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Volume Editor
Ryszard Wójcicki

D. Mundici

Advanced Łukasiewicz calculus and MV-algebras

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D. Mundici
Department of Mathematics "Ulisse Dini"
University of Florence
Viale Morgagni 67 A
50134 Florence
Italy
e-mail: mundici@math.unifi.it

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To Cecilia

Preface

This book is designed as a text for a second course in infinite-valued Łukasiewicz logic and its algebras, Chang's MV-algebras. It is also intended as a source of reference for the more advanced readers, and is a continuation of the monograph by Cignoli et al., "*Algebraic Foundations of Many-Valued Reasoning*," which may be used as a suitable text for a first course. I give complete versions of a compact body of recent results and techniques, virtually proving everything that is used throughout. So if I have accomplished my purpose, this book should be usable for individual study.

Modern Łukasiewicz logic and MV-algebra theory draw on three principal sources: polyhedral topology, functional analysis, and lattice-ordered abelian groups (ℓ-groups henceforth). This is so because

> Every free MV-algebra is an algebra of [0,1]-valued piecewise linear functions f over some unit cube, each linear piece of f having integer coefficients. Zerosets of these functions are, on the one hand, models of formulas in Łukasiewicz propositional logic $Ł_\infty$, and on the other hand, they are the most general rational polyhedra contained in some cube $[0,1]^n$.

> For any MV-algebra A, regular Borel probability measures on the maximal spectral space of A correspond to de Finetti's coherent probability assessments on the events represented by A, as an algebra of equivalence classes of formulas in Łukasiewicz logic.

> There is a categorical equivalence Γ between MV-algebras A and unital ℓ-groups $(G, 1)$, those ℓ-groups having a distinguished order unit.

Just as the $\mathbb{Z}$-module structure of $(G, 1)$ is missing in the MV-algebra $A = \Gamma(G, 1)$, several fundamental notions and constructs available in the framework of MV-algebras and Łukasiewicz logic hardly make any sense for unital ℓ-groups, despite the latter are categorically equivalent to MV-algebras. Thus, the equational definability of the class of MV-algebras gives us a way of introducing free and finitely presented objects—while the class of unital ℓ-groups is not even definable in first-order logic. Induction on the complexity of Łukasiewicz formulas, combined with their geometric representation as McNaughton functions, is a main tool to explore syntactic and semantic consequence in $Ł_\infty$, and the fundamental logic property of interpolation. Formulas in $Ł_\infty$ denote continuously valued events, just as boolean formulas denote yes-no events; coherent probability assessments on these events yield Rényi conditionals, which would make no sense for unital ℓ-groups. σ-complete MV-algebras provide a natural framework for generalizations of many classical results originally proved for σ-complete boolean algebras, such as the theorem of Loomis–Sikorski and Poincaré's recurrence theorem. Several main techniques and results of probability theory, that Carathéodory reformulated in the language of σ-complete boolean algebras, have nontrivial MV-algebraic generalizations. Bases originate as algebraically invariant counterparts of disjunctive Schauder normal forms in Łukasiewicz logic; an MV-algebra has a basis iff it is finitely presented. Classical first-order logic with identity has a generalization to a Łukasiewicz first-order logic $Ł_{\omega\omega}$ with [0,1]-valued identity. Models of $Ł_{\omega\omega}$ are suitable sets X of unit vectors in a Hilbert space $\mathscr{H}$, and the identity degree of any two vectors $u, v \in X$ is their scalar product; functions and relations on X satisfy suitable continuity properties.

Since this book is devoted to these genuine MV-algebraic and logical topics, its overlap with books on ℓ-groups, with or without unit, is negligible.

Every chapter in this book relies on a combination of classical, as well as of recent mathematical results, well beyond the traditional domain of algebraic logic.

The first prerequisite for a profitable reading is familiarity with the main theorems of Łukasiewicz logic and MV-algebra theory, notably Chang completeness theorem, McNaughton representation of free MV-algebras, Wójcicki's analysis of consequence in Łukasiewicz logic, and the properties of the Γ functor. Secondly, the reader is assumed to have some acquaintance with a few basic facts of polyhedral topology and functional analysis. As is often the case in the study of advanced mathematical topics, detailed knowledge of the proofs of all background results is less important than knowing a place in the literature where one can go and look—if the need ever arises to check a proof.

To help the reader, all background results used in the course of the book are collected in two final Appendices, together with references for their proofs. The notation (B21.50) will refer to entry 21.50 in Appendix B.

This book has grown out of lectures delivered at various universities and summer schools during the last ten years.

I have made much use of conversations and correspondence with many friends and colleagues. I owe a particular debt of gratitude to Ettore Casari, Roberto Cignoli, Janusz Czelakowski, Antonio Di Nola, Sergio Doplicher, Anatolij Dvurečenskij, László Fuchs, Andrew Glass, Marco Grandis, Petr Hájek, Charles Holland, Tomáš Kroupa, Ioana Leuştean, Jorge Martínez, Franco Montagna, Hiroakira Ono, Beloslav Riečan, Constantine Tsinakis, Hans Weber, and Ryszard Wójcicki. From the late Sauro Tulipani I learned that de Finetti's coherence criterion can be applied to events described by Łukasiewicz logic.

I am also grateful to my former students Stefano Aguzzoli, Agata Ciabattoni, Brunella Gerla, and Giovanni Panti.

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Florence, November 2010 D. Mundici

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Notation and Terminology

The symbol $\Rightarrow$ is to be read "implies". The symbol $\Leftrightarrow$ is to be read "iff", which is short for "if, and only if". The symbols $\exists$ and $\forall$ are to be read "there is an" and "for all", respectively.

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, respectively, denote the set of integer, rational and real numbers.

By a *countable* set we mean a set whose cardinality is either finite or equal to the cardinality of the set of integers.

A family $\mathcal{F}$ of subsets of a set X is said to have the *finite intersection property* if for every finite set $F_1, \ldots, F_k$ of members of $\mathcal{F}$ the intersection $F_1 \cap \ldots \cap F_k$ is nonempty.

