at $U^{(j)}$-stabilisers as well as U-stabilisers for $x \in X^0_{\min}$.

Recall that $\widehat{X}^{s,\widehat{U}}_{\min+} = \widehat{X}^0_{\min} \setminus UZ_{\min}(\widehat{X})$, where $Z_{\min}(\widehat{X})$ is defined as $Z_{\min}$ for $\widehat{X}$ instead of X. In order to describe $X^{\hat{s},\widehat{U}}_{\min+} = \widehat{\psi}(\widehat{X}^{s,\widehat{U}}_{\min+} \setminus \widehat{E})$ more precisely, we need to consider the construction of $\widehat{X}$ from X in more detail. Let $U = U^{(0)} > U^{(1)} > \ldots > U^{(s)} > U^{(s+1)} = \{e\}$ be the derived series for U, as before, and let $\widehat{U}^{(j)} = U^{(j)} \rtimes \lambda(\mathbb{G}_m)$ for $0 \leqslant j \leqslant s+1$; then $U^{(j)}$ and $\widehat{U}^{(j)}$ are normal in H. The first step in the construction is to take the largest j such that $(***)$ does not hold for $\widehat{U}^{(j)}$ and then blow X up along the closure of

$$C_j(X) = \{x \in X^0_{\min} : \dim \mathrm{Stab}_{U^{(j)}}(x) = d^{(j)}_{\max}\}$$

where $d^{(j)}_{\max}$ is the maximal value of $\dim \mathrm{Stab}_{U^{(j)}}(x)$ for $x \in X^0_{\min}$; we let $\psi_{(1)} : X_{(1)} \to X$ denote the resulting blow-up, with exceptional divisor denoted $E_{(1)}$. Here $C_j(X)$ is an H-invariant closed subscheme of $X^0_{\min}$, and $(X_{(1)})^0_{\min}$ is an open subscheme of the blow-up $(\psi_{(1)})^{-1}(X^0_{\min})$ of $X^0_{\min}$ along $C_j(X)$ [6]. Furthermore if $x \in Z_{\min}$ and L is very ample then $H^0(X, L)$ decomposes as a representation of $\mathrm{Stab}_U(x)$ as the direct sum of a trivial one-dimensional representation (which is a weight space for the action of $\lambda(\mathbb{G}_m)$) and the subspace

$$\{\sigma \in H^0(X, L) : \sigma_x = 0\}$$

which is isomorphic as a $\mathrm{Stab}_U(x) \rtimes \lambda(\mathbb{G}_m)$-module to the tangent space to x in $\mathbb{P}(H^0(X, L)^*)$. From this it follows that if $(***)$ does not hold for $\widehat{U}^{(j)}$, so that $C_j(X)$ is a proper closed subscheme of $X^0_{\min}$, then successively blowing up $X^0_{\min}$ along $C_j(X)$ strictly reduces the value of $d^{(j)}_{\max}$. Thus we can obtain $\widehat{X}^0_{\min}$ from $X^0_{\min}$ inductively by blowing up $X^0_{\min}$ along $C_j(X)$, removing the complement of $(X_{(1)})^0_{\min}$ in the result and repeating the process until $(***)$ is satisfied.

To describe $X^{\hat{s},\widehat{U}}_{\min+}$, we therefore need to understand $Z_{\min}(X_{(1)})$ and $(X_{(1)})^0_{\min}$, as well as the centre of the next blow-up. If $Z_{\min} \not\subseteq C_j(X)$ then $Z_{\min}(X_{(1)})$ is the proper transform of $Z_{\min}$ in $X_{(1)}$ and

$$\psi_{(1)}((X_{(1)})^0_{\min} \setminus E_{(1)}) = \{x \in X^0_{\min} : \dim \mathrm{Stab}_{U^{(j)}}(p(x)) < d^{(j)}_{\max}\}.$$

