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Shape-Preserving Approximation by Real and Complex Polynomials

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To the memory of my parents
Gheorghe and Ana

Preface

In many problems arising in engineering and science one requires approximation methods to reproduce physical reality as well as possible. Very schematically, if the input data represents a complicated discrete/continuous quantity of information, of "shape" S (S could mean, for example, that we have a "monotone/convex" collection of data), then one desires to represent it by the less-complicated output information, that "approximates well" the input data and, in addition, has the same "shape" S.

This kind of approximation is called "shape-preserving approximation" and arises in computer-aided geometric design, robotics, chemistry, etc.

Typically, the input data is represented by a real or complex function (of one or several variables), and the output data is chosen to be in one of the classes polynomial, spline, or rational functions.

The present monograph deals in Chapters 1–4 with shape-preserving approximation by real or complex polynomials in one or several variables. Chapter 5 is an exception and is devoted to some related important but non-polynomial and nonspline approximations preserving shape. The spline case is completely excluded in the present book, since on the one hand, many details concerning shape-preserving properties of splines can be found, for example, in the books of de Boor [49], Schumaker [344], Chui [69], DeVore–Lorentz [91], Kvasov [218] and in the surveys of Leviatan [229], Kocić–Milovanović [196], while on the other hand, we consider that shape-preserving approximation by splines deserves a complete study in a separate book.

The topic of shape-preserving approximation by real polynomials has a long history and probably begins with an earlier result of Pál [295] in 1925, which states that any convex function on an interval $[a, b]$ can be uniformly approximated on that interval by a sequence of convex polynomials.

The first constructive answer to the Pál's result seems to have been given by T. Popoviciu [317] in 1937, who proved that if f is convex (strictly convex) of order k on $[0, 1]$ (in the sense defined in Section 1.1), then the Bernstein polynomial $B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n})$ is convex (strictly convex, respectively) of order k on $[0, 1]$, for all $n \in \mathbb{N}$.

Over time, much effort has been expended by many mathematicians to contribute to this topic. As good examples of surveys concerning shape-preserving approximation of univariate real functions by real polynomials, we can mention those of Leviatan [229], [230] in 1996 and 2000, that of Kocić–Milovanović [197] in 1997, and that of Hu–Yu [178] in 2000.

Also, a few aspects in the univariate real case are presented in the following books:

Lorentz–v. Golitschek–Makozov [249] in 1996, see Chapter 2, Section 3, titled Monotone Approximation (pp. 43–49), and page 82, with Problem 9.4 and Notes 10.1, 10.2,

Shevchuk [349] in 1992, referring to some results in monotone and convex approximation of univariate real functions by real polynomials,

Lorentz [247](see p. 23) in 1986, DeVore–Lorentz [91] in 1993, (see Chapter 10, Section 3, from page 307 to page 309), concerning some shape-preserving properties of real Bernstein polynomials, and

Gal [123] in 2005, concerning shape-preserving properties of classical Hermite–Fejér and Grünwald interpolation polynomials.

For the situation in the case of one complex variable, it is worth noting that two books concerning the study of complex polynomials have recently been published. The first is that of Sheil–Small [346] in 2002, which studies many geometric properties of complex polynomials and rational functions. But except for two small sections on the complex convolution polynomials through Cesàro and de la Vallée-Poussin trigonometric kernels (Sections 4.5 and 4.6, from page 156 to page 166), in fact that book does not deal with the preservation of geometric properties of analytic functions by approximating complex polynomials. The second book mentioned above is that of Rahman–Schmeisser [320] in 2002, which refers to the critical points, zeros, and extremal properties of complex polynomials, which are regarded as analytic functions of a special kind. Although some of its results improve classical inequalities of great importance in approximation theory (of Nikolskii, Bernstein, Markov, etc.), this book again does not deal with the preservation of geometric properties of analytic functions by approximating complex polynomials.

In the cases of two/several real or complex variables, there are no books at all treating the subject of shape-preserving approximation.

Therefore, we may conclude that despite the very large numbers of papers in the literature, at present, none of the books has been dedicated entirely to shape-preserving approximation by real and complex polynomials.

The present monograph seeks to fill this gap in the mathematical literature and is, to the best of our knowledge, the first book entirely dedicated to this topic. It attempts to assemble the main results from the great variety of contributions spread across a large number of journals all over the world.

This monograph contains the work of the main researchers in this area, as well as the research of the author over the past five years in these subjects and many new contributions that have not previously been published.

Chapter 1 mainly studies shape-preserving approximation and interpolation of real functions of one real variable by real polynomials. The "shapes" taken into consideration are convexity of order k (which includes the usual positivity, monotonicity, and convexity), some variations of positivity as almost positivity, strongly/weakly almost positivity, copositivity (with its variations almost copositivity, strongly/weakly copositivity), comonotonicity, and coconvexity. A variation of copositive approximation, called intertwining approximation (with its two variations almost and nearly intertwining), also is presented.

Chapter 2 deals with shape-preserving approximation of real functions of two/several real variables by bivariate/multivariate real polynomials. A main characteristic of this chapter is that to one concept of shape in univariate case, several concepts of shapes of a bivariate/multivariate function may be associated. For example, monotonicity has as variations *bivariate monotonicity, axial monotonicity, strong monotonicity*; convexity has the variations *axial-convexity, polyhedral convexity, strong convexity, and subharmonicity*, and so on.

In Chapter 3 we consider shape-preserving approximation of analytic functions of one complex variable by complex polynomials in the unit disk. The concepts of "shapes" preserved through approximation by polynomials are those in geometric function theory: *univalence, starlikeness, convexity, close-to-convexity, spiralikeness, growth of coefficients*, etc. The construction of such polynomials is mainly based on the Shisha-type method and on the convolution method.

Chapter 4 contains extensions of some results in Chapter 3 to shape-preserving approximation of analytic functions of several complex variables on the unit ball or the unit polydisk by polynomials of several complex variables.

It is worth noting that three constructive methods are "red lines" of the book, that is, they work for real univariate variables, real multivariate variables, complex univariate variables, and complex multivariate variables. These are the methods of Bernstein, producing Bernstein-type polynomials; the Shisha-type method; and the convolution-type method. As a consequence, Chapters 1–4 use these three methods. Also, although the error estimates produced by the *tensor product method* are not always the best possible, because of its simplicity we use it intensively in order to extend the results from the univariate to the bivariate/multivariate case.

Chapters 1–5 begin with an introductory section, in which we describe in detail the corresponding chapter and introduce the main concepts.

The book ends with Chapter 5, which is an appendix containing some related topics of great interest in shape-preserving approximation. Shape-preserving approximation by splines is not included in this chapter for the reasons mentioned at the beginning of this preface.

Let us mention that systematic results in Chapters 2–5 have been obtained by the author of this monograph in a series of papers, singly or jointly written with other researchers (as can be seen in the bibliography), and many new results appear for the first time here. Also, many open questions suggested at the end of Chapters 1–5 might be of interest for future research.

The book is intended for use in the fields of approximation of functions, mathematical analysis, numerical analysis, computer-aided geometric design, data fitting, fluid mechanics, and engineering, robotics, and chemistry. It is also suitable for graduate courses in the above domains.

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Contents

1

Shape-Preserving Approximation by Real Univariate Polynomials

In this chapter we present the main results concerning shape-preserving approximation by polynomials for real functions of one real variable, defined on compact subintervals of the real axis. There is a very rich literature dedicated to this topic that would suffice to write a separate book. Due to this fact, it was impossible for me to avoid the more pronounced survey-like character of this chapter. Also, for the proofs of some main results that are very technical and long, we will present here only their most important ideas and steps.

1.1 Introduction

In this section we will introduce the history of the subject, followed by very brief descriptions of the next sections in the chapter.

Probably one of the first results on the topic is an earlier result of Pál [295] in 1925, which states that any convex function on an interval $[a, b]$ can be uniformly approximated on that interval by a sequence of convex polynomials.

