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Theory of Hypergeometric Functions

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Theory of Hypergeometric Functions

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Preface

One may say that the history of hypergeometric functions started practically with a paper by Gauss (cf. [Gau]). There, he presented most of the properties of hypergeometric functions that we see today, such as power series, a differential equation, contiguous relations, continued fractional expansion, special values and so on. The discovery of a hypergeometric function has since provided an intrinsic stimulation in the world of mathematics. It has also motivated the development of several domains such as complex functions, Riemann surfaces, differential equations, difference equations, arithmetic theory and so forth. The global structure of the Gauss hypergeometric function as a complex function, i.e., the properties of its monodromy and the analytic continuation, has been extensively studied by Riemann. His method is based on complex integrals. Moreover, when the parameters are rational numbers, its relation to the period integral of algebraic curves became clear, and a fascinating problem on the uniformization of a Riemann surface was proposed by Riemann and Schwarz. On the other hand, Kummer has contributed a lot to the research of arithmetic properties of hypergeometric functions. But there, the main object was the Gauss hypergeometric function of one variable.

In contrast, for more general hypergeometric functions, including the case of several variables, the question arises: *What in fact are hypergeometric functions ?* Since Gauss and Riemann, many researchers tried generalizing the Gauss hypergeometric function. Those which are known under the names of Goursat, Pochhammer, Barnes, Mellin, and Appell are such hypergeometric functions. Although these functions interested some researchers as special objects, they didn't attract many researchers and no significant result came about. If anything, those expressed with the aid of some properties of hypergeometric functions appeared interestingly in several situations, either in partial or another form. The orthogonal polynomials studied in Szegö's book, several formulas that we can find everywhere in Ramanujan's enormous notebooks, spherical functions on Lie groups, and applications to mathematical physics containing quantum mechanics, are such examples. Simply, they were not considered from a general viewpoint of hypergeometric functions.

In this book, hypergeometric functions of several variables will be treated. Our point of view is that the hypergeometric functions are complex integrals of complex powers of polynomials. Most of the properties of hypergeometric functions which have appeared in the literature up to now can be reconsidered from this point of view. In addition, it turns out that these functions establish interesting connections among several domains in mathematics.

One of the prominent properties of hypergeometric functions is the so-called contiguity relations. We understand them based on the classical paper by G. D. Birkhoff [Bir1] about difference equations and their generalization. This is an approach treating hypergeometric functions as solutions of difference equations with respect to shifts of parameters, and characterizing by analysis of asymptotic behaviors when the parameters tend to infinity. One sees a relation between the Padé approximation and the continued fractional expansion. For this purpose, we use either analytic or algebraic de Rham cohomology (twisted de Rham cohomology) as a natural form of complex integrals. In Chapter 2, several relations satisfied by hypergeometric functions will be derived and explained in terms of twisted de Rham cohomology. There, the reader may notice that the excellent idea due to J. Hadamard about a "*finite part of a divergent integral*" developed in his book [Had] will be naturally integrated into the theory. In Chapter 4, we will construct cycles via the saddle point method and apply the Morse theory on affine varieties to describe the global structure of an asymptotic behavior of solutions to difference equations.

Another prominent feature is a holonomic system of partial differential equations satisfied by hypergeometric functions, in particular, an infinitesimal concept called integrable connection (the Gauss−Manin connection) that has a form of partial differential equations of the first order, and a topological concept called monodromy that is its global realization. The latter means to provide a linear representation of a fundamental group, in other words, a local system on the underlying topological space. But here, what is important is not only the topological concept but the mathematical substance that provides it. Hypergeometric functions provide such typical examples. As a consequence, they also help us understand the fundamental group itself.

We will treat complex integrals of complex powers of polynomials, but the main point is not only to state general theorems in an abstract form but also to provide a concrete form of the statements. In Chapters 3 and 4, for linear polynomials, concrete formulas of differential equations, difference equations, integral representations, etc. will be derived, applying the idea from the invariant theory of general linear groups.

In the world surrounding hypergeometric functions, there are several subjects studying power series, orthogonal functions, spherical functions, differential equations, difference equations, etc. in a broad scope such as real (complex) analysis, arithmetic analysis, geometry, algebraic topololgy and combinatorics, which are mutually related and attract researchers. This book explains one such idea. In particular, micro-local analysis and the theory of holonomic $\mathcal{D}$-modules developed in Japan provided considerable impacts. In Chapter 3 of this book, we will treat a holonomic system of Fuchsian partial differential equations over Grassmannians satisfied by the hypergeometric functions, introduced by Gelfand et al., defined as integrals of complex powers of functions as described above. But there, we will explain them only by concrete computations. For a general theory of $\mathcal{D}$-modules, we propose that the reader consult the book written by Hotta and Tanisaki[1] in this series. Here, we will not treat either arithmetic aspects or the problem of the uniformization of complex manifolds. There are also several applications to mathematical physics such as conformal field theory, and solvable models in statistical mechanics. For these topics, the reader may consult Appendix D and the references in this book.

If this book serves as the first step to understanding hypergeometric functions and motivate the reader's interest towards further topics, we should say that our aim has been accomplished.