However if $Z_{\min} \subseteq C_j(X)$ then describing $Z_{\min}(X_{(1)})$ and $\psi_{(1)}((X_{(1)})^0_{\min} \setminus E_{(1)})$ is more complicated. In this situation $Z_{\min}(X_{(1)}) \subseteq (\psi_{(1)})^{-1}(Z_{\min})$ is the image under $p_{(1)}$ of

$$(\psi_{(1)})^{-1}\{x \in X^0_{\min} \setminus C_j(X) : \lambda(\mathbb{G}_m) \text{ acts on the fibre of } L^* \text{ over } \lim_{t\to\infty} \lambda(t)\cdot x \text{ with weight } r_{\min}\}$$

where $r_{\min}$ is minimal among those natural numbers r such that

$$\{x \in X^0_{\min} : \lambda(\mathbb{G}_m) \text{ acts on the fibre of } L^* \text{ over } \lim_{t\to\infty} \lambda(t) \cdot x \text{ with weight } r\} \not\subseteq C_j(X).$$

Remark 2.14 It follows from Theorem 2.12 and the construction described in Remark 2.13 that the open H-invariant subscheme $X^{\hat{s},\widehat{U}}_{\min+}$ of X is nonempty unless

$UZ_{\min}$ is dense in X. Furthermore, by construction, for each $1 \le j \le s$, the function $\dim \mathrm{Stab}_{U^{(j)}}(-)$ is constant on $X^{\hat{s},\widehat{U}}_{\min+}$.

We will see in the next section that by applying (the proof of) Theorem 2.12 to the closures of subschemes of X where the dimensions of the $U^{(j)}$-stabilisers take different values, and combining this with the partial desingularisation construction of [20] for reductive GIT quotients, X can be stratified so that each stratum is a locally closed H-invariant subscheme of X with a geometric quotient by the action of H.

3 Stratifying Quotient Stacks

Let $H = U \rtimes R$ be a linear algebraic group with internally graded unipotent radical U (cf. Definition 3) acting on a projective scheme X over an algebraically closed field $\Bbbk$ of characteristic 0 with respect to an ample linearisation $\mathscr{L}$. We fix an invariant inner product on the Lie algebra Lie R of the Levi factor, just as in the construction of the HKKN stratification (cf. Sect. 2.1.1).

The aim of this section is to prove Theorem 1.1 stated in Sect. 1. We will prove this result using a recursive argument involving the dimensions of X and of H and the number of irreducible components of X. The idea will be to start by defining a 'minimum' stratum, which will be a non-empty H-invariant open subscheme of X, and then proceed recursively.

We will assume that H is connected and that X is reduced.

Remark 3.1 Let us explain why these assumptions do not involve any significant loss of generality.

1. The HKKN stratification $\{S_\beta \mid \beta \in \mathscr{B}\}$ is usually indexed by a finite subset $\mathscr{B}$ of a positive Weyl chamber. Then the strata are not necessarily connected even when X is irreducible, and it is often useful to refine the stratification so that the strata are the connected components of S_β, or are unions of some but not all of these connected components (cf. [19]). There is a similar ambiguity in the construction of the refined stratification defined in this section: at some points we take connected components, but this is not crucial to the definition. Indeed if we wish to allow the group H to be disconnected then we cannot assume that the strata are connected since they are required to be H-invariant. Then instead of taking connected components (which will be invariant under the component H_0 of the identity in H), we can take their H-sweeps, which will be disjoint unions of at most $|H/H_0|$ of these connected components.
2. If X is non-reduced, then we can define the stratification on X by using a positive power of L to define an H-equivariant embedding of X into a projective space $\mathbb{P}^n$, and then take the fibre product of X with the stratification on $\mathbb{P}^n$. Indeed, we will see that the stratification is functorial for equivariant closed immersions (this is essentially the third statement in Theorem 1.1). This follows as the reductive

notions of GIT (semi)stability are functorial and since in the non-reductive case, we are assuming that H has internally graded unipotent radical. The stable loci when H has internally graded unipotent radical and adapted linearisation are also functorial, as they have Hilbert–Mumford style descriptions (see Definition 6). We note that in the more general non-reductive GIT set up described in Sect. 2.2 the notions of (semi)stability are not functorial, as taking H-invariants is not exact, and so there can be invariants which do not extend to the ambient space. The advantage of assuming that X is reduced is that the complement to the open stratum then has a canonical scheme structure; thus it is easier to recursively define the stratification.