The first constructive solution to Pál's result seems to have been given by T. Popoviciu [317] in 1937, who proved that if f is convex (strictly convex) of order k on $[0, 1]$ (in the sense defined below in this section), then the *Bernstein* polynomial $B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n})$ is convex (strictly convex, respectively) of order k on $[0, 1]$, for all $n \in \mathbb{N}$.

In the intervening years, a great deal of work has been done on this topic by many mathematicians. The aim of this chapter is to present this great effort in detail.

The topic of Chapter 1 might be divided into five main directions.

The first direction deals with the shape-preserving properties of interpolation polynomials, and this is the subject of Section 1.2. We mention here the contributions of (in alphabetical order) Deutch, Gal, Ivan, Kammerer, Kopotun, Lorentz, Morris, Nikolcheva, Passow, Popoviciu, Raymon, Roulier, Rubinstein, Szabados, Wolibner, Young, Zeller, and others.

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The second direction deals with the shape-preserving properties of the so-called Bernstein-type polynomial operators, (thus called because their constructions were suggested by the form of Bernstein's polynomials), representing the subjects of Section 1.3. We can mention here the contributions of (in alphabetical order) Berens, Butzer, Carnicer, Dahmen, Derrienic, DeVore, Gadzijev, Goodman, Ibikli, Ibragimov, Kocić, Lacković, Lupaş, Mastroianni, Micchelli, Munoz-Delgado, Müller, Nessel, Păltănea, Peña, Phillips, Ramirez-Gonzalez, Raşa, Sabloniére, Sauer, Stancu, Wood, and others.

Because of its close connection with the shape-preserving properties (see Section 5.1), the variation-diminishing property too is presented in Section 1.3.

The third direction deals with the so-called *Shisha*-type results, and it began with Shisha's paper of 1965. The method is, in general, based on polynomials of simultaneous approximation of a function and its derivatives, to which are added suitable polynomials (uniformly convergent to zero) in such a way that the new sum preserves some signs of the derivatives of the function. We mention here the contributions of (in chronological order) Shisha, Roulier, and Anastassiou–Shisha. It is contained in Section 1.4.

It is worth noting here the importance of Shisha's method, taking into account that because of its simplicity, it was extended to real functions of two real variables in Chapter 2, to complex functions of one complex variable in Chapter 3, and to complex functions of several complex variables in Chapter 4.

Note that the second direction of research produces rather weak degrees of approximation in terms of $\omega_k(f; \frac{1}{\sqrt{n}}), k = 1, 2$, while the third direction of research, although essentially improving the estimates of the second direction, has, however, the shortcoming that these estimates are given in terms of the moduli of smoothness of the derivatives of the function.

In order to obtain better estimates, that is, with respect to the moduli of smoothness (of various orders) of a function, one of the most used techniques (introduced for the first time in DeVore-Yu [86]) can be described as follows: first one approximates f by piecewise polynomials (splines) with the same shape as f, and then one replaces the piecewise polynomials by polynomials of the same shape. Estimates in terms of first- or higher-order moduli of smoothness in all the L^p-spaces, $0 < p \leq +\infty$, were found by (in alphabetical order) Beatson, DeVore, Ditzian, Dzyubenko, Hu, Iliev, Ivanov, Kopotun, Leviatan, Lorentz, Mhaskar, Newman, Operstein, Popov, Prymak, Shevchuk, Shvedov, Szabados, Wu, Yu, Zeller, Zhou, and others.

The main results are included in Sections 1.5, 1.6, 1.7 and are represented by the so-called *positive and copositive (with their variations like, almost, strongly/weakly, intertwining*) approximation, *monotone and comonotone* approximation (with the variation *nearly comonotone* approximation), and *convex and coconvex* approximation (with the variation *nearly coconvex* approximation), respectively. The above-mentioned variations of classical positive/copositive, comonotone, and coconvex approximations were introduced by the authors in order to improve the estimates, by requiring that the polynomials preserve the corresponding "shapes" in a major part of the interval, except for small neighborhoods of the endpoints and of the points where the approximated function changes the "shapes".

The shape-preserving approximation results in Sections 1.5, 1.6, and 1.7 can also be classified with respect to the type of error estimate, as follows:

(i) approximation results with respect to the L^p-norm and in terms of best approximation quantities $E_n(f^{(i)})_p$, $i = 0, 1, 2$, and $0 < p \leq +\infty$.
(ii) approximation results with respect to the L^p-norm and in terms of the L^p-Ditzian–Totik moduli of smoothness, $0 < p \leq +\infty$.
Note that in both cases (i) and (ii), the uniform cases (i.e., $p = +\infty$) are richer in results than the cases $0 < p < +\infty$ and will be separately treated.
(iii) pointwise approximation on $[-1, 1]$ with DeVore-Telyakokovskii-Gopengauz-type estimates, in terms of the usual moduli of smoothness and with respect to the increments $\frac{1}{n^2} + \frac{1}{n}(1 - x^2)^{1/2}$ and $\frac{1}{n}(1 - x^2)^{1/2}$.
(iv) approximation in terms of higher moduli of smoothness of higher derivatives of functions.

Notice that while in monotone and convex approximation, the methods that produces estimates in terms of second-order moduli of smoothness are linear, the methods in convex approximation that produce the best possible order, i.e., in terms of the third-order moduli of smoothness, together with those in copositive, comonotone, and coconvex approximations, are nonlinear. It is not known whether there exist corresponding linear methods of approximation for these last three cases too.

Section 1.8 deals with the fifth direction of research, based on convolution-type polynomials and on the Boolean-sum method. This method produces good approximation errors of DeVore–Gopengauz type, but with respect to the previous ones has the advantage that the constructed polynomials preserve even higher-order convexities too. We mention here the contributions of Jia-Ding Cao and Gonska.

Section 1.9 presents a constructive example of a nonconvolution, positive linear polynomial operator that reproduces the linear functions, gives an error estimate of DeVore–Gopengauz-type in terms of second-order modulus of smoothness and preserves convexities of higher-order of the approximated function. The contributions belong to Jia-Ding Cao, Cottin, Gavrea, Gonska, Kacsó, Lupaş and Zhou.

In what follows, we introduce well-known concepts of shapes (monotonicities, convexities, etc.) necessary for the next sections of Chapter 1. Denote by $C[a, b]$ the space of all real functions defined and continuous on $[a, b]$.

Definition 1.1.1. (i) $f : [a, b] \to \mathbb{R}$ is called j-convex on $[a, b]$ (or convex of order j), if all the jth forward differences $\Delta_h^j f(t), 0 \leq h \leq (b - a)/j, t \in [a, b - jh]$ are non-negative (i.e., ≥ 0). Here $\Delta_h^j f(t) = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} f(t + kh)$, for all $j = 0, 1, \ldots$. If there exists $f^{(j)}$ on $[a, b]$, a simple application of the mean value theorem shows that the condition $f^{(j)}(x) \geq 0$, for all x $[a, b]$, implies that f is j-convex on $[a, b]$. Recall that the usual convexity (2-convexity in the above sense) can also be defined by the inequality $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$, for all $\lambda \in [0, 1]$ and $x, y \in [a, b]$.

Also, f is called j-concave on $[a, b]$ if all the jth forward differences $\Delta_h^j f(t), 0 \leq h \leq (b-a)/j, t \in [a, b-jh]$ are nonpositive (i.e., ≤ 0).