We asked Toshitake Kohno to write Appendix D including his recent result. We express our gratitude to him.

Lastly, our friends Takeshi Sasaki, Keiji Matsumoto and Masaaki Yoshida gave us precious remarks and criticisms on this manuscript. We also express our gratitude to them.

June, 1994.

Kazuhiko Aomoto
Michitake Kita

[1] The translation is published as [H-T-T].

Preface to English Edtion

After the publication of the original Japanese edition, hypergeometric functions attracted researchers both domestic and abroad, and some aspects are now fairly developed, for example in relation to arrangement of hyperplanes, conformal field theory and random matrix theory. Some related books have also been published: those by M. Yoshida [Yos3], which treats the uniformaization via period matrix, by M. Saito, B. Sturmfels and N. Takayama [S-S-T], which treats algebraic $\mathcal{D}$-modules satisfied by hypergeometric functions, and by P. Orlik and H. Terao [Or-Te3], which sheds light on hypergeometric functions from viewpoint of arrangements of hyperplanes, are particularly related to the contents of this book.

In this English edition, the contents are almost the same as the original except for a minor revision. In particular, in spite of its importance, hypergeometric functions of confluent type are not treated in this book (they can be treated in the framework of twisted de Rham theory but the situation becomes much more complicated). As for the references, we just added several that are directly related to the contents of this book. For more detailed and up-to-date references, the reader may consult the book cited above, etc.

The co-author of this book, who had been going to produce outstanding results unfortunately passed away in 1995. May his soul rest in peace.

Finally, I am indebted to Dr. Kenji Iohara, who has taken the trouble to translate the original version into English.

August, 2010. *Kazuhiko Aomoto*

Contents

Notation

$\Omega^p(\mathbb{C}^n)$: the space of p-forms with polynomial coefficients

$U(u) = \prod_{j=1}^{m} P_j(u)^{\alpha_j}, P_j(u) \in \mathbb{C}[u_1, \ldots, u_n]$

D: the divisor defined by $P := P_1 \cdots P_m$

$M := \mathbb{C}^n \setminus D$: a variety related to an integral representation of hypergeometric functions

d : the exterior derivative

$\omega := dU/U = \sum_{j=1}^{m} \alpha_j \frac{dP_j}{P_j}$: a completely integrable holomorphic connection form on M

$\nabla_\omega := d + \omega_\wedge$: the covariant differential operator with respect to ω

$\mathcal{L}_\omega$: the complex local system of rank 1 generated by solutions of $\nabla_\omega h = 0$

$\mathcal{L}_\omega^\vee$: the dual local system of $\mathcal{L}_\omega$

$\Omega^p(*D)$: the space of rational p-forms having poles along D

$\Omega^p(\log D)$: the space of logarithmic p-forms having poles along D

$\mathcal{O}_M$: the sheaf of germs of holomorphic functions on M

Ω_M^p: the sheaf of germs of holomorphic p-forms on M

$\mathcal{A}_M^p$: the sheaf of germs of C^∞ p-forms on M

$\mathcal{K}_M^p$: the sheaf of germs of currents of degree p on M

$\mathcal{S}(M)$: the space of sections of $\mathcal{S}$ over M

$\mathcal{S}_c(M)$: the space of sections with compact support of $\mathcal{S}$ over M

$H^p(M, \mathcal{S})$: the cohomology with coefficients in the sheaf $\mathcal{S}$

$H_p(M, \mathcal{L}_\omega)$: the homology with coefficients in the local system $\mathcal{L}_\omega$

$H_p^{\ell f}(M, \mathcal{L}_\omega)$: the locally finite homology with coefficient $\mathcal{L}_\omega$

$r := \dim H^n(M, \mathcal{L}_\omega) = \binom{m-1}{n}$

a lattice $L = \mathbb{Z}^\ell$

$e_1, \ldots, e_\ell$: a standard basis of L $e_i = (0, \ldots, \overset{i}{\smile}{1}, 0, \ldots, 0), 1 \le i \le \ell$.

$\nu = (\nu_1, \ldots, \nu_\ell) \in L, \nu! = \prod_{i=1}^{\ell} \nu_i!, |\nu| = \sum_{i=1}^{\ell} \nu_i$

$|J| = n$ for a set of indices $\{j_1, \ldots, j_n\}$

For $x = (x_1, \ldots, x_\ell) \in \mathbb{C}^\ell$, $x^\nu = x_1^{\nu_1} \cdots x_\ell^{\nu_\ell}$

$\sum_\nu$: the sum is taken over $\nu \in \mathbb{Z}_{\geq 0}^\ell$

$(\gamma; c) := \Gamma(\gamma + c)/\Gamma(\gamma),\ \gamma + c \notin \mathbb{Z}_{\leq 0}$

$L^\vee := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$: the dual lattice of L

$G(n+1, m+1)$: Grassmannian of $(n+1)$-dimensional subspaces of $\mathbb{C}^{m+1}$