Let us describe the recursive construction when H is connected and X is reduced. For each linear action on X of $H = U \rtimes R$ with internal grading $\lambda : \mathbb{G}_m \to R$ and linearisation $\mathscr{L}$ with underlying ample line bundle L, we will first use recursion to define a *nonempty* H-invariant open subscheme $\mathscr{S}_0(X, H, \lambda, \mathscr{L})$ of X that admits a geometric quotient $\mathscr{S}_0(X, H, \lambda, \mathscr{L})/H$; this will be done by considering seven different cases. After defining the open stratum, we will define the stratification $\{\mathscr{S}_\gamma | \gamma \in \Gamma\}$ of X with strata $\mathscr{S}_\gamma = \mathscr{S}_\gamma(X, H, \lambda, \mathscr{L})$ and index set $\Gamma = \Gamma(X, H, \lambda, \mathscr{L})$ by letting $X_1, \ldots, X_k$ be the connected components of the projective subscheme of X equal to the complement of $\mathscr{S}_0(X, H, \lambda, \mathscr{L})$, letting $\mathscr{S}_{0,i}(X, H, \lambda, \mathscr{L})$ for $1 \leqslant i \leqslant m$ be the connected components of $\mathscr{S}_0(X, H, \lambda, \mathscr{L})$, and setting

$$\Gamma(X, H, \lambda, \mathscr{L}) := \{0\} \times \{1, \ldots, m\} \cup \bigcup_{1 \leqslant j \leqslant k} \{X_j\} \times \Gamma(X_j, H, \lambda, \mathscr{L}|_{X_j}). \tag{2}$$

The strata indexed by $(0, i) \in \Gamma(X, H, \lambda, \mathscr{L})$ for $1 \leqslant i \leqslant m$ are then the connected components of the open subscheme $\mathscr{S}_0(X, H, \lambda, \mathscr{L})$ constructed using the case by case argument below, whereas the stratum indexed by an element (X_j, γ) for $1 \leq j \leq k$ and $\gamma \in \Gamma(X_j, H, \lambda, \mathscr{L}|_{X_j})$ is

$$\mathscr{S}_{(X_j,\gamma)}(X, H, \lambda, \mathscr{L}) := \mathscr{S}_\gamma(X_j, H, \lambda, \mathscr{L}|_{X_j}), \tag{3}$$

where the strata $\mathscr{S}_\gamma(X_j, H, \lambda, \mathscr{L}|_{X_j})$ are constructed by induction. The partial order on Γ then naturally comes from the partial orders on each $\Gamma(X_j, H, \lambda, \mathscr{L}|_{X_j})$ with $(0, i) < (X_j, \gamma)$ for all $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k$ and $\gamma \in \Gamma(X_j, H, \lambda, \mathscr{L}|_{X_j})$.

Let us now describe how to define $\mathscr{S}_0(X, H, \lambda, \mathscr{L})$ in the seven different cases. Let

$$U = U^{(0)} \geqslant U^{(1)} = [U, U] \geqslant \ldots \geqslant U^{(s)} \geqslant \{e\}$$

be the derived series of U. Let

$$Z_{\min}^{d_0} = \{x \in Z_{\min} | \dim(\mathrm{Stab}_U(x)) = d_0\}$$

where d_0 is the minimal value of $\dim(\mathrm{Stab}_U(x))$ for $x \in Z_{\min}$. Then $Z_{\min}^{d_0}$ is a nonempty H-invariant open subscheme of $Z_{\min}$, and $UZ_{\min}^{d_0}$ is a nonempty locally closed subscheme of X.