(ii) A function $f : [0, 1] \to \mathbb{R}$ is called starshaped on $[0, 1]$ if $f(\lambda x) \leq \lambda f(x)$, for all $\lambda \in [0, 1]$, $x \in [0, 1]$. If the above inequality is strict for all $\lambda \in (0, 1)$ then f is called strictly starshaped. Also, if there exists $f'(x)$ on $[0, 1]$, $f(0) = 0, f(x) \geq 0, x \in [0, 1]$, then the starshapedness (it is equivalent to) can be expressed by the differential inequality $xf'(x) - f(x) \geq 0$, for all $x \in (0, 1]$ (see, e.g., L. Lupaş [254]);

A function $f : [0, 1] \to \mathbb{R}$ is called α-star-convex on $[0, 1]$, where $\alpha \in [0, 1]$, if $f(\lambda x + (1-\lambda)\alpha y) \leq \lambda f(x) + (1-\lambda)\alpha f(y)$, for all $x, y \in [0, 1]$, $\lambda \in [0, 1]$ (see Toader [386]).

(iii) A function $f : [a, b] \to \mathbb{R}$, $f(x) > 0$, for all $x \in [a, b]$, is called logarithmic-convex on $[a, b]$, if $\log[f(x)]$ is a 2-convex function on $[a, b]$;

(iv) A function $f : [a, b] \to \mathbb{R}$ is called quasiconvex on $[a, b]$ if it satisfies the inequality $f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$, for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. It is known that f is quasiconvex on $[a, b]$ if and only if for any $c \in \mathbb{R}$, $\{x \in [a, b]; f(x) \leq c\}$ is a convex set;

More generally, a function $f : [a, b] \to \mathbb{R}$ is called j-quasiconvex on $[a, b]$, $j \in \mathbb{N}$, if it satisfies the inequality

$$[x_2, \ldots, x_{j+1}; f] \leq \max\{[x_1, \ldots, x_j; f], [x_3, \ldots, x_{j+2}; f]\},$$

for every system of distinct points $x_1 < \cdots < x_{j+2}$ in $[a, b]$. Here

$$[x_1, \ldots, x_j; f] = \sum_{k=1}^{j} \frac{f(x_k)}{u_k(x_k)}$$

$\left(\text{with } u_k(x) = \frac{\Pi_{i=1}^j (x - x_i)}{x - x_k}\right)$ denotes the divided difference of f on the points $x_1, \ldots, x_j$, and $j = 1, 2, \ldots$. Note that for $j = 1$ we obtain again the usual quasi-convexity.

(v) Let $f, u \in C[a, b]$, $u(x) > 0$, for all $x \in [a, b]$. We say that f is u-monotone if $u(x_1) f(x_2) - u(x_2) f(x_1) \geq 0$, for all $a \leq x_1 < x_2 \leq b$.

(vi) For $(x_k)_{k=0}^n$, $0 \leq x_0 < x_1 < \cdots < x_n \leq 1$, let us denote by $S_{[0,1]}[f; (x_k)_k]$ the number of changes of sign in the finite sequence $f(x_0), f(x_1), \ldots, f(x_n)$, where zeros are disregarded. Also, define the number of changes of sign for f on $[0, 1]$ by $S_{[0,1]}[f] = \sup\{S_{[0,1]}[f; (x_k)_k]; (x_k)_{k=0}^n, n \in \mathbb{N}\}$. One says that the linear operator $L : C[0, 1] \to C[0, 1]$ is strongly variation-diminishing on $[0, 1]$, if $S_{[0,1]}[L(f)] \leq S_{[0,1]}[f]$, for all $f \in C[0, 1]$.

Remarks. (1) The concept of j-quasiconvexity belongs to E. Popoviciu (see, e.g., [311]) and the concept of u-monotonicity was introduced by Kocić–Lacković [195].

(2) The j-convexity introduced in Definition 1.1.1 (i) is sometimes called Jensen convexity of order j. A slightly more general concept of convexity, called Popoviciu convexity of order j, was introduced by Popoviciu [315] (in a slightly different denomination), as follows: one says that $f : [a, b] \to \mathbb{R}$ is Popoviciu convex of order j, if for all systems of distinct points (not necessarily equidistant) $a \le x_0 < \cdots < x_j \le b$, we have $[x_0, \ldots, x_j; f] \ge 0$. But, according to a result stated without proof by Popoviciu [318] in 1959, and completely proved in 1997 in, e.g., Ivan–Raşa [184], if f is continuous on $[a, b]$, then for any system of distinct points $a \le x_0 < \cdots < x_j \le b$, there are the points $c, c + h, \ldots, c + jh \in [a, b], h \ge 0$, such that $[x_0, \ldots, x_j; f] = \frac{1}{j!h^j} \Delta_h^j f(c)$.

This immediately implies that for continuous functions, the Jensen and Popoviciu convexities coincide and because in approximation, most of the time the functions considered are at least continuous, in those cases we will simply refer to j-convexity.

(3) The concept of an α-star-convex function $\alpha \in [0, 1]$ is an intermediate concept between the concept of usual convex and that of starshaped function. Indeed, in Definition 1.1.1 (ii), for $\alpha = 1$ we get the concept of usual convex function, while for $\alpha = 0$ we get the concept of starshaped function.

It is worthwhile to point out here the following main properties of an α-star-convex function $f : [0, 1] \to \mathbb{R}$, with $\alpha \in (0, 1]$ (see Mocanu–Şerb–Toader [274]): f is starshaped on $[0, 1]$ (for $f(0) \le 0$), continuous on $(0, \alpha)$, bounded on $[0, 1]$, and Lipschitz in each compact subinterval of $(0, \alpha)$.

Also, we need the following.

Definition 1.1.2. (i) (see e.g. DeVore–Lorentz [91], p. 44) The modulus of smoothness of $f \in L^p[-1, 1]$, $0 < p \le +\infty$, denoted by $\omega_k(f, t)_p$, $k \in \{0, 1, \ldots, \}$ is defined by $\omega_0(f, t)_p := \|f\|_{L^p[a,b]} := \|f\|_p$ and for $k \ge 1$ by

$$\omega_k(f, t)_p = \sup_{0 \le h \le t} \{\|\overline{\Delta}_h^k f(\cdot)\|_p\},$$

where $\overline{\Delta}_h^k f(x) = \Delta_h^k f(x)$ if $x, x + kh \in [-1, 1]$, $\overline{\Delta}_h^k f(x) = 0$; otherwise, $\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + ih)$. Here $L^\infty[-1, 1] = C[-1, 1]$, the space of all continuous functions on $[-1, 1]$.

(ii) (see Ditzian–Totik [98]) Set $\varphi(x) := \sqrt{1 - x^2}$ and define the kth symmetric difference

$$\Delta_{h\varphi}^k f(x) := \begin{cases} \sum\limits_{i=0}^{k} (-1)^i \binom{k}{i} f(x + (i - \frac{k}{2}) h\varphi(x)), & x \pm \frac{k}{2} h\varphi(x) \in [-1, 1], \\ 0 & \text{otherwise,} \end{cases}$$

where $\Delta^0_{h\varphi} f(x) := f(x)$. Then the Ditzian–Totik modulus of smoothness of order k is given by

$$\omega_k^\varphi(f,t)_p := \sup_{0<h\le t} \|\Delta^k_{h\varphi} f\|_p.$$

(iii) (Sendov–Popov [345]) The kth averaged modulus of smoothness (called τ-modulus too) defined for a measurable bounded real function defined on $[a,b]$ is given by

$$\tau_k(f,t,[a,b])_p = \|\omega_k(f,\cdot,t)\|_{L^p[a,b]},$$

where $1 \le p \le \infty$, Δ^k_h is the kth symmetric difference from the above point (ii), and

$$\omega_k(f,x,t) = \sup\{|\Delta^k_h|; y \pm mh/2 \in [x - mt/2, x + mt/2] \cap [a,b]\}.$$

Remark. For $p = \infty$ one can modify these moduli by taking into account not only the position of x in the interval when setting $\Delta^k_{h\varphi} f$, but also how far the endpoints of the interval $[x - \frac{k}{2}h\varphi(x), x + \frac{k}{2}h\varphi(x)]$ are from the endpoints of $[-1,1]$. Thus, one can introduce the following.