For $x = \begin{pmatrix} x_{00} & \cdots & x_{0m} \\ \vdots & & \vdots \\ x_{n0} & \cdots & x_{nm} \end{pmatrix}$, $x(j_0 \cdots j_n) := \det \begin{pmatrix} x_{0j_0} & \cdots & x_{0j_n} \\ \vdots & & \vdots \\ x_{nj_0} & \cdots & x_{nj_n} \end{pmatrix}$

$Y := \bigcup_{0 \leq j_0 < \cdots < j_n \leq m} \{x(j_0 \cdots j_n) = 0\}$

$E(n+1, m+1; \alpha_0, \ldots, \alpha_m) = E(n+1, m+1; \alpha)$: the system of hypergeometric differential equations of type $(n+1, m+1; \alpha)$

$\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_m]$: the ring of polynomials

$\mathbb{C}(z) = \mathbb{C}(z_1, \ldots, z_m)$: the rational function field of $z_1, \ldots, z_m$

$\mathbb{C}[[z]] = \mathbb{C}[[z_1, \ldots, z_m]]$: the ring of formal power series

$\mathbb{C}((z)) = \mathbb{C}((z_1, \ldots, z_m))$: the ring of formal Laurent series

$GL_m(\mathbb{C}(z))$: the group of the regular matrices of order n with components in $\mathbb{C}(z)$

$GL_m(\mathbb{C}((z)))$: the group of the regular matrices of order n with components in $\mathbb{C}((z))$

$M_m(\mathbb{C}(z))$: the algebra of the matrices of order m with components in $\mathbb{C}(z)$

$\mathcal{B}_m = \left\{ A = \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{mm} \end{pmatrix} \in GL_m(\mathbb{C}) \right\}$: a Borel subgroup of $GL_m(\mathbb{C})$

$\mathcal{U}_m = \left\{ A = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in GL_m(\mathbb{C}) \right\}$: a maximal unipotent Lie subgroup of $GL_m(\mathbb{C})$

$\mathcal{F}_m := GL_m(\mathbb{C})/\mathcal{B}_m$: a flag manifold

$\mathcal{A}_m := GL_m(\mathbb{C})/\mathcal{U}_m$: a principal affine space

$\triangle(0; \varepsilon) = \{z \in \mathbb{C} |\ |z| < \varepsilon\}$: the open disk of center at the origin with radius ε

$\Re z, \Im z$: the real and imaginary part of a complex number z

$\arg z$: an argument of z

$\mathfrak{S}_m$: the mth symmetric group

$\chi(M)$: the Euler characteristic of M

$\hat{\varphi}(x; \alpha) = \int_\gamma U \cdot \varphi$

T, T_j: shift operators

Chapter 1
Introduction: the Euler–Gauss Hypergeometric Function

The binomial series

$$(1+x)^{\alpha}=\sum_{n=0}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n,\quad |x|<1$$

is the generating function of binomial coefficients $\binom{\alpha}{n}=\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. A hypergeometric function can be regarded as a generating analytic function of more complicated combinatorial numbers which generalizes the binomial series. By studying its analytic structure, it provides us with information such as relations among combinatorial numbers and their growth. The aim of this book is to treat hypergeometric functions of several variables as complex analytic functions. Hence, we assume that the reader is familiar with basic facts about complex functions.

$\Gamma(n)=(n-1)!=1\cdot 2\cdot\cdots\cdot(n-1)$, $n=1,2,3,\cdots$ satisfies the recurrence formula $\Gamma(n+1)=n\Gamma(n)$ and $\Gamma(1)=1$. Conversely, these two properties determine $\Gamma(n)$ uniquely. A question arises "Can we extend the function $\Gamma(z)$ for all $z\in\mathbb{C}$?" The answer is "No," if we do not restrict ourselves to $z\in\mathbb{Z}$. But, if the behavior of $\Gamma(z+m)$ is given as $m\mapsto+\infty$, $\Gamma(z)$ itself can be determined by considering $\Gamma(z+1)$, $\Gamma(z+2),\cdots,\Gamma(z+m),\cdots$ $(m\in\mathbb{Z}_{>0})$. That is, $\Gamma(z)$ can be determined by its behavior at infinity. As a phenomenon in analysis, it sometimes happens that a function or a vector is determined by its behavior at infinity. Such a situation is called a limit point and this is our basic idea to treat hypergeometric functions in this book.

In this Introduction, we shall study basic properties of the Euler–Gauss hypergeometric functions from several viewpoints. For detailed subjects, we may refer to the well-known books like [AAR], [Ca], [Er1], [Mag], [Ol], [Sh], [W-W], [Wat] etc. See also [I-K-S-Y] for a historical overview of analytic differential equations. First, we start from an infinite-product representation of the Γ-function.