For any two sets $E \subseteq F$ we let χ_E denote the *characteristic function* of E in F, i.e., the function $\chi_E: F \to \{0, 1\}$ defined by $\chi_E^{-1}(1) = E$. The ambient set F will always be clear from the context.

For every function $f: F \to G$ and $E \subseteq F$ we let $f \restriction E$ denote the restriction of f to E. For any two sets D and V we let V^D be the set of all functions $f: D \to V$. The notation $f: x \mapsto y$ stands for $f(x) = y$. Given functions $f: X \to Y$ and $g: Y \to Z$ we denote by $gf: X \to Z$ the composite function defined by $(gf)(x) = g(f(x))$ for all $x \in X$.

For any topological space Y and subset X of Y, we denote by $\mathrm{cl}(X)$ the closure of X in Y (the latter being always clear from the context). Similarly, $\mathrm{int}(X)$ denotes the interior of X.

Unless otherwise specified, the adjective *linear* is understood in the affine sense.

For each $n = 1, 2, \ldots$ we let $\mathbb{R}^n$ be n-dimensional euclidean space. We further let $e_1, \ldots, e_n$ be the standard basis vectors of $\mathbb{R}^n$, and $\pi_1, \ldots, \pi_n$ the coordinate (= identity = projection) functions restricted to the unit n-cube $[0, 1]^n$.

For any subset S of $\mathbb{R}^n$ we denote by $\mathrm{conv}(S)$ the set of all convex combinations of elements of S. Thus $x \in \mathrm{conv}(S)$ iff there are $x_1, \ldots, x_k \in S$ and real numbers $\lambda_1, \ldots, \lambda_k \geq 0$ such that $\lambda_1 + \cdots + \lambda_k = 1$ and $x = \lambda_1 x_1 + \cdots + \lambda_k x_k$. The set S is said to be *convex* if $S = \mathrm{conv}(S)$.

A hyperplane H is a *supporting hyperplane* of a closed convex set $T \subseteq \mathbb{R}^n$ if $H \cap T \neq \emptyset$ and $T \subseteq H^+$ or $T \subseteq H^-$, where $H^\pm$ are the two closed half-spaces bounded by H. The set $T \cap H$ is said to be a *face* of T. By convention, $\emptyset$ and T are called the *improper* faces of T. All other faces of T are said to be *proper*.

For any subset S of $\mathbb{R}^n$ we denote by aff(S) the *affine hull* of S, i.e., the set of all *affine combinations* in $\mathbb{R}^n$ of elements of S. Thus $x \in \mathrm{aff}(S)$ iff there are $x_1, \ldots, x_k \in S$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\lambda_1 + \cdots + \lambda_k = 1$ and $x = \lambda_1 x_1 + \cdots + \lambda_k x_k$. A set $\{y_1, \ldots, y_m\}$ of points in $\mathbb{R}^n$ is said to be *affinely independent* if none of its elements is an affine combination of the remaining elements.

The *relative interior* $\mathrm{relint}(S)$ of a convex set S in $\mathbb{R}^n$ is the interior of S in the affine hull of S.

As usual, gcd and lcm denote greatest common divisor and least common multiple.

Unless otherwise specified, in every MV-algebra considered in this book the unit and the zero element will be distinct.

We let $\hom(A)$ denote the set of homomorphisms of the MV-algebra A into the MV-algebra [0,1]. For every homomorphism η of A into an MV-algebra B, the *kernel* $\ker(\eta)$ of η, is defined by $\ker(\eta) = \eta^{-1}(0)$.

For each $k = 1, 2, \ldots$, we denote by L_k the $(k+1)$-element Łukasiewicz chain $\{0, 1/k, \ldots, (k-1)/k, 1\}$. This is denoted $Ł_{k+1}$ in [1, p. 8].

Reference

1. Cignoli, R. L. O., D'Ottaviano, I. M. L., Mundici, D. (2000). *Algebraic foundations of many-valued reasoning*. Volume 7 of Trends in Logic. Dordrecht: Kluwer.

Chapter 1
Prologue: de Finetti Coherence Criterion and Łukasiewicz Logic

In this chapter we will see that coherent probability assessments on (not necessarily yes–no) events, such as those given by the measurement of physical observables, are convex combinations of valuations in Łukasiewicz propositional logic $Ł_\infty$. Besides familiarity with [1], the only prerequisite for this chapter is some acquaintance with the very basic properties of convex sets in euclidean space.

1.1 Events, Possible Worlds and de Finetti Coherence Criterion

Just as the measurement of an observable of a physical system in a given state outputs a real number x—and after a suitable normalization, x can be assumed to lie in the unit interval [0,1]—similarly a possible world assigns a (truth-)value $x \in [0, 1]$ to any event. In particular, the value x assigned to a yes–no event X is 1 if X occurs, and 0 otherwise. If X has a continuous spectrum, our expectation "X has a large value" is made precise by the result of the measurement/observation of X. More details will be given in Sect. 1.6 of this chapter.

Stripping away all inessentials, given an integer $n > 0$ and two sets $E = \{X_1, \ldots, X_n\}$ and $W \subseteq [0, 1]^E$, let us imagine two players, Ada and Blaise, waging money on the possible occurrence of the "events" of E in the future "possible worlds" of W. Ada, who is a mathematical bookmaker, proclaims her "betting odd" $\beta(X_i) \in [0, 1]$, and Blaise, the bettor, chooses a "stake" σ_i for each $X_i \in E$. Then Blaise pays Ada $\sigma_i \cdot \beta(X_i)$ euros $(i = 1, \ldots, n)$,with the stipulation that Ada will pay back $\sigma_i \cdot w(X_i)$ euros in the possible world $w \in W$ where the value $w(X_i)$ is made known. Ada is so confident in her "book" β that Blaise is allowed to put down a *negative stake* σ_i, should he rate $\beta(X_i)$ excessive. The result is a "reverse bet": Ada now pays Blaise $|\sigma_i| \cdot \beta(X_i)$ euros, to receive $|\sigma_i| \cdot w(X_i)$ in the possible world w. The total balance of this bet on events $X_1, \ldots, X_n$, with stakes $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$, in the possible world w is

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$$\sum_{i=1}^{n} \sigma_i(\beta(X_i) - w(X_i)),$$

where money transfers are conventionally oriented so that "positive" means Blaise-to-Ada. Ada's book $\beta\colon E \to [0, 1]$ would quickly lead her to financial disaster if in every possible world this total balance is < 0. For, assuming the set W of possible worlds is closed in $[0, 1]^n$, by suitably rescaling his stakes, Blaise might ensure a net profit of at least one zillion euros whatever happens. Adopting the understatements which are so common in contemporary economic theory, we give the following

Definition 1.1 Fix an integer $n > 0$. For any two sets $E = \{X_1, \ldots, X_n\}$ and $W \subseteq [0, 1]^E$, we say that a map $\beta\colon E \to [0, 1]$ is *W-incoherent* if for some $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$ the inequality $\sum_{i=1}^{n} \sigma_i(\beta(X_i) - w(X_i)) < 0$ holds for all $w \in W$. Otherwise, β is *W-coherent*.

