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p-Adic Lie Groups

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Introduction

This book presents a complete account of the foundations of the theory of p-adic Lie groups. It moves on to some of the important more advanced aspects. Although most of the material is not new, it is only in recent years that p-adic Lie groups have found important applications in number theory and representation theory. These applications constitute, in fact, an increasingly active area of research. The book is designed to give to the advanced, but not necessarily graduate, student a streamlined access to the basics of the theory. It is almost self contained. Only a few technical computations which are well covered in the literature will not be repeated. My hope is that researchers who see the need to take up p-adic methods also will find this book helpful for quickly mastering the necessary notions and techniques. The book comes in two parts. Part A on the analytic side grew out of a course which I gave at Münster for the first time during the summer term 2001, whereas part B on the algebraic side is the content of a course given at the Newton Institute during September 2009.

The original and proper context of p-adic Lie groups is p-adic analysis. This is the point of view in Part A. Of course, in a formal sense the notion of a p-adic Lie group is completely parallel to the classical notion of a real or complex Lie group. It is a manifold over a nonarchimedean field which carries a compatible group structure. The fundamental difference is that the p-adic notion has no geometric content. As we will see, a paracompact p-adic manifold is topologically a disjoint union of charts and therefore is, from a geometric perspective, completely uninteresting. The point instead is that, like for real Lie groups, manifolds and Lie groups in the p-adic world are a rich source, through spaces of functions and distributions, of interesting group representations as well as various kinds of important topological group algebras. We nevertheless find the geometric language very intuitive and therefore will use it systematically. In the first chapter we recall what a nonarchimedean field is and quickly discuss the elementary analysis over such fields. In particular, we carefully introduce the notion of a locally analytic function which is at the base for everything to follow. The second chapter then defines manifolds and establishes the formalism of their tangent spaces. As a more advanced topic we include the construction of the natural topology on vector spaces of locally analytic functions. This is due to Féaux de Lacroix in his thesis. It is the starting point for the representation theoretic applications of the theory. In the third chapter we finally introduce p-adic Lie groups and we construct the corresponding Lie algebras. The main purpose of this chapter then is to understand how much information about the Lie group can be recovered from its Lie algebra. Here again lies a crucial difference to Lie groups over the real numbers. Since p-adic Lie groups topologically are totally disconnected they contain arbitrarily small open subgroups. Hence the Lie algebra determines the Lie group only locally around the unit element which is formalized by the notion of a Lie group germ. As the length of the chapter indicates this relation between Lie groups and Lie algebras is technically rather involved. It requires a whole range of algebraic concepts which we all will introduce. As said before, only for a few computations the reader will be referred to the literature. The key result is contained in the discussion of the convergence of the Hausdorff series.

There are three existing books on the material in Part A: "Variétés différentielles et analytiques. Fascicule de résultats" and "Lie Groups and Lie Algebras" by Bourbaki and Serre's lecture notes on "Lie Algebras and Lie groups". The first one contains no proofs, the nature of the second one is encyclopedic, and the last one some times is a bit short on details. All three develop the real and p-adic case alongside each other which has advantages but makes a quick grasp of the p-adic case alone more difficult. The presentation in the present book places its emphasis instead on a streamlined but still essentially self contained introduction to exclusively the p-adic case.