Definition 1.1.3. (Shevchuk [349]) Let us define

$$\varphi_\delta(x) := \sqrt{(1 - x - \frac{\delta}{2}\varphi(x))(1 + x - \frac{\delta}{2}\varphi(x))}, \qquad x \pm \frac{\delta}{2}\varphi(x) \in [-1,1],$$

and by C^r_φ the set of functions $f \in C^r(-1,1) \cap C[-1,1]$, such that $\lim_{x\to\pm 1} \varphi^r(x) f^{(r)}(x) = 0$.

The modified Ditzian–Totik modulus of smoothness of order (k,r) is given by

$$\omega^\varphi_{k,r}(f^{(r)},t) := \sup_{0\le h\le t} \sup_x |\varphi^r_{kh}(x) \Delta^k_{h\varphi(x)} f^{(r)}(x)|, \quad t \ge 0,$$

where $\Delta^k_h f(x)$ denotes the kth symmetric difference and the inner supremum is taken over all x such that

$$x \pm \frac{k}{2} h\varphi(x) \in (-1,1).$$

Remarks. (1) For $k = 0$ we have

$$\omega^\varphi_{0,r}(f^{(r)},t) = \|\varphi^r f^{(r)}\|_\infty,$$

while for $r = 0$ we have

$$\omega^\varphi_{k,0}(f^{(0)},t) := \omega^\varphi_k(f,t).$$

The above condition guarantees that for $k \geq 1$, it follows that $\omega_{k,r}^{\varphi}(f^{(r)}, t) \to 0$, as $t \to 0$. Also, if $f \in C_{\varphi}^{r}$ and $0 \leq m < r$, then

$$\omega_{k+r-m,m}^{\varphi}(f^{(m)}, t) \leq C(k,r)t^{r-m}\omega_{k,r}^{\varphi}(f^{(r)}, t), \quad t \geq 0.$$

Conversely, if $f \in C[-1,1],\ m < \alpha < k$, and $\omega_{k}^{\varphi}(f,t) \leq t^{\alpha}$, then $f \in C_{\varphi}^{m}$ and

$$\omega_{k-m,m}^{\varphi}(f^{(m)}, t) \leq C(\alpha, k)t^{\alpha-m}, \quad t \geq 0.$$

(2) If $f \in C_{\varphi}^{m}$ and $\omega_{r-m,m}^{\varphi}(f^{(m)}, t) \leq t^{r-m}$, then

$$\|\varphi^{r} f^{(r)}\|_{\infty} \leq C(r).$$

If we denote the class of all functions satisfying this last inequality by $\mathbb{B}^{r}$, then the converse is valid too, that is, if $f \in \mathbb{B}^{r}$ and $0 \leq m < r$, then $f \in C_{\varphi}^{m}$ and

$$\omega_{r-m,m}^{\varphi}(f^{(m)}, t) \leq C(r)t^{r-m}\|\varphi^{r} f^{(r)}\|_{\infty}, \quad t \geq 0.$$

1.2 Shape-Preserving Interpolation by Polynomials

The existence of interpolating polynomials that are monotone with the interpolated data was established by Wolibner [399] and independently by Kammerer [189] and Young [404], as follows.

Theorem 1.2.1. (see Wolibner [399], Young [404], Kammerer [189]) *Let $(x_i, y_i), i = 1, \ldots, n$ be a set of data such that $x_1 < x_2 < \cdots < x_n$ and $y_i \neq y_{i+1}, i = 1, \ldots, n-1$, then there exists an algebraic polynomial p with the following properties:*

$$p(x_i) = y_i, i = 1, \ldots, n,\ \operatorname{sgn}[p'(x)] = \operatorname{sgn}[\Delta y_i],\ x \in [x_i, x_{i+1}],\ i = 1, \ldots, n-1,$$

where $\Delta y_i = y_{i+1} - y_i$.

Proof. We follow here the ideas in the proof of Wolibner [399]. Denote by $\phi(x)$ the continuous piecewise linear function defined on $[x_1, x_n]$ and passing through all the points (x_k, y_k). It is evident that we can define a twice differentiable function $f : [x_1, x_n] \to \mathbb{R}$ such that $f(x_k) = y_k, k = 1, \ldots, n$, f is comonotone with ϕ, (i.e., $f(x)$ is of the same monotonicity with $\phi(x)$ on each subinterval $[x_k, x_{k+1}]$), the monotonicity is given by the sign of the difference $f(x_{k+1}) - f(x_k)$), and, in addition f is strictly monotonic on each subinterval $[x_k, x_{k+1}]$.

It follows that f' can have only simple zeros. Denote by $c_j,\ j = 1, \ldots, m$, $m \leq n$, the x_j that are simple zeros. Then $F(x) = \frac{f'(x)}{\Pi_{k=1}^{m}(x - x_k)}$ cannot be zero on $[x_1, x_n]$, i.e., by the continuity of $F(x)$, we get that $F(x) > 0, \forall x \in [x_1, x_n]$ or $F(x) < 0, \forall x \in [x_1, x_n]$. In both cases, for any positive $\varepsilon > 0$, there exists an approximation polynomial P attached to F such that $\|F - P\|_{\infty} < \varepsilon$ and

P is strictly positive or strictly negative on $[x_1, x_n]$, as is F. Here $\|\cdot\|_\infty$ denotes the uniform norm on $[x_1, x_n]$.

Defining $Q(x) = f(x_1) + \int_{x_1}^x P(t)\Pi_{k=1}^m (t - x_k)dt$, it is easily seen that $Q(x)$ imitates the monotonicity of f on each subinterval $[x_k, x_{k+1}]$. Also, we get

$$\begin{aligned}|Q(x) - f(x)| &= \left| \int_{x_1}^x P(t)\Pi_{k=1}^m (t - x_k)dt - [f(x) - f(x_1)] \right| \\ &= \left| \int_{x_1}^x P(t)\Pi_{k=1}^m (t - x_k)dt - \int_{x_1}^x f'(t)dt \right| \\ &\le \varepsilon \int_{x_1}^{x_n} |\Pi_{k=1}^m (t - x_k)|dt \le (x_n - x_1)^{m+1}\varepsilon,\end{aligned}$$

for all $x \in [x_1, x_n]$. So for sufficiently small ε, we have that the $Q(x_k)$ are sufficiently close to $y_k = f(x_k),\ k = 1, \dots, n$.

Now, for any $\varepsilon > 0$ and $s = 2, 3, \dots, n$, choose $y_{k-1,\varepsilon}^{(s)} \ne y_{k,\varepsilon}^{(s)},\ k = 1, \dots, n$, such that $|y_{k,\varepsilon}^{(s)}| < \varepsilon,\ k = 1, \dots, s-1, |y_{k,\varepsilon}^{(s)} - 1| < \varepsilon, k = s, \dots, n$, the points $x_1 < \cdots < x_n$ remaining the same. Also, the corresponding linear piecewise function passing through all the points $(x_k, y_{k,\varepsilon}^{(s)})$ is denoted by $\phi^{(s)}(x)$.

According to the above reasonings there exist the polynomials $Q_\varepsilon^{(s)}(x),\ s = 2, \dots, n$, such that they are comonotone with $\phi^{(s)}(x)$ and satisfy

$$|Q_\varepsilon^{(s)}(x_k)| < \varepsilon,\ k = 1, \dots, s-1,$$

and

$$|Q_\varepsilon^{(s)}(x_k) - 1| < \varepsilon,\ k = s, \dots, n.$$

Also, by convention define $Q_\varepsilon^{(1)}(x) = 1$.