1.2 Coherence and Valuations in Łukasiewicz Logic

Theorem 1.4 will establish a first connection between coherent assessments and valuations in Łukasiewicz logic. The theorem will be continued in Theorem 10.7.

In preparation for these results, for each $n = 1, 2, \ldots,$ we let FORM_n denote the set of formulas $\psi(X_1, \ldots, X_n)$ whose variables are contained in the set $\{X_1, \ldots, X_n\}$; ψ is the same as a formula in boolean logic, except that conjunction and disjunction are written as $\odot$ and $\oplus$ instead of $\wedge$ and $\vee$. As is well known, the lattice connectives $\wedge$ and $\vee$ in $Ł_\infty$ are different from the basic connectives $\odot$ and $\oplus$.

We will write $\alpha \to \beta$ as an abbreviation of $\neg\alpha \oplus \beta$. As usual, $\alpha \leftrightarrow \beta$ stands for $(\alpha \to \beta) \odot (\beta \to \alpha)$.

More generally, for any set $\mathcal{X}$ of variables, we denote by $\mathsf{FORM}_{\mathcal{X}}$ the set of formulas whose variables are among those of $\mathcal{X}$. For each formula ϕ we let $\mathrm{var}(\phi)$ be the set of variables occurring in ϕ. For any set $\Phi \subseteq \mathsf{FORM}_{\mathcal{X}}$ we also use the notation $\mathrm{var}(\Phi)$ for $\bigcup\{\mathrm{var}(\psi) \mid \psi \in \Phi\}$.

As usual, when writing formulas we assume that $\neg$ is more binding than $\odot$, and the latter is more binding than $\oplus$.

Definition 1.2 A *valuation* (of FORM_n in $Ł_\infty$) is a function $V\colon \mathsf{FORM}_n \to [0, 1]$ such that

$$V(\neg\phi) = 1 - V(\phi)$$
$$V(\phi \oplus \psi) = V(\phi) \oplus V(\psi) = \min(1, V(\phi) + V(\psi))$$
$$V(\phi \odot \psi) = V(\phi) \odot V(\psi) = \max(0, V(\phi) + V(\psi) - 1).$$

We let VAL_n denote the set of valuations of FORM_n. For each $w = (w_1, \ldots, w_n) \in [0, 1]^n = [0, 1]^{\{X_1,\ldots,X_n\}}$ we let V_w be the only valuation of VAL_n such that $V_w(X_i) = w_i$ for all $i = 1, \ldots, n$. Thus, $w = V_w \restriction \{X_1, \ldots, X_n\}$.

More generally, for any set $\mathcal{X}$ of variables, $\mathsf{VAL}_{\mathcal{X}}$ denotes the set of valuations $V\colon \mathsf{FORM}_{\mathcal{X}} \to [0,1]$. A formula ϕ is a *tautology* if $V(\phi) = 1$ for all valuations $V \in \mathsf{VAL}_{\mathrm{var}(\phi)}$. To signify that ψ is a tautology we write $\vdash \psi$.

Definition 1.3 For every $n = 1, 2, \ldots$ and nonempty set $Y \subseteq [0,1]^n$ we define

$$\mathrm{Th}\, Y = \{\psi \in \mathsf{FORM}_n \mid V_w(\psi) = 1 \quad \text{for all} \quad w \in Y\}. \tag{1.1}$$

For any set $\Phi \subseteq \mathsf{FORM}_{\mathcal{X}}$ and $V \in \mathsf{VAL}_{\mathcal{X}}$ we say that V *satisfies* Φ if $V(\psi) = 1$ for all $\psi \in \Phi$. If there is a valuation V satisfying Φ we say that Φ is *satisfiable*. Otherwise Φ is *unsatisfiable*. When Φ is a singleton $\{\phi\}$ we define the (un)satisfiability of formula ϕ in the obvious way.

As usual, by a *convex combination* C of valuations $V_1, \ldots, V_r \in \mathsf{VAL}_n$ we mean a function $C \in \mathbb{R}^{\mathsf{FORM}_n}$ of the form

$$C(\psi) = \lambda_1 V_1(\psi) + \cdots + \lambda_r V_r(\psi) \quad \text{for all} \quad \psi \in \mathsf{FORM}_n,$$

where $\lambda_1, \ldots, \lambda_r$ are real coefficients ≥ 0 whose sum is 1. In general, C is not a valuation.

For any $n = 1, 2, \ldots$ and set $E = \{X_1, \ldots, X_n\}$ we will freely identify

$$[0,1]^E = [0,1]^{\{1,\ldots,n\}} = [0,1]^n.$$

Theorem 1.4 *For any set $E = \{X_1, \ldots, X_n\}$, closed nonempty set $W \subseteq [0,1]^E$, and map $\beta\colon E \to [0,1]$ the following conditions are equivalent:*

(i) *β is W-coherent.*
(ii) *There do not exist $\sigma_1, \ldots, \sigma_n \in \mathbb{R}$ such that $\sum_{i=1}^n \sigma_i(\beta(X_i) - \upsilon(X_i)) < -1$ for all $\upsilon \in W$.*
(iii) *β is a convex combination of points in W, in symbols, $\beta \in \mathrm{conv}(W)$, (equivalently, β is a convex combination of at most $n+1$ points in W).*
(iv) *$\beta = C \restriction \{X_1, \ldots, X_n\}$ for some convex combination C in $\mathbb{R}^{\mathsf{FORM}_n}$ of valuations, all satisfying* $\mathrm{Th}\, W$.
(v) *$\beta = D \restriction \{X_1, \ldots, X_n\}$ for some convex combination D of at most $n+1$ valuations $V_0, \ldots, V_n \in \mathsf{VAL}_n$, each V_i satisfying* $\mathrm{Th}\, W$.