Lazard discovered in the 1960s a purely algebraic approach to p-adic Lie groups. Unfortunately his seminal paper is notoriously difficult to read. Part B of this book undertakes the attempt to give an account of Lazard's work again in a streamlined form which is stripped of all inessential generalities and ramifications. Lazard proceeds in an axiomatic way starting from the notion of a p-valuation ω on a pro-p-group G. After some preliminaries in the fourth chapter this concept is explained in chapter five. It will not be too difficult to show that any p-adic Lie group has an open subgroup which carries a p-valuation. Lazard realized that, vice versa, any pro-p-group with a p-valuation (and satisfying an additional mild condition of being "of finite rank") is a compact p-adic Lie group in a natural way. The technical tool to achieve this important result is the so called completed group ring $\Lambda(G)$ of a profinite group G. It is the appropriate analog of the algebraic group ring of a finite (or, more generally, discrete) group in the context of profinite groups. In the presence of a p-valuation ω Lazard develops a technique of computation in $\Lambda(G)$, which as such is a highly complicated and in general noncommutative algebra. All of this will be presented in the sixth chapter. In the last chapter seven we go back to Lie algebras. Being a p-adic Lie group a pro-p-group G with a p-valuation of finite rank ω has a Lie algebra $\mathrm{Lie}(G)$ over the field of p-adic numbers $\mathbb{Q}_p$. By inverting p and a further completion process the completed group ring $\Lambda(G)$ can be enlarged to a $\mathbb{Q}_p$-Banach algebra $\Lambda_{\mathbb{Q}_p}(G,\omega)$ which turns out to be naturally isomorphic to a certain completion of the universal enveloping algebra of Lie(G). This is another one of Lazard's important results. It provides us with a different route to construct Lie(G) which is independent of any analysis. In fact, it does better than that since it leads to a natural Lie algebra over the ring over p-adic integers $\mathbb{Z}_p$ associated with the pair (G,ω). This means that the algebraic theory, via this notion of a p-valuation, makes the connection between Lie group and Lie algebra much more precise than the analytic theory was able to do. The final question addressed in the last chapter is the question on the possibility of varying the p-valuation on the same group G. Using the newly established direct connection to the Lie algebra this problem can be transferred to the latter. There it eventually becomes a problem of convexity theory which is much easier to solve. This, in particular, allows to prove the very useful technical fact that there always exists a p-valuation with rational values. Its most important consequence is the result that the completed group ring $\Lambda(G)$ of any (G,ω) of finite rank is a noetherian ring of finite global dimension. This is why completed group rings of p-adic Lie groups have become important in number theory (where they are applied to Galois groups G), and why they deserve further systematic study in noncommutative algebra.

This is the first textbook in the proper sense on Lazard's work. The book "Analytic Pro-p-Groups" by Dixon, du Sautoy, Mann, and Segal has a completely different perspective. It is written entirely from the point of view of abstract group theory. Moreover, it does not mention Lazard's concept of a p-valuation at all but replaces it by an alternative axiomatic approach based on the notion of a uniformly powerful pro-p-group. This approach is very conceptual as well but also less flexible and more restrictive than the one by Lazard which we follow.

It is a pleasure to thank J. Coates for persuading me to undertake this lecture series at the Newton Institute and to write it up in this book, the audience for the valuable feedback, the Newton Institute for its hospitality and support, and T. Schoeneberg for a careful reading of Part B.

Münster, February 2011 *Peter Schneider*

Contents

Part A

p-Adic Analysis and Lie Groups

Chapter I
Foundations

1 Ultrametric Spaces

We begin by establishing some very basic and elementary notions.

Definition. *A metric space* (X, d) *is called ultrametric if the strict triangle inequality*

$$d(x, z) \leq \max(d(x, y), d(y, z)) \quad \textit{for any } x, y, z \in X$$

is satisfied.

Examples will be given later on.

Remark. i. *If* (X, d) *is ultrametric then* $(Y, d\,|Y \times Y)$*, for any subset* $Y \subseteq X$*, is ultrametric as well.*

ii. *If* $(X_1, d_1), \ldots, (X_m, d_m)$ *are ultrametric spaces then the cartesian product* $X_1 \times \cdots \times X_m$ *is ultrametric with respect to*

$$d((x_1, \ldots, x_m), (y_1, \ldots, y_m)) := \max(d_1(x_1, y_1), \ldots, d_m(x_m, y_m)).$$

Let (X, d) be an ultrametric space in the following.

Lemma 1.1. *For any three points* $x, y, z \in X$ *such that* $d(x, y) \neq d(y, z)$ *we have*

$$d(x, z) = \max(d(x, y), d(y, z)).$$

Proof. We may assume that $d(x, y) < d(y, z)$. Then

$$d(x, y) < d(y, z) \leq \max(d(y, x), d(x, z)) = \max(d(x, y), d(x, z)).$$

The maximum in question therefore necessarily is equal to $d(x, z)$ so that

$$d(x, y) < d(y, z) \leq d(x, z).$$

We deduce that

$$d(x, z) \leq \max(d(x, y), d(y, z)) \leq d(x, z).$$

$\square$

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Let $a \in X$ be a point and $\varepsilon > 0$ be a positive real number. We call

$$B_\varepsilon(a) := \{x \in X : d(a,x) \leq \varepsilon\}$$

the *closed ball* and

$$B_\varepsilon^-(a) := \{x \in X : d(a,x) < \varepsilon\}$$

the *open ball* around a of radius ε. Any subset in X of one of these two kinds is simply referred to as a *ball*. As the following facts show this language has to be used with some care.