Denote by A_ε the value of the determinant $Q_\varepsilon^{(s)}(x_k),\ k, s = 1, \dots, n$, and by $B_\varepsilon^{(s)}$ the value of the determinant obtained from the above one by replacing the sth column with $y_{k,\varepsilon}^{(s)},\ s = 1, \dots, n$. Obviously, we have $\lim_{\varepsilon\to 0} A_\varepsilon = 1$ and $\lim_{\varepsilon\to 0} B_\varepsilon^{(s)} = y_s - y_{s-1}, s = 2, \dots, n$. Therefore, for an ε_0 sufficiently small, we have $A_{\varepsilon_0} > 0$ and $\mathrm{sign}(B_{\varepsilon_0}^{(s)}) = \mathrm{sign}(y_s - y_{s-1}), s = 2, \dots, n$.

Then the polynomial $W(x) = \sum_{s=1}^n \frac{B_{\varepsilon_0}^{(s)}}{A_{\varepsilon_0}} Q_{\varepsilon_0}^{(s)}(x)$ will satisfy the conditions in the statement. □

Remarks. (1) For generalizations of Wolibner's result see, e.g., Ivan [183].

(2) A direct consequence of the above theorem is the following result in Deutch–Morris [80], called SAIN (i.e., simultaneous approximation and interpolation-preserving norm)-type result: if $f \in C[a, b]$ and $x_0 < \cdots < x_n$ are distinct points in $[a, b]$, then for any $\varepsilon > 0$, there exists a polynomial p such that

$$p(x_i) = f(x_i), i = 0, \dots, n, \|f - p\|_\infty < \varepsilon, \|p\|_\infty = \|f\|_\infty$$

(here $\|\cdot\|_\infty$ denotes the uniform norm on $C[a, b]$). In some particular cases, this result can also be considered to belong to the topic of approximation and interpolation by polynomials preserving positivity or positive bounds. Indeed, suppose $0 < f(x) \leq \|f\|_\infty$, for all $x \in [a, b]$ (obviously, the second inequality is always valid). From the continuity of f, there exists $c > 0$ such that $f(x) \geq c > 0$, for all $x \in [a, b]$, and therefore for any sufficiently small ε (more exactly for $0 < \varepsilon < c$), the approximating and interpolating polynomial p also satisfies $0 < p(x) \leq \|f\|_\infty$, for all $x \in [a, b]$.

(3) The Wolibner's theorem does not provide any information about the degree of the polynomial p. If we denote by s the smallest degree of p that still satisfies Theorem 1.2.1, then the first result concerning s was obtained by Rubinstein [330], but only for the particular case $n = 2$ and $y_0 < y_1 < y_2$. In Nikolcheva [285], for equidistant nodes in $[0, 1]$ and for the hypothesis $\Delta y_i \geq cm^\alpha$, one obtains the best estimate, $s = O(\alpha \cdot \log(n))$. Similar results were obtained in Passow–Raymon [300] and Passow [299].

Another direction of research concerning the shape–preserving interpolation by polynomials was discovered by T. Popoviciu in a series of papers published between 1960 and 1962, see [312], [313], [314], and can be described as follows. First let us consider the following simple definition.

Definition 1.2.2. Let $f \in C[a, b]$ and $a \leq x_1 < x_2 < \cdots < x_n \leq b$ be fixed nodes. A linear operator $U : C[a, b] \to C[a, b]$ is said to be of interpolation type (on the nodes $x_i,\ i = 1, \ldots, n$) if for any $f \in C[a, b]$ we have

$$U(f)(x_i) = f(x_i),\ \forall\ i = 1, \ldots, n.$$

Remark. Important particular cases of U are of the form

$$U_n(f)(x) = \sum_{k=1}^{n} f(x_k)P_k(x), \quad n \in \mathbf{N},$$

where $P_k \in C[a, b]$ satisfy $P_k(x_i) = 0$ if $k \neq i$ and $P_k(x_i) = 1$ if $k = i$, and contain the classical Lagrange interpolation polynomials and Hermite–Fejér interpolation polynomials.

Now, if $f \in C[a, b]$ is, for example, monotone (or convex) on $[a, b]$, it is easy to note that because of the interpolation conditions, in general $U(f)$ cannot be monotone (or convex) on $[a, b]$.

However, it is a natural question whether $U(f)$ remains monotone (or convex) on neighborhoods of some points in $[a, b]$. In this sense, we can introduce the following definition.

Definition 1.2.3. Let $U : C[a, b] \to C[a, b]$ be a linear operator of interpolation type on the nodes $a \leq x_1 < \cdots < x_n \leq b$.

Let $y_0 \in (a, b)$. If for any $f \in C[a, b]$, nondecreasing on $[a, b]$, there exists a neighborhood of y_0, $V_f(y_0) = (y_0 - \varepsilon_f, y_0 + \varepsilon_f) \subset [a, b]$, $\varepsilon_f > 0$ (i.e., depending on f) such that $U(f)$ is nondecreasing on $V_f(y_0)$, then y_0 is called a point of weak preservation of partial monotonicity and correspondingly, U is said to have the property of weak preservation of partial monotonicity (about y_0).

If the above neighborhood $V(y_0)$ does not depend on f, then y_0 is called a point of strong preservation.

Similar definitions hold if monotonicity is replaced by, e.g., convexity (of any order).

For example, we present the following two results below concerning the Hermite–Fejér polynomials based on some special Jacobi nodes.

Theorem 1.2.4. (Gal–Szabados [139], Theorem 2.2; see also Gal [123], p. 46, Theorem 2.2.2) *For $n \in \mathbb{N}$, let $H_n(f)(x) = \sum_{i=1}^{n} h_{i,n}(x) f(x_{i,n})$ be the classical Hermite–Fejér polynomial based on the roots $-1 < x_{n,n} < x_{n-1,n} < \cdots < x_{1,n} < 1$ of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, where $\alpha, \beta \in (-1, 0]$ and*

$$h_{i,n}(x) = l_i^2(x)\left[1 - \frac{l''(x_{i,n})}{l'(x_{i,n})}(x - x_{i,n})\right],$$

$$l_i(x) = l(x)/[(x - x_{i,n})l'(x_{i,n})], \quad l(x) = \prod_{i=1}^{n}(x - x_{i,n}).$$

If $f : [-1, 1] \to \mathbb{R}$ is monotone on $[-1, 1]$, then for any root ξ of the polynomial $l'(x)$, there is a constant $c > 0$ (independent of n and of f) such that $H_n(f)(x)$ is of the same monotonicity with f in $\left(\xi - \frac{c_\xi}{n^{7+2\gamma}}, \xi + \frac{c_\xi}{n^{7+2\gamma}}\right) \subset (-1, 1)$, where $c_\xi = \frac{c}{(1-\xi^2)^{5/2+\delta}}$, $\gamma = \max\{\alpha, \beta\}$, and

$$\delta = \begin{cases} \alpha, \text{ if } 0 \le \xi < 1, \\ \beta, \text{ if } -1 < \xi \le 0. \end{cases}$$

Proof. Let us denote $H_n(f)(x) = \sum_{i=1}^{n} h_{i,n}(x) f(x_{i,n})$, where

$$h_{i,n}(x) = l_i^2(x)\left[1 - \frac{l''(x_{i,n})}{l'(x_{i,n})}(x - x_{i,n})\right],$$

$$l_i(x) = l(x)/[(x - x_{i,n})l'(x_{i,n})], \quad l(x) = \prod_{i=1}^{n}(x - x_{i,n}).$$

By, e.g., Popoviciu [312] we have

$$h_{i,n}(0) = l^2(0)[2 - (1-\lambda)x_{i,n}^2]/[l'(x_{i,n})^2(1 - x_{i,n}^2)x_{i,n}^3],$$

for all $i = 1, \dots, n$, and

$$H'_n(f)(x) = \sum_{i=1}^{n-1}[Q_i(x)][f(x_{i,n}) - f(x_{i+1,n})],$$

where $Q_i(x) = \sum_{j=1}^{i} h'_{j,n}(x),\ i = 1, \dots, n-1$.