Proof of (i$\Leftrightarrow$ ii$\Leftrightarrow$ iii) The implication (i$\Rightarrow$ii) is trivial. For the converse, let us assume condition (i) fails for $\beta = (\beta(X_1), \ldots, \beta(X_n)) \in \mathbb{R}^n = \mathbb{R}^E$. Using the notation $\circ$ for scalar product in $\mathbb{R}^n$, for some $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ we have $c \circ (\beta - \upsilon) < 0$ for all $\upsilon = (\upsilon(X_1), \ldots, \upsilon(X_n)) \in W$. Since W is closed, for some $\epsilon > 0$ the continuous function $x \mapsto c \circ (\beta - x)$ attains its maximum value $-\epsilon$ at some point in W. Then the n-tuple $(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$ given by $\sigma_i = 2c_i/\epsilon$ is a counterexample to (ii).

(iii$\Rightarrow$i) Evidently, each $\upsilon \in W$ is a W-coherent map. We *claim* that W-coherence is preserved by convex combinations in $\mathbb{R}^E$ of elements $\upsilon_1, \ldots, \upsilon_m \in W$.

Otherwise, (absurdum hypothesis) for some $0 \leq \lambda_1, \ldots, \lambda_m$ with $\sum_{j=1}^m \lambda_j = 1$ the convex combination $c = \sum_{j=1}^m \lambda_j \upsilon_j$ is W-incoherent. There are real numbers

$\sigma_1, \ldots, \sigma_n$ such that $\sum_{i=1}^n \sigma_i(c(X_i) - v(X_i)) < 0$, for all $v \in W$. In particular, for each $j = 1, \ldots, m$, $\sum_{i=1}^n \sigma_i(c(X_i) - v_j(X_i)) < 0$, i.e., $\sum_{i=1}^n \sigma_i c(X_i) < \sum_{i=1}^n \sigma_i v_j(X_i)$. It follows that

$$\sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} \sigma_i c(X_i) \right) < \sum_{j=1}^{m} \lambda_j \left(\sum_{i=1}^{n} \sigma_i v_j(X_i) \right)$$

i.e.,

$$\sum_{i=1}^{n} \sigma_i c(X_i) < \sum_{i=1}^{n} \sigma_i \sum_{j=1}^{n} \lambda_j v_j(X_i) = \sum_{i=1}^{n} \sigma_i c(X_i),$$

which is impossible. Having thus settled our claim, we have proved (iii⇒i).

(i⇒iii) Let us suppose $\beta \notin \mathrm{conv}(W)$. Since W is compact, by (B21.50) so is its convex hull $\mathrm{conv}(W)$. Then the classical separation argument (B21.51) yields a real number $\xi > 0$, together with vectors $a, b \in \mathbb{R}^n$ and a hyperplane $H = a^{\perp} + b \subseteq \mathbb{R}^n$ such that $a \circ \beta - a \circ w < -\xi$ for all $w \in \mathrm{conv}(W)$. Here $a \neq 0$ and $a^{\perp}$ denotes the hyperplane $\{x \in \mathbb{R}^n \mid a \circ x = 0\}$, i.e., the orthogonal complement of a. The stakes $\sigma_i = a_i/\xi$, $(i = 1, \ldots, n)$ show that (ii) is false, whence a fortiori, β is not W-coherent. The parenthetical remark in (iii) follows from Carathéodory theorem (B21.55).

Also the equivalence (iv⇔v) follows from Carathéodory theorem.

The proof that (iv⇔iii) requires the introduction of additional material on Łukasiewicz logic Ł_∞ and free MV-algebras, to be used throughout the book.

1.3 McNaughton Functions and Free MV-Algebras

For $n = 1, 2, \ldots$, a *McNaughton function* $f : [0, 1]^n \to [0, 1]$ is a continuous piecewise linear function all of whose linear pieces have integer coefficients. In other words, f is continuous and there are linear (affine) polynomials $l_1, \ldots, l_k$ with integer coefficients such that for each $x \in [0, 1]^n$ there is $i = 1, \ldots, k$ with $f(x) = l_i(x)$. We denote by $\mathcal{M}([0, 1]^n)$ the MV-algebra of all McNaughton functions over $[0, 1]^n$. More generally, for any nonempty $Y \subseteq [0, 1]^n$ we will denote by $\mathcal{M}(Y)$ the MV-algebra of restrictions to Y of the functions in $\mathcal{M}([0, 1]^n)$.

We say that formulas $\phi, \psi \in \mathsf{FORM}_n$ are *equivalent*, in symbols, $\phi \equiv \psi$, if $V(\phi) = V(\psi)$ for all valuations $V \in \mathsf{VAL}_n$. We denote by $\psi/\!\equiv$ the equivalence class of ψ. The dependence on $n = 1, 2, \ldots$ will always be clear from the context.

Theorem 1.5 *For each* $i = 1, \ldots, n$, *let* $\pi_i : [0, 1]^n \to [0, 1]$ *be the* i*th coordinate function.*

(i) *The set* $\mathsf{FORM}_n/\!\equiv$ *of equivalence classes of formulas of* FORM_n *equipped with the MV-algebraic operations inherited from the connectives* $\neg, \odot$ *and*

$\oplus$, *coincides with the free MV-algebra* FREE_n *over the free generating set* $\{X_1/\equiv, \ldots, X_n/\equiv\}$.

(ii) *The map* $X_i/\equiv \mapsto \pi_i$ *uniquely extends to a homomorphism* $\boldsymbol{\iota}$ *of* FREE_n *into* $\mathcal{M}([0,1]^n)$.

(iii) *In fact,* $\boldsymbol{\iota}$ *is an isomorphism of* FREE_n *onto* $\mathcal{M}([0,1]^n)$.