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1.1 Γ-Function

1.1.1 Infinite-Product Representation Due to Euler

Consider a meromorphic function $\varphi(z)$ over $\mathbb{C}$ satisfying the difference equation

$$\varphi(z+1) = z\varphi(z), \quad z \in \mathbb{C}. \tag{1.1}$$

From this, we obtain

$$\begin{aligned}\varphi(z) &= z^{-1}\varphi(z+1) \\ &= z^{-1}(z+1)^{-1}\cdots(z+N-1)^{-1}\varphi(z+N),\end{aligned} \tag{1.2}$$

for any natural number N. Take the limit $N \mapsto +\infty$: if we assume that an asymptotic expansion of $\varphi(z)$ as $|z| \mapsto +\infty$ has the form

$$\varphi(z) = e^{-z}z^{z-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\left\{1 + O\left(\frac{1}{|z|}\right)\right\} \tag{1.3}$$

in the sector $-\pi + \delta < \arg z < \pi - \delta$ $(0 < \delta < \frac{\pi}{2})$ (here $O\left(\frac{1}{|z|}\right)$, called the Landau symbol, is a function asymptotically at most equivalent to $\frac{1}{|z|}$), applying (1.3) to $\varphi(z+N)$, we obtain

$$\begin{aligned}\varphi(z) = (2\pi)^{\frac{1}{2}} \lim_{N\mapsto+\infty} & z^{-1}(z+1)^{-1} \\ & \cdots(z+N-1)^{-1}e^{-z-N}(z+N)^{z+N-\frac{1}{2}}.\end{aligned} \tag{1.4}$$

Now, an asymptotic expansion of the Γ-function $\varphi(z) = \Gamma(z)$ as $\Re z \mapsto +\infty$ is given by the Stirling formula ([W-W] Chap12 or [Er2])

$$\begin{aligned}\Gamma(z) = e^{-z}z^{z-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\left\{1 + \frac{1}{12z} + \cdots\right\}, \\ -\pi + \delta < \arg z < \pi - \delta.\end{aligned} \tag{1.5}$$

In particular, we have

$$\Gamma(N) = (N-1)! = e^{-N} N^{N-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \left\{ 1 + O\left(\frac{1}{N}\right) \right\}, \tag{1.6}$$

$$\begin{aligned}(z+N)^{z+N-\frac{1}{2}} &= N^{z+N-\frac{1}{2}} \left(1 + \frac{z}{N}\right)^{z+N-\frac{1}{2}} \\ &= N^{z+N-\frac{1}{2}} e^z \left(1 + O\left(\frac{1}{N}\right)\right).\end{aligned} \tag{1.7}$$

By the formula

$$\lim_{N \mapsto +\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{N-1} - \log N\right) = \gamma \tag{1.8}$$

(γ is Euler's constant), the right-hand side of (1.4) becomes

$$\begin{aligned}&(2\pi)^{\frac{1}{2}} \lim_{N \mapsto +\infty} z^{-1}(z+1)^{-1} \cdots (z+N-1)^{-1} e^{-z-N} (z+N)^{z+N-\frac{1}{2}} \\ &= \lim_{N \mapsto +\infty} \frac{\Gamma(N)}{\prod_{j=0}^{N-1}(z+j)} N^z = e^{-\gamma z} \left\{ z \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right) e^{-\frac{z}{j}} \right\}^{-1} = \Gamma(z),\end{aligned} \tag{1.9}$$

which coincides with an infinite-product representation of $\Gamma(z)$.

1.1.2 Γ-Function as Meromorphic Function

Similarly, if a meromorphic solution $\psi(z)$ of the difference equation (1.1) has an asymptotic expansion as $|z| \mapsto +\infty$

$$\psi(z) = e^{-z} (-z)^{z-\frac{1}{2}} e^{\pi\sqrt{-1}z} (2\pi)^{-\frac{1}{2}} \left\{ 1 + O\left(\frac{1}{|z|}\right) \right\} \tag{1.10}$$

in the sector $\delta < \arg z < 2\pi - \delta$, by the formulas

$$\psi(z) = (z-1) \cdots (z-N) \psi(z-N), \tag{1.11}$$

$$\begin{aligned}(N-z)^{\frac{1}{2}+z-N} &= N^{\frac{1}{2}+z-N} \left(1 - \frac{z}{N}\right)^{\frac{1}{2}+z-N} \\ &= N^{\frac{1}{2}+z-N} e^z \left(1 + O\left(\frac{1}{N}\right)\right),\end{aligned}$$

we obtain

$$\psi(z) = e^{-\gamma z} e^{\pi\sqrt{-1}z} \prod_{j=1}^{\infty} \left(1 - \frac{z}{j}\right) e^{\frac{z}{j}} \tag{1.12}$$

$$= \frac{e^{\pi\sqrt{-1}z}}{\Gamma(1-z)}.$$

In this way, the function $e^{\pi\sqrt{-1}z}/\Gamma(1-z)$ can be characterized as a meromorphic solution of (1.1) having the asymptotic behavior (1.10).

1.1.3 Connection Formula

Now, the ratio $P(z) = \psi(z)/\varphi(z)$ is a periodic function satisfying $P(z+1) = P(z)$ that can be expressed by the Gauss formula

$$P(z) = \frac{e^{\pi\sqrt{-1}z}}{\Gamma(z)\Gamma(1-z)} = e^{\pi\sqrt{-1}z}\frac{\sin \pi z}{\pi} \tag{1.13}$$

$$= e^{2\pi\sqrt{-1}z} z \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{j^2}\right).$$

The relation

$$\psi(z) = P(z)\varphi(z) \tag{1.14}$$

provides a linear relation between two solutions of (1.1) that each of them has an asymptotic expansion as $\Re z \mapsto +\infty$ or $-\infty$, called a connection relation, and the problem to find this relation is called a connection problem, and $P(z)$ is called a connection coefficient or a connection function. Any connection function is periodic.