Reasoning as in the proof of Lemma 3 in Popoviciu [314], we get

$$Q_i(\xi) > \min\{h'_{1,n}(\xi), -h'_{n,n}(\xi)\} > 0, \text{ for all } i = 1, \dots, n-1.$$

Let $a_n, b_n \in (0,1),\ a_n, b_n \searrow 0$ (when $n \to +\infty$) be such that $|h'_{1,n}(\xi)| \geq c_1 a_n,\ |h'_{n,n}(\xi)| \geq c_2 b_n$, and $s_n = \min\{a_n, b_n\}$.

It easily follows that $Q_i(\xi) \geq c_3 s_n, i = 1, \ldots, n-1$. By Szegö [383], Theorem 14.5, we have

$$\sum_{j=1}^{n} h_{j,n}(x) = 1,\ \forall\, x \in [-1,1],$$

where $h_{j,n}(x) \geq 0,\ \forall\, x \in [-1,1],\ j = 1, \ldots, n$.

Applying the Bernstein's inequality twice we obtain

$$Q_i(\xi) \leq c_1 |d_i - \xi| n^2/(1-\xi^2), \quad i = 1, \ldots, n-1,$$

where d_i is the nearest root of $Q_i(x)$ to ξ, and therefore

$$\max_{|x-\xi| \leq a_\xi \frac{s_n}{n^2}} Q_i(x) > 0, \quad i = 1, \ldots, n-1$$

with $a_\xi = c_2(1-\xi^2)$.

It remains to find a (lower) estimate for s_n. First we have

$$|P_n^{(\alpha,\beta)}(\xi)| \geq \frac{c_3 n^{-1/2}}{(1-\xi)^{\delta/2+1/4}},$$

(see Theorem 8.21.8 in Szegö [383]).

By Popoviciu [314], p. 79, relation (27),

$$h'_{1,n}(\xi) = \frac{l^2(\xi)}{(x_{1,n}-\xi)^3 [l' x_{1,n})]^2} \left[2 + (x_{1,n} - \xi) \frac{l''(x_{1,n})}{l'(x_{1,n})} \right] > 0,$$
$$h'_{n,n}(\xi) = \frac{l^2(\xi)}{(x_{n,n}-\xi)^3 [l' x_{n,n})]^2} \left[2 + (x_{n,n} - \xi) \frac{l''(x_{n,n})}{l'(x_{n,n})} \right] < 0.$$

By Szegö [383], Theorem 14.5, $2 + (x_{i,n} - \xi) \dfrac{l''(x_{i,n})}{l'(x_{i,n})} \geq 1$ and by Szegö [383], (7.32.11),

$$h'_{1,n}(\xi) \geq \frac{l^2(\xi)}{(x_{1,n}-\xi)^3 [l'(x_{1,n})]^2} = \frac{[P_n^{(\alpha,\beta)}(\xi)]^2}{(x_{1,n}-\xi)^3 [P_n^{(\alpha,\beta)'}(x_{1,n})]^2}$$
$$\geq \frac{c_4 [P_n^{(\alpha,\beta)}(\xi)]^2}{n^{2q}(1-\xi)^3},$$

(where $q = \max\{2+\alpha, 2+\beta\}$).

Also,

$$-h'_{n,n}(\xi) = |h'_{n,n}(\xi)| \geq \frac{c_5 [P_n^{(\alpha,\beta)}(\xi)]^2}{n^{2q}(1+\xi)^3}.$$

Thus we obtain

$$Q_i(\xi) \geq \frac{c_8}{n^{5+2\gamma}(1-\xi^2)^{7/2+\delta}}, \quad i = 1, \dots, n-1.$$

Finally, taking $s_n = \dfrac{c_8}{n^{5+2\gamma}(1-\xi^2)^{7/2+\delta}}$ we easily obtain the theorem. □

For $n \geq 3$ odd, let $H_n(f)(x)$ be the Hermite–Fejér interpolation polynomial based on the roots $x_{i,n} \in (-1,1),\ i = 1, \dots, n$, of λ-ultraspherical polynomials of degree $n,\ \lambda > -1,\ \lambda \neq 0$. Also, consider the Côtes–Christoffel numbers of the Gauss–Jacobi quadrature given by

$$\lambda_{i,n} := 2^{2-\lambda}\pi \left[\Gamma\left(\frac{\lambda}{2}\right)\right]^{-2} \frac{\Gamma(n+\lambda)}{\Gamma(n+1)}(1-x_{i,n}^2)^{-1}[P_n^{(\lambda)'}(x_{i,n})]^{-2}, \quad i = 1, \dots, n,$$

and define

$$\Delta_h^2 f(0) = f(h) - 2f(0) + f(-h).$$

We also have the following result:

Theorem 1.2.5. (Gal–Szabados [139], Theorem 2.3; see also Gal [123], p. 49, Theorem 2.2.3) *Let $f \in C[-1,1]$ satisfy*

$$\sum_{i=1}^{n} [\lambda_{i,n} \Delta_{x_{i,n}}^2 f(0)]/x_{i,n}^2 > 0$$

(if f is strictly convex on $[-1,1]$ then obviously it satisfies this condition). Then $H_n(f)(x)$ is strictly convex in $[-|d_n|, |d_n|]$, with

$$|d_n| \geq \frac{c(\lambda) \sum_{i=1}^{\frac{n-1}{2}} [\lambda_{i,n} \Delta_{x_{i,n}}^2 f(0)]/x_{i,n}^2}{n^2 \left[\omega_1\left(f; \frac{1}{n}\right) + \|f - H_n(f)\|\right]_I},$$

where $c(\lambda) > 0$ is independent of f and n, $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\omega_1\left(f; \frac{1}{n}\right)_{\left[-\frac{1}{2}, \frac{1}{2}\right]}$ is the first-order modulus of continuity on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, and $\|\cdot\|_{\left[-\frac{1}{2}, \frac{1}{2}\right]}$ is the uniform norm on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Proof. Denote $H_n(f)(x) = \sum_{i=1}^{n} h_{i,n}(x) f(x_{i,n})$, where

$$h_{i,n}''(x) = -4\frac{l''(x_{i,n})}{l'(x_{i,n})} l_i(x) l_i'(x) + 2[(l_i'(x))^2 + l_i(x) l_i''(x)]$$
$$\left[1 - \frac{l''(x_{i,n})}{l'(x_{i,n})}(x - x_{i,n})\right].$$

But $l_i(0) = 0$ and $l_i'(0) = -\dfrac{l'(0)}{x_{i,n}l'(x_{i,n})}$, for $i \neq (n+1)/2$ and

$$1 + x_{i,n}\frac{l''(x_{i,n})}{l'(x_{i,n})} = \frac{1+\lambda x_{i,n}^2}{1-x_{i,n}^2}, \quad i = 1, \ldots, n \quad \text{(see, e.g., \textit{Popoviciu} [312]).}$$

We obtain

$$h_{i,n}''(0) = \frac{2(l'(0))^2}{(l'(x_{i,n}))^2} \cdot \frac{1}{x_{i,n}^2}\left(\frac{1+\lambda x_{i,n}^2}{1-x_{i,n}^2}\right) > 0, \ \forall\ i \neq (n+1)/2.$$

Also, because $x_{i,n} = -x_{n+1-i,n}$, $i = 1, \ldots, n$, $l'(x_{i,n}) = l'(x_{n+1-i,n})$ (since n is odd) we easily get

$$h_{i,n}''(0) = h_{n+1-i,n}''(0).$$

But $\lambda_{i,n} = \dfrac{c_1\lambda\Gamma(n+\lambda)}{\Gamma(n+1)} \cdot \dfrac{1}{(l'(x_{i,n}))^2} \cdot \dfrac{1}{1-x_{i,n}^2}$ and $(l'(0))^2 \sim n^\lambda$, which together with the above inequality implies

$$h_{i,n}''(0) \geq c_2\lambda n\lambda_{i,n}/x_{i,n}^2, \text{ for all } i \neq (n+1)/2.$$