(iv) *For each* $\psi \in \mathsf{FORM}_n$, *let*

$$\hat{\psi} = \boldsymbol{\iota}(\psi/\equiv) \tag{1.2}$$

denote the McNaughton function represented by ψ. *Then*

$$\hat{\psi}(w) = V_w(\psi) \quad \textit{for all } w \in [0,1]^n. \tag{1.3}$$

(v) *For every nonempty set* $Y \subseteq [0,1]^n$ *and formula* $\psi \in \mathsf{FORM}_n$,

$$\psi \in \mathrm{Th}\, Y \Leftrightarrow \hat{\psi}(y) = 1 \quad \textit{for all } y \in Y. \tag{1.4}$$

Proof A proof of (i) is obtainable from [1, 4.4.4, 4.5.5]. The universal property of free MV-algebras immediately yields (ii). By Chang completeness theorem (A21.17), $\boldsymbol{\iota}$ is one–one. By McNaughton theorem (A21.48), $\boldsymbol{\iota}$ is onto $\mathcal{M}([0,1]^n)$. This proves (iii). Arguing by induction on the number of connectives in ψ, one routinely verifies (1.3) and settles (iv). Finally, (v) is a direct consequence of (iv). □

Definition 1.6 For any set of formulas $\Phi \subseteq \mathsf{FORM}_n$, the set $\mathrm{Mod}(\Phi) \subseteq [0,1]^n$ is defined by $\mathrm{Mod}(\Phi) = \{w \in [0,1]^n \mid V_w(\phi) = 1 \text{ for all } \phi \in \Phi\}$. For $\theta \in \mathsf{FORM}_n$, instead of $\mathrm{Mod}(\{\theta\})$ we write $\mathrm{Mod}(\theta)$, or even $\mathrm{Mod}_{\{X_1,\ldots,X_n\}}(\theta)$, if clarity so demands.

Thus a formula $\phi \in \mathsf{FORM}_n$ is satisfiable iff $\mathrm{Mod}_{\{X_1,\ldots,X_n\}}(\phi)$ is nonempty; ϕ is a tautology iff $\mathrm{Mod}_{\{X_1,\ldots,X_n\}}(\phi) = [0,1]^n$. Further, from (1.3) we have

$$\mathrm{Mod}(\theta) = \hat{\theta}^{-1}(1). \tag{1.5}$$

1.4 $\Phi \vdash \psi$, ψ is a Consequence of Φ

The following fundamental result will find repeated use throughout this book:

Theorem 1.7 *For all* $n = 1, 2, \ldots$ *and* $\theta, \phi \in \mathsf{FORM}_n$ *the following conditions are equivalent*:

(i) *Every valuation* $V \in \mathsf{VAL}_n$ *satisfying* θ *also satisfies* ϕ;

(ii) $\mathrm{Mod}(\theta) \subseteq \mathrm{Mod}(\phi)$;

(iii) *For some integer* $k > 0$ *the formula* $\theta^k \to \phi$ *is a tautology, where* θ^k *is short for* $\underbrace{\theta \odot \cdots \odot \theta}_{k \text{ occurrences of } \theta}$.

(iv) *For some integer* $k > 0$ *the formula*

$$\underbrace{\theta \to (\theta \to (\theta \to \cdots \to (\theta \to (\theta \to \phi))\cdots))}_{k \text{ occurrences of } \theta} \tag{1.6}$$

is a tautology.

(v) *For some integer* $k > 0$ *there is a sequence of formulas* $\alpha_0, \ldots, \alpha_{k+1}$ *such that* $\alpha_0 = \theta$, $\alpha_{k+1} = \phi$, *and for each* $i = 1, \ldots, k+1$ *either* α_i *is a tautology, or there are* $p, q \in \{0, \ldots, i-1\}$ *such that* α_q *is the formula* $\alpha_p \to \alpha_i$.

(vi) *For some integer* $k > 0$ *there is a sequence of formulas* $\alpha_0, \ldots, \alpha_{k+1}$ *such that* $\alpha_0 = \theta$, $\alpha_{k+1} = \phi$, *and for each* $i = 1, \ldots, k+1$ *either* α_i *is a tautology in* FORM_n, *or there are* $p, q \in \{0, \ldots, i-1\}$ *such that* α_q *is the formula* $\alpha_p \to \alpha_i$.

Proof (i⇔ii) By Definition 1.6.

(iii⇔iv) One promptly verifies that the two formulas (1.6) and $\theta^k \to \phi$ are equivalent.

(v⇔i) Follows from [1, 4.5.2, 4.6.7].

(v⇔iv) Follows from [1, 4.6.4].

(vi⇒v) Trivial.

(iv⇒vi) By induction on k, one verifies that ϕ can be obtained as the final formula α_{k+1} of a sequence $\alpha_0, \ldots, \alpha_{k+1}$ as in (v), only needing the assumed tautology (1.6). □

Definition 1.8 For $\mathcal{X}$ a set of variables, $\emptyset \neq \Phi \subseteq \mathsf{FORM}_{\mathcal{X}}$, and ψ a formula, we write $\Phi \vdash \psi$ [read: "ψ is a (*syntactic*) *consequence* of Φ"] if there is an integer $k > 0$ and a set $\{\phi_1, \ldots, \phi_l\} \subseteq \Phi$ such that $(\phi_1 \odot \cdots \odot \phi_l)^k \to \psi$ is a tautology.

By Chang completeness theorem together with Theorem 1.7(iii⇔v), this definition is promptly seen to agree with the definition of syntactic consequence given in [1, 4.3.2].

Corollary 1.9 *For every set* $\mathcal{X}$ *of variables, nonempty set* $\Phi \subseteq \mathsf{FORM}_{\mathcal{X}}$, *and arbitrary formula* ψ, *the following conditions are equivalent*:

(a) $\Phi \vdash \psi$.