1.2 Power Series and Higher Logarithmic Expansion

1.2.1 Hypergeometric Series

Consider the convergent series

$${}_2F_1(\alpha, \beta, \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha; n)(\beta; n)}{(\gamma; n) n!} x^n, \;\; \gamma \neq 0, -1, -2, \ldots \tag{1.15}$$

on the unit disk $\Delta(0;1) = \{x \in \mathbb{C}; |x| < 1\}$ $\Big((\alpha;n) = \alpha(\alpha+1)\cdots(\alpha+n-1)$, the Pochhammer symbol[1]$\Big)$. Here and after, we abbreviate ${}_2F_1(\alpha,\beta,\gamma;x)$ as $F(\alpha,\beta,\gamma;x)$, which is called the Euler–Gauss hypergeometric function, or simply Gauss' hypergeometric function. Setting $\beta = \gamma$, we get

$$F(\alpha,\beta,\beta;x) = \sum_{n=0}^{\infty} \frac{(\alpha;n)}{n!} x^n = (1-x)^{-\alpha},$$

and the specialization $\alpha = \beta = 1$, $\gamma = 2$ yields

$$xF(1,1,2;x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x).$$

Moreover, setting $\alpha = \gamma = 1$ and taking the limit $\beta \mapsto \infty$, we obtain an elementary analytic function

$$e^x = \lim_{\beta \mapsto \infty} F\left(1,\beta,1;\frac{x}{\beta}\right).$$

1.2.2 Gauss' Differential Equation

$y = F(\alpha,\beta,\gamma;x)$ satisfies the following second-order Fuchsian linear differential equation:

$$Ey = x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dy}{dx} - \alpha\beta y = 0. \tag{1.16}$$

In fact, it is sufficient to substitute y in (1.16) by (1.15). This is called Gauss' differential equation. Here, $E = E(x, \frac{d}{dx})$ is a second-order Fuchsian differential operator. Conversely, assume that y can be expanded as a holomorphic power series around the origin in the form

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad a_0 = 1. \tag{1.17}$$

Comparing the coefficients of x^n, for y to be a solution of (1.16), we get the recurrence relation

$$(n+1)(\gamma+n)a_{n+1} = (n+\alpha)(n+\beta)a_n,$$

i.e.,

[1] For $n < 0$, we set $(\alpha;n) = (\alpha+n)^{-1}\cdots(\alpha-1)^{-1}$.

$$a_{n+1} = \frac{(n+\alpha)(n+\beta)}{(n+1)(n+\gamma)} a_n, \quad n \geq 0, \tag{1.18}$$

which implies that y coincides with $F(\alpha, \beta, \gamma; x)$. In this way, $F(\alpha, \beta, \gamma; x)$ can be characterized as a unique solution of (1.16) which is holomorphic around the origin and which gives 1 at $x = 0$.

1.2.3 First-Order Fuchsian Equation

Setting $y = y_1$, $x\frac{dy}{dx} = y_2\beta$, (1.16) can be transformed into the autonomous first-order Fuchsian equation

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(\frac{A_0}{x} + \frac{A_1}{x-1}\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{1.19}$$

where we set

$$A_0 = \begin{pmatrix} 0 & \beta \\ 0 & 1-\gamma \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -\alpha & \gamma-\alpha-\beta-1 \end{pmatrix}.$$

This equation describes the horizontal direction of an integrable connection ([De]), called the Gauss–Manin connection, over the complex projective line $\mathbb{P}^1(\mathbb{C}) = \mathbb{C}^{\cup}\{\infty\}$ which admits singularities at $x = 0, 1, \infty$.

1.2.4 Logarithmic Connection

Let a, b be two different points of $\mathbb{C} \setminus \{0, 1\}$. By a theory of linear ordinary differential equations, one can extend the solutions of equation (1.16) analytically along a path σ connecting a and b. Moreover, this extension depends only on the homotopy class of the path connecting a and b. This is the property known as the uniqueness of analytic continuation. From this, it follows that the function $F(\alpha, \beta, \gamma; x)$, a priori defined on $\Delta(0; 1)$, is extended analytically to a single-valued analytic function on the universal covering $\tilde{X}$ of the complex one-dimensional manifold $X = \mathbb{C} \setminus \{0, 1\}$. Let us think about expressing this extension more explicitly.