Therefore

$$H_n''(f)(0) = \sum_{i=1}^{(n-1)/(2)} h_{i,n}''(0)\Delta_{x_{i,n}}^2 f(0) \geq c_3\lambda n\sum_{i=1}^{n}\lambda_{i,n}\Delta_{x_{i,n}}^2 f(0)/x_{i,n}^2 > 0.$$

By this last relationship it follows that $H_n(f)$ is strictly convex in a neighborhood of 0. Let d_n be the nearest root of $H_n''(f)$ to 0. We may assume that $|d_n| \leq \dfrac{c}{n}$ (since otherwise there is nothing to prove, the interval of convexity cannot be larger than $\left[-\frac{c}{n}, \frac{c}{n}\right]$). Then by the mean value theorem, Bernstein's inequality and Stechkin's inequality (see, e.g., Szabados–Vértesi [381], p. 284) we get

$$\begin{aligned} H_n''(f)(0) &= |H_n''(f)(0) - H_n''(f)(d_n)| = |d_n| \cdot |H_n'''(f)(y)| \\ &\leq |d_n|c_4n^2\|H_n'(f)\|_J \leq c_5|d_n|n^3\omega_1\left(H_n(f); \frac{1}{n}\right)_I \\ &\leq c_5|d_n|n^3\left[\omega_1\left(f; \frac{1}{n}\right) + \omega_1\left(H_n(f) - f; \frac{1}{n}\right)\right]_I \\ &\leq c_5|d_n|n^3\left[\omega_1\left(f; \frac{1}{n}\right) + \|f - H_n(f)\|\right]_I, \end{aligned}$$

where $J = \left[-\frac{1}{4}, \frac{1}{4}\right]$, $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Combining the last inequality with the previous inequality satisfied by $H''_n(f)(0)$, the proof of the theorem is immediate. □

Remark. All the details concerning this direction of research can be found in Chapter 2 of the recent monograph Gal [123], where a deep and extensive study concerning shape-preserving interpolation by classical univariate interpolation polynomials (of Lagrange, Grünwald, or Hermite–Fejér type) is made.

For the error estimate in shape-preserving interpolation, we mention here the following four results.

The first two results show the existence of such polynomials with good approximation properties and can be stated as follows.

Theorem 1.2.6. (Ford–Roulier [120]) *Let $p \in \mathbb{N}$, $1 \le r_1 < r_2 < \cdots < r_s \le p$ with $r_i, i = 1, \ldots, s$, natural numbers, $\varepsilon_j = \pm 1, j = 1, \ldots, s$, and $a \le x_0 < \cdots x_m \le b$ interpolation nodes. For any $f \in C^p[a, b]$ satisfying*

$$\varepsilon_i f^{(r_i)}(x) > 0, \forall x \in [a, b], \quad i = 1, \ldots, s,$$

there exists a sequence of polynomials $(P_n(x))_n$, degree$(P_n) \le n$, such that for sufficiently large n we have

$$\varepsilon_i P_n^{(r_i)}(x) > 0, \forall x \in [a, b], \quad i = 1, \ldots, s, \text{ with } P_n(x_j) = f(x_j), \quad j = 0, \ldots, m,$$

and the estimate

$$\|f - P_n\|_\infty \le C n^{-p} \omega_1(f^{(p)}; 1/n)_\infty$$

holds, where $C > 0$ is independent of f and n. Here $\|\cdot\|_\infty$ denotes the uniform norm on $C[a, b]$.

Proof. Let us sketch the proof. According to a result in the doctoral thesis of Roulier [325], f can be extended to a function $F \in C^p[a-1, b+1]$ such that $\omega_1(F^{(p)}; h)_\infty \le \omega_1(f^{(p)}; h)_\infty$, for all $h \in [0, b-a]$. Denote by Q_n the polynomial of best approximation of degree $\le n$ attached to F on $[a-1, b+1]$. Jackson's theorem implies

$$\|Q_n - F\|_{C[a-1,b+1]} \le C n^{-p} \omega_1(F^{(p)}; 1/n)_\infty,$$

where ω_1 is the uniform modulus of continuity on $[a-1, b+1]$.

Now let L_m be the Lagrange's interpolation polynomial of degree $\le m$ satisfying $L_m(x_i) = \delta_i = F(x_i) - Q_n(x_i), i = 0, \ldots, m$.

Since $|\delta_i| \le C n^{-p} \omega_1(F^{(p)}; 1/n)_\infty$, for all $i = 0, \ldots, m$, it is easy to derive that $|L_m(x)| \le C_1 n^{-p} \omega_1(F^{(p)}; 1/n)_\infty$ for all $x \in [a-1, b+1]$, where C_1 depends only on m and the points x_i, $i = 0, \ldots, m$.

Setting $P_n(x) = Q_n(x) + L_m(x)$, it is easy to see that $P_n(x_i) = F(x_i) = f(x_i)$, $i = 0, \ldots, m$, and $\|P_n - f\|_{C[a-1,b+1]} \le C_2 n^{-p} \omega_1(f^{(p)}; 1/n)_\infty$, by the above mentioned result in Roulier [325] (here $\omega_1(f^{(p)}, 1/n)_\infty$ denotes the uniform modulus of continuity on $[a, b]$).

Also, according to another result in Roulier [325], $\|P_n - f\|_{C[a-1,b+1]} \leq C_2 n^{-p}\omega_1(f^{(p)}; 1/n)_\infty$ implies $\|P_n^{(i)} - f^{(i)}\|_{C[a,b]} \leq C_3 n^{i-p}\omega_1(f^{(p)}; 1/n)_\infty$, $i = 0, \ldots, p$.

This means that $P_n^{(r_i)} \to f^{(r_i)}$, uniformly on $[a, b]$, for all $i = 1, \ldots, s$, which because of the strict inequalities

$$\varepsilon_i f^{(r_i)}(x) > 0, \forall x \in [a, b],\ i = 1, \ldots, s,$$

immediately implies the conclusion in the statement. □

Remark. A similar result to that of Theorem 1.2.6, but in the more general setting of nondifferentiable functions, has been considered by Szabados [379], who obtained estimates in terms of $\omega_1(f; \frac{\log n}{n})_\infty$.

Theorem 1.2.6 can be slightly refined by combining it with approximation by monotone sequences of polynomials, as follows. For simplicity, we consider the problem on $[0, 1]$.

Theorem 1.2.7. (Gal [130]) *Let $p \in \mathbb{N}$, $1 \leq r_1 < r_2 < \cdots < r_s \leq p$ with $r_i, i = 1, \ldots, s$ natural numbers, $\varepsilon_j = \pm 1, j = 1, \ldots, s$ and $0 \leq x_1 < \cdots < x_m \leq 1$ interpolation nodes. For any $f \in C^p[0, 1]$ satisfying*

$$\varepsilon_i f^{(r_i)}(x) > 0, \forall x \in [0, 1],\ i = 1, \ldots, s,$$

there exist sequences of polynomials $(P_n(x))_n, (Q_n(x))_n$, degree$(P_n) \leq n$, degree$(Q_n) \leq n$, such that for sufficiently large n, we have

$$\varepsilon_i P_n^{(r_i)}(x) > 0,\ \varepsilon_i Q_n^{(r_i)}(x) > 0,\ \forall x \in [0, 1],\ i = 1, \ldots, s,$$
$$P_n(x_j) = Q_n(x_j) = f(x_j),\ j = 0, \ldots, m,$$

the estimate

$$\|P_n - Q_n\|_\infty \leq C n^{-p}\omega_1(f^{(p)}; 1/n)_\infty$$

holds, where $C > 0$ is independent of f and n, and in addition,

$$Q_n(x) \leq Q_{n+1}(x) \leq f(x) \leq P_{n+1}(x) \leq P_n(x),\ \forall x \in [0, 1],\ n \in \mathbb{N}.$$

Proof. From the proofs of the Theorem and Corollary 1 in Gal–Szabados [140], we distinguish two steps.