(b) *There is an integer* $k > 0$ *and a sequence* $\phi_1, \ldots, \phi_k \in \Phi$ *such that the formula* $\phi_1 \to (\phi_2 \to (\phi_3 \to \cdots \to (\phi_{k-1} \to (\phi_k \to \psi))\cdots))$ *is a tautology.*

(c) *For some integer* $t > 0$ *there is a sequence of formulas* $\beta_1, \ldots, \beta_t$ *such that* $\beta_t = \psi$ *and for each* $i = 1, \ldots, t$ *either* $\beta_i \in \Phi$, *or* β_i *is a tautology, or there are* $p, q \in \{1, \ldots, i-1\}$ *such that* β_q *is the formula* $\beta_p \to \beta_i$.

(d) *For some integer* $t > 0$ *there is a sequence of formulas* $\beta_1, \ldots, \beta_t$ *such that* $\beta_t = \psi$ *and for each* $i = 1, \ldots, t$ *either* $\beta_i \in \Phi$, *or* β_i *is a tautology in* $\mathsf{FORM}_{\mathcal{X} \cup \mathrm{var}(\psi)}$, *or there are* $p, q \in \{1, \ldots, i-1\}$ *such that* β_q *is the formula* $\beta_p \to \beta_i$.

Proof This easily follows from Theorem 1.7(iii$\Leftrightarrow \cdots \Leftrightarrow$vi). □

In the particular case when $\Phi = \{\phi\}$ we write $\phi \vdash \psi$ instead of $\{\phi\} \vdash \psi$. If $\Phi = \{\phi_1, \ldots, \phi_r\}$, then $\Phi \vdash \psi$ iff $\phi_1 \odot \cdots \odot \phi_r \vdash \psi$ iff $\phi_1 \wedge \cdots \wedge \phi_r \vdash \psi$.

Corollary 1.10 *For all $n = 1, 2, \ldots$ and $\phi, \psi, \theta \in \mathsf{FORM}_n$,*

$$\theta \vdash \phi \leftrightarrow \psi \quad \textit{iff} \ \hat{\phi} \restriction \mathrm{Mod}(\theta) = \hat{\psi} \restriction \mathrm{Mod}(\theta).$$

Proof Combine (1.2–1.5) with Theorem 1.7, and note that, by definition of $\to$, $\theta \vdash \phi \to \psi$ iff $\hat{\phi} \restriction \mathrm{Mod}(\theta) \leq \hat{\psi} \restriction \mathrm{Mod}(\theta)$. □

1.5 Lindenbaum Algebras, End of Proof of Theorem 1.4

Definition 1.11 Fix $n = 1, 2, \ldots$ and suppose $\Phi \subseteq \mathsf{FORM}_n$ is satisfiable. Then for any $\phi, \psi \in \mathsf{FORM}_n$ we write $\phi \equiv_\Phi \psi$ iff $\Phi \vdash \phi \leftrightarrow \psi$. For each formula $\varphi \in \mathsf{FORM}_n$ we denote by $\varphi/{\equiv_\Phi}$ the $\equiv_\Phi$-equivalence class of φ. The set of $\equiv_\Phi$-equivalence classes forms an MV-algebra $\mathsf{FORM}_n/{\equiv_\Phi}$, called the *Lindenbaum algebra* of Φ and denoted LIND_Φ. Thus,

$$\mathsf{LIND}_\Phi = \left\{ \frac{\psi}{\equiv_\Phi} \mid \psi \in \mathsf{FORM}_n \right\}. \tag{1.7}$$

In case $\Phi = \{\theta\}$ for some $\theta \in \mathsf{FORM}_n$, we write LIND_θ instead of $\mathsf{LIND}_{\{\theta\}}$, and $\equiv_\theta$ instead of $\equiv_{\{\theta\}}$.

Lemma 1.12 *Let $\theta = \theta(X_1, \ldots, X_n)$ be a satisfiable formula. Then the map $\lambda\colon \varphi/{\equiv_\theta} \mapsto \hat{\varphi} \restriction \mathrm{Mod}(\theta)$ is an isomorphism of LIND_θ onto $\mathcal{M}(\mathrm{Mod}(\theta))$.*

Proof If $\hat{\psi} \restriction \mathrm{Mod}(\theta) \neq \hat{\varphi} \restriction \mathrm{Mod}(\theta)$ then by (1.5), $V_w(\theta) = 1$ and $V_w(\psi) \neq V_w(\varphi)$ for some $w \in [0, 1]^n$, whence $\mathrm{Mod}(\theta) \not\subseteq \mathrm{Mod}(\psi \leftrightarrow \varphi)$. By Theorem 1.7, $\theta \nvdash \psi \leftrightarrow \varphi$, and hence $\psi/{\equiv_\theta} \neq \varphi/{\equiv_\theta}$, thus showing that λ is a homomorphism. Next suppose $\psi/{\equiv_\theta} \neq 0$, i.e., $\theta \nvdash \neg\psi$. Again by Theorem 1.7, $\mathrm{Mod}(\theta) \not\subseteq \mathrm{Mod}(\neg\psi)$. In other words, for some valuation V_w we have $V_w(\theta) = 1$ and $V_w(\neg\psi) \neq 1$. Thus $V_w(\psi) > 0$, whence $\hat{\theta}(w) = 1$ and $\hat{\psi}(w) > 0$. As a consequence, $\hat{\psi} \restriction \mathrm{Mod}(\theta) \neq 0$, and λ is one–one. Finally, let $g \in \mathcal{M}(\mathrm{Mod}(\theta))$, and write $g = h \restriction \mathrm{Mod}(\theta)$ for some $h \in \mathcal{M}([0, 1]^n)$. By McNaughton theorem (A21.48), we can write $h = \hat{\psi}$ for some formula $\psi(X_1, \ldots, X_n)$, and conclude that $g = \lambda(\psi/{\equiv_\theta})$, which shows that λ is onto $\mathcal{M}(\mathrm{Mod}(\theta))$. □

Lemma 1.13 *For any $z \in [0, 1]^n$ and open neighborhood N of z there is a function $g \in \mathcal{M}([0, 1]^n)$ such that $g(z) = 0$ and $g(y) = 1$ for each $y \in [0, 1]^n \setminus N$.*