Now, we introduce parameters $\lambda_1 = \beta$, $\lambda_2 = \gamma - \alpha - 1$, $\lambda_3 = -\alpha$. Then, (1.19) can be rewritten in the form of a linear Pfaff system

$$d\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (\lambda_1\theta_1 + \lambda_2\theta_2 + \lambda_3\theta_3)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{1.20}$$

Here,

$$\theta_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} d\log x + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} d\log(x-1), \tag{1.21}$$

$$\theta_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} d\log x + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} d\log(x-1), \tag{1.22}$$

$$\theta_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} d\log x + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} d\log(x-1), \tag{1.23}$$

are logarithmic differential 1-forms. We integrate (1.20) along a smooth curve σ connecting 0 and x. By the fact that $y_1 = 1$ and $y_2 = 0$ at $x = 0$, we obtain an integral form of (1.20):

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^x (\lambda_1\theta_1 + \lambda_2\theta_2 + \lambda_3\theta_3) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{1.24}$$

1.2.5 Higher Logarithmic Expansion

Solving (1.24) by Picard's iterative methods (cf. [In]), y_1 and y_2 can be expressed as convergent series of $\lambda_1, \lambda_2, \lambda_3$ around the origin of $\mathbb{C}^3$:

$$y_1 = L_\phi(x) + \sum_{r=1}^{\infty} \sum_{1 \leqslant i_1,\ldots,i_r \leqslant 3} \lambda_{i_1} \cdots \lambda_{i_r} L_{i_1\cdots i_r}(x), \tag{1.25}$$

$$y_2 = \sum_{r=1}^{\infty} \sum_{1 \leqslant i_1,\ldots,i_r \leqslant 3} \lambda_{i_1} \cdots \lambda_{i_r} L'_{i_1\cdots i_r}(x). \tag{1.26}$$

Here, $L_{i_1\cdots i_r}(x)$, $L'_{i_1\cdots i_r}(x)$ are the analytic functions on $\tilde{X}$ defined, along the path σ, by the recurrence relations:

$$L_\phi(x) = 1, \qquad L'_\phi(x) = 0 \tag{1.27}$$

$$L_{1i_2\cdots i_r}(x) = \int_0^x d\log x \cdot L'_{i_2\cdots i_r}(x), \quad r \geq 1 \tag{1.28}$$

$$L_{2i_2\cdots i_r}(x) = L_{3i_2\cdots i_r}(x) = 0, \quad r \geq 1 \tag{1.29}$$

$$L'_{1i_2\cdots i_r}(x) = -\int_0^x d\log(x-1) L_{i_2\cdots i_r}(x), \quad r \geq 1 \tag{1.30}$$

$$L'_{2i_2\cdots i_r}(x) = \int_0^x d\log\left(\frac{x-1}{x}\right) L'_{i_2\cdots i_r}(x), \quad r \geq 1 \tag{1.31}$$

$$L'_{3i_2\cdots i_r}(x) = -\int_0^x d\log(x-1) L_{i_2\cdots i_r}(x) + \int_0^x d\log x \cdot L'_{i_2\cdots i_r}(x), \quad r \geq 1. \tag{1.32}$$

The series of functions $L_{i_1\cdots i_r}(x)$, $L'_{i_1\cdots i_r}(x)$ appearing here has been studied by Lappo-Danilevsky and Smirnov ([La], [Sm]) in detail, and the functions are called hyper logarithms by them. Today, these functions, which are analytic functions on $\tilde{X}$, are also called polylogarithms or higher logarithms. By the way, as we have

$$L_1(x) = L_2(x) = L_3(x) = 0$$

$$L_1'(x) = L_2'(x) = 0, \quad L_3'(x) = -\log(1-x)$$

$$L_{11}(x) = L_{21}(x) = L_{31}(x) = L_{12}(x) = L_{22}(x)$$

$$= L_{32}(x) = L_{23}(x) = L_{33}(x) = 0,$$

$$L_{13}(x) = -\int_0^x \frac{\log(1-x)}{x} dx,$$

$$L_{11}'(x) = L_{21}'(x) = L_{31}'(x) = L_{12}'(x) = L_{22}'(x) = L_{32}'(x) = 0,$$

$$L_{13}'(x) = \frac{1}{2}\log^2(1-x), \quad L_{23}'(x) = -\frac{1}{2}\log^2(1-x) + \int_0^x \frac{\log(1-x)}{x} dx,$$

$$L_{33}'(x) = -\int_0^x \frac{\log(1-x)}{x} dx,$$

$$L_{113}(x) = \frac{1}{2}\int_0^x \frac{\log^2(1-x)}{x} dx,$$

$$L_{123}(x) = -\frac{1}{2}\int_0^x \frac{\log^2(1-x)}{x} dx + \int_0^x \frac{dx}{x}\left(\int_0^x \frac{\log(1-x)}{x} dx\right),$$

$$L_{133}(x) = -\int_0^x \frac{dx}{x}\left(\int_0^x \frac{\log(1-x)}{x} dx\right)$$

etc., from (1.25), we obtain the expansion

$$\begin{aligned} F(\alpha,\beta,\gamma;x) &= 1 - \lambda_1\lambda_3 \int_0^x \frac{\log(1-x)}{x} dx \\ &\quad + \frac{1}{2}\lambda_1\lambda_3(\lambda_1-\lambda_2)\int_0^x \frac{\log^2(1-x)}{x} dx \\ &\quad + \lambda_1\lambda_3(\lambda_2-\lambda_3)\int_0^x \frac{dx}{x}\left(\int_0^x \frac{\log(1-x)}{x} dx\right) + \cdots \end{aligned} \tag{1.33}$$