Step 1. We start with the polynomial sequence $(p_k)_k$, degree$(p_k) \leq k$, satisfying Theorem 1.2.6, i.e., for sufficiently large n we have

$$\varepsilon_i p_k^{(r_i)}(x) > 0,\ \forall x \in [0, 1],\ i = 1, \ldots, s,\ \text{where } p_k(x_j) = f(x_j),\ j = 0, \ldots, m,$$

and the estimate

$$\|f - p_k\|_\infty \leq C k^{-p}\omega_1(f^{(p)}; 1/k)_\infty$$

holds.

Step 2. With the aid of $(p_k)_k$, one construct the polynomials P_n and Q_n satisfying the relationships (5) and (8), respectively, in Gal–Szabados [140] (where P_n and Q_n are defined as special arithmetic means of p_k), replacing there $E_k(f)_\infty$ by the expression $Ck^{-p}\omega_1(f^{(k)}; 1/k)_\infty$.

By the mentioned proof, for all $n \geq 4$ we get

$$Q_n(x) \leq Q_{n+1}(x) \leq f(x) \leq P_{n+1}(x) \leq P_n(x), \ \forall x \in [0,1], \ n \in \mathbb{N},$$

and

$$\|P_n - Q_n\|_\infty \leq Cn^{-p}\omega_1(f^{(p)}; 1/n)_\infty.$$

Since the polynomials P_n and Q_n are arithmetic means of p_k, it is immediate that

$$\varepsilon_i P_n^{(r_i)}(x) > 0, \quad \varepsilon_i Q_n^{(r_i)}(x) > 0, \quad \forall x \in [0,1], i = 1, \ldots, s.$$

Now, in order to get the interpolation conditions too, let us redefine Q_n and P_n by $Q_n := Q_n + L_m^{(1)}$, $P_n := P_n + L_m^{(2)}$, where $L_m^{(1)}$ and $L_m^{(2)}$ are the Lagrange polynomials of degrees $\leq m$ satisfying the conditions $L_m^{(1)}(x_j) = f(x_j) - Q_n(x_j)$, $L_m^{(2)}(x_j) = f(x_j) - P_n(x_j)$, $j = 0, \ldots, m$.

Reasoning as in the proof of Theorem 1.2.6, for the redefined Q_n and P_n, we get

$$P_n(x_j) = Q_n(x_j) = f(x_j), \ j = 0, \ldots, m$$

and

$$\|Q_n - f\|_\infty \leq Cn^{-p}\omega_1(f^{(p)}; 1/n)_\infty,$$
$$\|P_n - f\|_\infty \leq Cn^{-p}\omega_1(f^{(p)}; 1/n)_\infty,$$

which by $\|Q_n - p_n\|_\infty \leq \|Q_n - f\|_\infty + \|f - P_n\|_\infty$, immediately implies

$$\|P_n - Q_n\|_\infty \leq Cn^{-p}\omega_1(f^{(p)}; 1/n)_\infty.$$

Also, as in the proof of Theorem 1.2.6, we get the uniform convergence of $Q_n^{(r_i)}$ and $P_n^{(r_i)}$ to $f^{(r_i)}$, $i = 0, \ldots, s$, which for sufficiently large n also implies

$$\varepsilon_i P_n^{(r_i)}(x) > 0, \quad \varepsilon_i Q_n^{(r_i)}(x) > 0, \quad \forall x \in [0,1], \quad i = 1, \ldots, s.$$

Obviously, the monotonicity properties of the redefined sequences $(Q_n)_n$ and $(P_n)_n$ (with respect to n) become non-strict, because of interpolation conditions. The theorem is proved. □

Remark. Two recent results in Kopotun [202], Kopotun [203], give the approximation estimates necessarily verified by interpolation j-convex polynomials (interpolating a function which is not necessarily j-convex), in the case that the interpolation nodes are not close to the endpoints. These results remaining valid for j-convex functions too, it is clear that they can be considered to belong to the shape-preserving interpolation topic.

Now, since by the alternating Chebyshev theorem, the best approximation polynomial of degree $\leq n$ interpolates the function on at least $n+1$ points, in this section we also present three results concerning the preservation of j-convexity by the best approximation polynomials.

Theorem 1.2.8. (Roulier [327]) *Let $m \in \mathbb{N}$, $f \in C^{2m-1}[-1,1]$, $1 \leq i_1 < i_2 < \cdots < i_q < m$ be fixed integers and $\varepsilon_j, j = 1, \ldots, q$ be fixed signs. For any $n \in \{0, 1, \ldots, \}$, denote by Q_n the best approximation polynomial of degree $\leq n$ of f on $[-1,1]$. If $\varepsilon_j f^{(i_j)}(x) > 0$, for all $x \in [-1,1]$ and all $j = 1, \ldots, q$ and if $\sum_{k=1}^{+\infty} \frac{1}{k}\omega_1(f^{(2m-1)}; \frac{1}{k})_\infty < +\infty$, then for sufficiently large n, we have $\varepsilon_j Q_n^{(i_j)}(x) > 0$, for all $x \in [-1,1]$ and all $j = 1, \ldots, q$.*

Proof. We sketch here the proof using the ideas in Roulier [327]. In fact, it is based on two lemmas. The first one is well known (see, e.g., G.G. Lorentz's monograph [248]) and can be stated as follows.

Lemma (A). (Lorentz [248], p. 74) *There exist constants $M_p > 0$, $p = 1, 2, \ldots,$ such that if w is any modulus of continuity for which $\sum_{k=1}^{+\infty} \frac{1}{k} w(1/k) < +\infty$ and if for $f \in C[-1,1]$ and polynomials $q_n(x)$ of degree $\leq n$ we have the estimate*

$$|f(x) - q_n(x)| \leq C[\Delta_n(x)]^p w(\Delta_n(x)),$$

then f has continuous derivative $f^{(p)}$ and

$$|f^{(p)}(x) - q_n^{(p)}(x)| \leq M_p \sum_{k \geq [(\Delta_n(x))^{-1}]} \frac{1}{k} w(1/k), \forall x \in [-1,1].$$

Here $\Delta_n(x) = \max\{n^{-1}(1-x^2)^{1/2}, n^{1/2}\}$, $\Delta_0(x) = 1$.

Proof of Lemma A. Because of its importance in approximation theory, let us sketch its proof below. It is easy to see that we can write $f(x) = q_n(x) + \sum_{j=1}^{\infty}(q_{2^j n}(x) - q_{2^{j-1}n}(x))$, where by the hypothesis it follows that

$$|q_{2^j n}(x) - q_{2^{j-1}n}(x)| \leq 2[\Delta_{2^{j-1}n}(x)]^p w(\Delta_{2^{j-1}n}(x)).$$

This implies the uniform convergence of the series (on $[-1,1]$), that is, the differentiated series (of any order) is also uniformly convergent and we get

$$f^{(p)}(x) = q_n^{(p)}(x) + \sum_{j=1}^{\infty}(q_{2^j n}^{(p)}(x) - q_{2^{j-1}n}^{(p)}(x)).$$

Taking into account the elementary inequalities $\frac{1}{4}\Delta_n(y) \leq \Delta_{2n}(y) \leq \frac{1}{2}\Delta_n(y)$, valid for all $y \in [-1,1]$ and applying a well known Markov-type inequality in terms of the modulus of continuity (i.e., $|q_n(x)| \leq [\Delta_n(x)]^r w(\Delta_n(x))$, $|x| \leq 1$, implies $|q_n'(x)| \leq M_r[\Delta_n(x)]^{r-1} w(\Delta_n(x))$, $|x| \leq 1$, for its proof see, e.g., Theorem 3 in Lorentz [248], p. 71) p-times, we obtain

$$|q_{2^j n}^{(p)}(x) - q_{2^{j-1}n}^{(p)}(x)| \leq M_p w(\Delta_{2^j n}(x)).$$