Proof There is an open cube N' such that $z \in N' \cap [0, 1]^n \subseteq N$, and whose faces are given by equations of the form $x_i = r_i$ for suitable rationals r_i. By (A21.18) there is a function $f \in \mathcal{M}([0, 1]^n)$ vanishing precisely on the closure of N'. The continuity of f ensures that for some $\epsilon > 0$ we constantly have $f \geq \epsilon$ over the compact set $[0, 1]^n \setminus N$. For all suitably large m the McNaughton function $g = f \oplus \cdots \oplus f$ (m times) will constantly take the value 1 on $[0, 1]^n \setminus N$. □

Lemma 1.14 *For every nonempty closed subset W of $[0, 1]^n$, $W = \mathrm{Mod}(\mathrm{Th}\, W)$.*

Proof Since $\mathrm{Mod}(\mathrm{Th}\, W) = \{x \in [0, 1]^n \mid V_x(\theta) = 1 \text{ for all } \theta \in \mathrm{Th}\, W\}$, the inclusion $W \subseteq \mathrm{Mod}(\mathrm{Th}\, W)$ is trivial.

For the converse inclusion, arguing by induction on the number of connectives in $\psi \in \mathsf{FORM}_n$ and using Lemma 1.12 together with (1.3 and 1.4), we obtain

$$\begin{aligned} \psi \in \mathrm{Th}\, W &\Leftrightarrow V_x(\psi) = 1 \quad \text{for all} \quad x \in W \\ &\Leftrightarrow \hat{\psi}(x) = 1 \quad \text{for all} \quad x \in W \\ &\Leftrightarrow \hat{\psi} \restriction W = 1, \end{aligned}$$

whence $\mathrm{Th}\, W = \{\psi \in \mathsf{FORM}_n \mid \hat{\psi} \restriction W = 1\}$. Therefore,

$$\begin{aligned} x \in \mathrm{Mod}(\mathrm{Th}\, W) &\Leftrightarrow V_x(\theta) = 1 \quad \text{for all} \quad \theta \in \mathrm{Th}\, W \\ &\Leftrightarrow \hat{\theta}(x) = 1 \quad \text{for all} \quad \theta \in \mathrm{Th}\, W \\ &\Leftrightarrow x \in \bigcap\{\hat{\theta}^{-1}(1) \mid \theta \in \mathrm{Th}\, W\}. \end{aligned}$$

Suppose $x \in [0, 1]^n \setminus W$. Since W is closed, Lemma 1.13 yields an open neighborhood N of x disjoint from W, together with a formula $\psi \in \mathsf{FORM}_n$ such that $\hat{\psi}(y) = 1$ for all $y \in [0, 1]^n \setminus N \supseteq W$. Thus $\psi \in \mathrm{Th}\, W$ and $x \notin \mathrm{Mod}(\psi) = \{y \in [0, 1]^n \mid \hat{\psi}(y) = 1\}$. A fortiori, $x \notin \mathrm{Mod}(\mathrm{Th}\, W)$, whence $W \supseteq \mathrm{Mod}(\mathrm{Th}\, W)$, as required to complete the proof. □

Conclusion of the proof of Theorem 1.4.: By Lemma 1.14, $\beta \in \mathrm{conv}(W)$ iff β is the restriction to $\{X_1, \ldots, X_n\}$ of a convex combination of valuations all satisfying Th W. We have proved (iii⇔ iv) in Theorem 1.4. The proof of the theorem is complete.

□

1.6 Remarks

Definition 1.1 is a generalization of de Finetti's notion of coherent assessment for yes–no events, ([2, Sect. 7, p. 308], [3, pp. 6–7]). Item (ii) in Theorem 1.4 is de Finetti's alternative definition of incoherent assessment [4, footnote page 87]. Lemma 1.13 was first proved in [5, 4.17]. Theorem 1.4 was first proved in [6].

Theorem 1.7 and Corollary 1.9 combine results by Hay [7] and Wójcicki [8] with Pogorzelski's Local Deduction Theorem [9]. Wójcicki proved that the equivalence (i⇔iii) no longer holds in general if θ is replaced by an infinite set of formulas [8, Theorem 2]. The generalization of Theorem 1.7 for arbitrary sets of formulas in each finite-valued Łukasiewicz logic was also established by Wójcicki (see [8, Lemma 1], [10, 4.3.3]).

See [11] for a geometric representation of the consequence relation $\vdash$.

Note that in [1, 4.6.8], Lindenbaum algebras are only defined for sets of formulas in the set of variables $\{X_1, X_2, \ldots\}$.

Events and possible worlds from physical systems. As anticipated in Sect. 1.1, the final part of this chapter is devoted to giving "events" and "possible worlds" a sufficiently general definition within the commutative C^*-algebraic formulation of classical physical systems. This also works for quantum physical systems, by just removing the commutativity axiom [12, pp. 362, 378].

Readers not interested in the C^*-algebraic sources of events and possible worlds may safely skip the remainder of this section.

Let $\mathcal{C}$ be the C^*-algebra of a classical physical system $\mathtt{S}$. We denote by $\mathcal{C}_{\mathrm{sa}}$ the set of self-adjoint elements of $\mathcal{C}$. Any element of $\mathcal{C}_{\mathrm{sa}}$ represents an observable of $\mathtt{S}$. Further, we write $\mathcal{S}$ for the set of real-valued normalized positive linear functionals on $\mathcal{C}_{\mathrm{sa}}$. By [13, VIII, 2.1 and pp. 224–225], $\mathcal{C}$ can be identified with the C^*-algebra $C(\Sigma)$ of all complex-valued continuous functions over the compact Hausdorff space Σ of maximal ideals of $\mathcal{C}$. Under this identification, Σ is the space of all possible phases of $\mathtt{S}$, and $\mathcal{C}_{\mathrm{sa}}$ is the set of all real-valued continuous functions on Σ.

$\mathcal{C}_{\mathrm{sa}}$ typically includes such observables as position, energy, momentum, and each element of $\mathcal{S}$ is thought of as a convenient mathematical counterpart of a "mode of preparation" of $\mathtt{S}$. The Riesz representation theorem (B21.64) yields a one–one correspondence between $\mathcal{S}$ and the set of regular Borel probability measures on Σ. For any $\rho \in \mathcal{S}$ and $A \in \mathcal{C}_{\mathrm{sa}}$ the real number $\rho(A)$ is said to be the expectation value of the observable A whenever $\mathtt{S}$ is prepared in mode ρ.