In particular, the function

$$-\int_0^x \frac{\log(1-x)}{x} dx = \sum_{n=1}^{\infty} \frac{x^n}{n^2} \tag{1.34}$$

is called a di-logarithm or the Abel–Rogers–Spence function ([Lew]) and some interesting arithmetic properties are known. In general, $L_{i_1\cdots i_r}(x)$ and $L'_{i_1\cdots i_r}(x)$ are expressed in terms of iterated integrals à la K.T.-Chen with logarithmic differential 1-forms $d\log x$ and $d\log(x-1)$, and are extended to higher-dimensional projective spaces ([Ch], [Ao5]). See also Remark 3.12 in § 3.8 of Chapter 3.

1.2.6 $\mathcal{D}$-Module

One can re-interpret (1.16) as follows. Denoting the sheaf of holomorphic functions on X by $\mathcal{O}$ and of the ring of holomorphic differential operators by $\mathcal{D}$ (for sheaves, see, e.g., [Ka]), y gives a local section of $\mathcal{O}$ around a neighborhood of each point x_0 of X. Now, one can apply to y a partial differential operator $P(x, \frac{d}{dx})$ which is a section of $\mathcal{D}_{x_0}$, a germ of $\mathcal{D}$ at x_0, and one obtains a morphism

$$\begin{array}{ccc} \mathcal{D}_{x_0} & \longmapsto & \mathcal{O}_{x_0} \\ \Cup & & \Cup \\ P\left(x, \dfrac{d}{dx}\right) & \longmapsto & P\left(x, \dfrac{d}{dx}\right) y. \end{array} \tag{1.35}$$

A necessary and sufficient condition for the equality

$$P\left(x, \frac{d}{dx}\right) y = 0 \tag{1.36}$$

to be satisfied is that P can be rewritten in the form

$$P\left(x, \frac{d}{dx}\right) = Q\left(x, \frac{d}{dx}\right) E\left(x, \frac{d}{dx}\right) \tag{1.37}$$

$\left(Q\left(x, \frac{d}{dx}\right) \in \mathcal{D}_{x_0}\right)$ in a neighborhood of x_0, and a morphism (1.35) induces a homomorphism of $\mathcal{D}_{x_0}$-modules

$$\begin{array}{ccc} \mathcal{D}_{x_0}/\mathcal{D}_{x_0}E & \longmapsto & \mathcal{O}_{x_0} \\ \Cup & & \Cup \\ P\left(x, \dfrac{d}{dx}\right) & \longmapsto & P\left(x, \dfrac{d}{dx}\right) y, \end{array} \tag{1.38}$$

i.e., an element of $\mathrm{Hom}_{\mathcal{D}_{x_0}}(\mathcal{D}_{x_0}/\mathcal{D}_{x_0}E, \mathcal{O}_{x_0})$. From such a viewpoint, one obtains a structure of $\mathcal{D}$-modules satisfied by hypergeometric functions. But here, we do not discuss such a structure further. For a more systematic treatment, see, e.g., [Pha2], [Kas2], [Hot].

1.3 Integral Representation Due to Euler and Riemann ([AAR], [W-W])

1.3.1 Kummer's Method

Let us rewrite some of the factors which have appeared in coefficients of the power series (1.15) as the Euler integral representation. Assuming $\Re\alpha > 0$, $\Re(\gamma-\alpha) > 0$, we have

$$\frac{(\alpha;n)}{(\gamma;n)} = \frac{\Gamma(\gamma)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(\gamma+n)} \tag{1.39}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha+n-1}(1-u)^{\gamma-\alpha-1}du,$$

and by the binomial expansion

$$\sum_{n=0}^{\infty}\frac{(\lambda;n)}{n!}u^n = (1-u)^{-\lambda}, \quad |u|<1,$$

from (1.15), we obtain

$$F(\alpha,\beta,\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1} \tag{1.40}$$

$$\left\{\sum_{n=0}^{\infty}u^n x^n\frac{(\beta;n)}{n!}\right\}du$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}du$$

with the interchange of limit and integral. Here, for that the domain of the integral can avoid the singularity of $(1-ux)^{-\beta}$, we assume $|x|<1$. This method of finding an elementary integral representation is by Kummer. The integral (1.40) has been studied by Riemann in detail ([Ri1], [Ri2]), and one of our purposes is to extend this integral to higher-dimensional cases and to reveal systematically the structure of generalized hypergeometric functions.