Case 1 First assume that the central one-parameter subgroup $\lambda : \mathbb{G}_m \to R$ of R has at least two distinct weights for the linear action of H on X; equivalently $Z_{\min}$ is a projective subscheme of X with $Z_{\min} \neq X$. Under this assumption we have two possibilities to consider.

Case 1(a) Suppose that $UZ_{\min}^{d_0}$ is open in X. Then we define

$$\mathscr{S}_0(X, H, \lambda, \mathscr{L}) = U\{x \in \mathscr{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathscr{L}|_{Z_{\min}}) \mid \dim(\mathrm{Stab}_{U_j}(x)) = d_j \text{ for } 1 \leqslant j \leqslant s\}$$

where λ_0 is the trivial one-parameter subgroup that grades the trivial unipotent radical of the reductive group $R/\lambda(\mathbb{G}_m)$ and $d_j := \min\{\dim(\mathrm{Stab}_{U_j}(x)) : x \in \mathscr{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathscr{L}|_{Z_{\min}})\}$.

By induction we can assume that $\mathscr{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathscr{L}|_{Z_{\min}})$ is a nonempty R-invariant open subscheme of $Z_{\min}$ with a geometric quotient by the action of $R/\lambda(\mathbb{G}_m)$ (or equivalently by the action of R, since the central one-parameter subgroup $\lambda(\mathbb{G}_m)$ of R acts trivially on $Z_{\min}$). Indeed, we can construct such an open subscheme as in Case 2 described below. Thus $\mathscr{S}_0(X, H, \lambda, \mathscr{L})$ is an H-invariant nonempty open subscheme of X, and by the proof of Theorem 2.12 (see [6, Remark 2.10]) it has a geometric quotient

$$\begin{aligned}\mathscr{S}_0(X, H, \lambda, \mathscr{L})/H &\cong \mathscr{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathscr{L}|_{Z_{\min}})/R \\ &= \mathscr{S}_0(Z_{\min}, R/\lambda(\mathbb{G}_m), \lambda_0, \mathscr{L}|_{Z_{\min}})/(R/\lambda(\mathbb{G}_m)).\end{aligned}$$

Case 1(b) Suppose that $UZ_{\min}^{d_0}$ is not open in X. Then the locus $X_{\min+}^{\hat{s},\widehat{U}}$ defined at Definition 7 is a nonempty H-invariant open subset of $X_{\min}^0 \setminus UZ_{\min}$ (by Remark 2.14), which has a geometric quotient $X_{\min+}^{\hat{s},\widehat{U}}/\widehat{U}$ with a projective completion

$$\overline{X_{\min+}^{\hat{s},\widehat{U}}/\widehat{U}} \subseteq \widehat{X}/\!/\widehat{U},$$

where $\widehat{X} \to X$ is the blow-up given by Theorem 2.12, and on which $R/\lambda(\mathbb{G}_m)$ acts linearly with respect to an induced linearisation $\widehat{\mathscr{L}}$ (see Remark 2.14). We set

$$\mathscr{S}_0(X, H, \lambda, \mathscr{L}) = \left\{x \in X_{\min+}^{\hat{s},\widehat{U}} : \widehat{U}x \in \mathscr{S}_0(X_{\min+}^{\hat{s},\widehat{U}}/\widehat{U}, R/\lambda(\mathbb{G}_m), \lambda_0, \widehat{\mathscr{L}})\right\}.$$

Then $\mathscr{S}_0(X, H, \lambda, \mathscr{L})$ is an H-invariant nonempty open subscheme of X with a geometric quotient $\mathscr{S}_0(X, H, \lambda, \mathscr{L})/\widehat{U}$ by $\widehat{U}$, and by the inductive construction a