Let now $E = \{X_1, \ldots, X_m\}$ be a set of nonzero positive elements of $\mathcal{C}_{\mathrm{sa}}$. Under our standing identification, each X_i is a continuous function over Σ such that $f(x) \geq 0$ for all $x \in \Sigma$. Let $\sup X_i$ denote the sup norm of X_i. As explained in [12, pp. 363–369], each preparation mode $\rho \in \mathcal{S}$ determines a map $w_\rho \colon E \to [0, 1]$ by the stipulation $w_\rho(X_i) = \rho(X_i)/(\sup X_i)$. The set $W = \{w_\rho \mid \rho \in \mathcal{S}\}$ is closed in the m-cube $[0, 1]^E$. Intuitively, the "event" X_i occurs if "the value of the observable X_i is high," and the "possible world" $w_\rho \in W$ gives a precise "truth-value" in [0, 1] to this event.

Altogether, C^*-algebras have extensive capabilities to model events and possible worlds.

References

1. Cignoli, R. L. O., D'Ottaviano, I. M. L., Mundici, D. (2000). *Algebraic foundations of many-valued reasoning*. Volume 7 of Trends in Logic. Dordrecht: Kluwer
2. de Finetti, B. (1993). Sul significato soggettivo della probabilitá. (in Italian), *Fundamenta Mathematicae, 17*, 298–329, 1931. Translated into English as "On the Subjective Meaning of Probability (pp. 291–321)". In P. Monari & D. Cocchi (Eds.), *Probabilitá e Induzione*. Bologna: Clueb.
3. de Finetti, B. (1980). La prévision: ses lois logiques, ses sources subjectives. *Annales de l'Institut H. Poincaré, 7* (pp. 1–68), 1937. Translated into English by Henry E. Kyburg Jr., as

"Foresight: Its Logical Laws, its Subjective Sources". In H. E. Kyburg Jr. & H. E. Smokler (Eds.), *Studies in subjective probability*. Wiley: New York, 1964. Second edition published by Krieger, New York, pp. 53–118.
4. de Finetti, B. (1974). *Theory of Probability*, Vol. 1. Chichester: Wiley.
5. Mundici, D. (1986). Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus. *Journal of Functional Analysis*, *65*, 15–63.
6. Mundici, D. (2009). Interpretation of De Finetti coherence criterion in Łukasiewicz logic. *Annals of Pure and Applied Logic, 161*, 235–245.
7. Hay, L.S. (1963). Axiomatization of the infinite-valued predicate calculus. *Journal of Symbolic Logic, 28*, 77–86.
8. Wójcicki, R. (1973). On matrix representations of consequence operations of Łukasiewicz sentential calculi, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, *19*, 239–247.
9. Pogorzelski, W. A. (1964). A survey of deduction theorems for sentential calculi, (Polish) *Studia Logica, 15*, 163–178.
10. Wójcicki, R. (1988). *Theory of Logical Calculi: Basic Theory of Consequence Operations*, Synthese Library, Vol. 199. Dordrecht: Kluwer.
11. Mundici, D. (1988). The derivative of truth in Łukasiewicz sentential calculus, *Contemporary Mathematics*, American Mathematical Society, *69*, 209–227.
12. Emch, G. G. (1984). *Mathematical and conceptual foundations of 20th century physics*, Notas de Matemática, Vol. 100, Amsterdam: North-Holland.
13. Conway, J. B. (1985). *A course in functional analysis*, Graduate Texts in Mathematics, Vol. 96, New York: Springer.

Chapter 2
Rational Polyhedra, Interpolation, Amalgamation

One can hardly understand the fine structure of finitely presented (especially of finitely generated free and projective) MV-algebras without a working knowledge of the basic properties of rational polyhedra and their regular triangulations. The simplexes of these triangulations provide the volume elements of the integrals that evaluate the average truth-value of formulas and compute the invariant Rényi conditional introduced later in this book. Rational polyhedra are the genuine algebraic varieties of the formulas of Łukasiewicz logic: for, the zeroset of a McNaughton function of n variables is the most general possible rational polyhedron P contained in $[0, 1]^n$, $n = 1, 2, \ldots$. This chapter is an elementary introduction to rational polyhedra and their subdivisions into regular triangulations. The observation that rational polyhedra are preserved under projections onto rational hyperplanes gives us a way of proving the (deductive) interpolation property of $Ł_\infty$ and the amalgamation property of MV-algebras.

2.1 Rational Polyhedra, Complexes, Fans

Fix $n = 1, 2, \ldots$. A point $y \in \mathbb{R}^n$ is said to be *rational* if all its coordinates are rational numbers. A *rational hyperplane* H in $\mathbb{R}^n$ is a set $H = \{x \in \mathbb{R}^n \mid h \circ x = k\}$, for some nonzero $h \in \mathbb{Q}^n$ and $k \in \mathbb{Q}$. Equivalently, $0 \neq h \in \mathbb{Z}^n$ and $k \in \mathbb{Z}$. When $k = 0$ we say that H is *homogeneous*. The two closed halfspaces H^+ and H^- of $\mathbb{R}^n$ determined by H are said to be *rational*.

For any rational point $y = (y_1, \ldots, y_n) \in \mathbb{Q}^n$ we denote by $\mathrm{den}(y)$ the least common denominator of its coordinates, and we say that $\mathrm{den}(y)$ is the *denominator* of y. The integer vector

$$\tilde{y} = (\mathrm{den}(y) \cdot y_1, \ldots, \mathrm{den}(y) \cdot y_n, \mathrm{den}(y)) = \mathrm{den}(y)(y, 1) \in \mathbb{Z}^{n+1}$$

is called the *homogeneous correspondent* of y. Then $\tilde{y}$ is *primitive*, that is, minimal (as a nonzero integer vector) along its *ray* $\langle \tilde{y} \rangle = \{\lambda \tilde{y} \in \mathbb{R}^{n+1} \mid \lambda \geq 0\} = \mathbb{R}_{\geq 0}\tilde{y}$.

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