1.4 Gauss' Contiguous Relations and Continued Fraction Expansion

1.4.1 Gauss' Contiguous Relation

The functions $F(\alpha,\beta,\gamma;x)$ and $F(\alpha+l_1,\beta+l_2,\gamma+l_3;x)$ obtained by shifting the parameters α,β,γ by integers ($l_1,l_2,l_3\in\mathbb{Z}$) are linearly related. In particular, the following formulas are referred to as Gauss' contiguous relations [Pe]:

$$F(\alpha,\beta,\gamma;x)=F(\alpha,\beta+1,\gamma+1;x)-\frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}xF(\alpha+1,\beta+1,\gamma+2;x), \tag{1.41}$$

$$F(\alpha,\beta,\gamma;x)=F(\alpha+1,\beta,\gamma+1;x)-\frac{\beta(\gamma-\alpha)}{\gamma(\gamma+1)}xF(\alpha+1,\beta+1,\gamma+2;x). \tag{1.42}$$

Indeed, one can check these formulas by expanding $F(\alpha,\beta,\gamma;x)$, $F(\alpha,\beta+1,\gamma+1;x)$, $F(\alpha+1,\beta,\gamma+1;x)$, $F(\alpha+1,\beta+1,\gamma+2;x)$ as power series in x with the aid of (1.15) and comparing the coefficients of each term. For example, by (1.15), the coefficient of x^n ($n\geq 1$) in the right-hand side of (1.41) is given by

$$\begin{aligned}&\frac{(\alpha;n)(\beta+1;n)}{(\gamma+1;n)n!}-\frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}\frac{(\alpha+1;n-1)(\beta+1;n-1)}{(\gamma+2;n-1)(n-1)!}\\&=\frac{(\alpha;n)(\beta+1;n-1)}{(\gamma+1;n)n!}\left\{\beta+n-\frac{n(\gamma-\beta)}{\gamma}\right\}\\&=\frac{(\alpha;n)(\beta;n)}{(\gamma;n)n!}\end{aligned}$$

and as the constant term is 1, the right-hand side of (1.41) is equal to the left-hand side of (1.41). By (1.41), we have

$$\frac{F(\alpha,\beta,\gamma;x)}{F(\alpha,\beta+1,\gamma+1;x)}=1-\frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}x\frac{F(\alpha+1,\beta+1,\gamma+2;x)}{F(\alpha,\beta+1,\gamma+1;x)}. \tag{1.43}$$

On the other hand, by (1.42), we obtain

$$\frac{F(\alpha,\beta+1,\gamma+1;x)}{F(\alpha+1,\beta+1,\gamma+2;x)} \tag{1.44}$$
$$= 1 - \frac{(\beta+1)(\gamma-\alpha+1)}{(\gamma+1)(\gamma+2)} x \frac{F(\alpha+1,\beta+2,\gamma+3;x)}{F(\alpha+1,\beta+1,\gamma+2;x)}.$$

Using symbols expressing continued fractions such as $a \pm \frac{b}{c} = a \pm \frac{b|}{|c}$, $a \pm \frac{b}{c \pm \frac{d}{e}} = a \pm \frac{b|}{|c} \pm \frac{d|}{|e}$, it follows from (1.43), (1.44) that

$$\frac{F(\alpha,\beta,\gamma;x)}{F(\alpha,\beta+1,\gamma+1;x)} = 1 - \frac{\frac{\alpha(\gamma-\beta)x}{\gamma(\gamma+1)}|}{|1} - \frac{\frac{(\beta+1)(\gamma-\alpha+1)}{(\gamma+1)(\gamma+2)}x|}{|\frac{F(\alpha+1,\beta+1,\gamma+2;x)}{F(\alpha+1,\beta+2,\gamma+3;x)}}. \tag{1.45}$$

By the shift $(\alpha,\beta,\gamma) \mapsto (\alpha+1,\beta+1,\gamma+2)$, we can repeat this transformation several times, namely, we obtain the finite continued fraction expansion

$$\frac{F(\alpha,\beta,\gamma;x)}{F(\alpha,\beta+1,\gamma+1;x)} = 1 + \frac{a_1 x|}{|1} \tag{1.46}$$
$$+ \cdots + \frac{a_{2\nu} x|}{|\frac{F(\alpha+\nu,\beta+\nu,\gamma+2\nu;x)}{F(\alpha+\nu,\beta+\nu+1,\gamma+2\nu+1;x)}},$$

$$\begin{cases} a_{2\nu} &= -\dfrac{(\beta+\nu)(\gamma-\alpha+\nu)}{(\gamma+2\nu-1)(\gamma+2\nu)} \\ a_{2\nu+1} &= -\dfrac{(\alpha+\nu)(\gamma-\beta+\nu)}{(\gamma+2\nu)(\gamma+2\nu+1)} \end{cases}. \tag{1.47}$$

Since the left-hand side of (1.46) is holomorphic in a neighborhood of $x=0$, it can be expanded as a power series in x. The infinite continued fraction expansion

$$1 + \frac{a_1 x|}{|1} + \frac{a_2 x|}{|1} + \cdots = 1 + a_1 x - a_1 a_2 x^2 + \cdots \tag{1.48}$$

makes sense as a formal power series at $x=0$.

1.4.2 Continued Fraction Expansion

The identity

$$\frac{F(\alpha,\beta,\gamma;x)}{F(\alpha,\beta+1,\gamma+1;x)} = 1 + \frac{a_1 x|}{|1} + \frac{a_2 x|}{|1} + \cdots, \tag{1.49}$$