Lemma 1.2. i. *Every ball is open and closed in X.*

ii. *For $b \in B_\varepsilon(a)$, resp. $b \in B_\varepsilon^-(a)$, we have $B_\varepsilon(b) = B_\varepsilon(a)$, resp. $B_\varepsilon^-(b) = B_\varepsilon^-(a)$.*

Proof. Obviously $B_\varepsilon^-(a)$ is open and $B_\varepsilon(a)$ is closed in X. We first consider the equivalence relation $x \sim y$ on X defined by $d(x,y) < \varepsilon$. The corresponding equivalence class of b is equal to $B_\varepsilon^-(b)$ and hence is open. Since equivalence classes are disjoint or equal this implies $B_\varepsilon^-(b) = B_\varepsilon^-(a)$ whenever $b \in B_\varepsilon^-(a)$. It also shows that $B_\varepsilon^-(a)$ as the complement of the other open equivalence classes is closed in X.

Analogously we may consider the equivalence relation $x \approx y$ on X defined by $d(x,y) \leq \varepsilon$. Its equivalence classes are the closed balls $B_\varepsilon(b)$, and we obtain in the same way as before the assertion ii. for closed balls. It remains to show that $B_\varepsilon(a)$ is open in X. But by what we have established already with any point $b \in B_\varepsilon(a)$ its open neighbourhood $B_\varepsilon^-(b)$ is contained in $B_\varepsilon(b) = B_\varepsilon(a)$. □

The assertion ii. in the above lemma can be viewed as saying that any point of a ball can serve as its midpoint. By way of an example we will see later on that also the notion of a radius is not well determined.

Lemma 1.3. *For any two balls B and B' in X such that $B \cap B' \neq \emptyset$ we have $B \subseteq B'$ or $B' \subseteq B$.*

Proof. Pick a point $a \in B \cap B'$. As a consequence of Lemma 1.2.ii. the following four cases have to be distinguished:

1. $B = B_\varepsilon^-(a), B' = B_\delta^-(a)$,
2. $B = B_\varepsilon^-(a), B' = B_\delta(a)$,

3. $B = B_\varepsilon(a), B' = B^-_\delta(a)$,

4. $B = B_\varepsilon(a), B' = B_\delta(a)$.

Without loss of generality we may assume that $\varepsilon \leq \delta$. In cases 1, 2, and 4 we then obviously have $B \subseteq B'$. In case 3 we obtain $B \subseteq B'$ if $\varepsilon < \delta$ and $B' \subseteq B$ if $\varepsilon = \delta$. □

Remark. *If the ultrametric space X is connected then it is empty or consists of one point.*

Proof. Assuming that X is nonempty we pick a point $a \in X$. Lemma 1.2.i. then implies that $X = B_\varepsilon(a)$ for any $\varepsilon > 0$ and hence that $X = \{a\}$. □

Lemma 1.4. *Let $U = \bigcup_{i\in I} U_i$ be a covering of an open subset $U \subseteq X$ by open subsets $U_i \subseteq X$; moreover let $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ be a strictly descending sequence of positive real numbers which converges to zero; then there is a decomposition*

$$U = \bigcup_{j\in J} B_j$$

of U into pairwise disjoint balls B_j such that:

(a) *$B_j = B_{\varepsilon_{n(j)}}(a_j)$ for appropriate $a_j \in X$ and $n(j) \in \mathbb{N}$,*

(b) *$B_j \subseteq U_{i(j)}$ for some $i(j) \in I$.*

Proof. For $a \in U$ we put

$$n(a) := \min\{n \in \mathbb{N} : B_{\varepsilon_n}(a) \subseteq U_i \text{ for some } i \in I\}.$$

The family of balls $J := \{B_{\varepsilon_{n(a)}}(a) : a \in U\}$ by construction has the properties (a) and (b) and covers U (observe that for any point a in the open set U_i we find some sufficiently big $n \in \mathbb{N}$ such that $B_{\varepsilon_n}(a) \subseteq U_i$). The balls in this family indeed are pairwise disjoint: Suppose that

$$B_{\varepsilon_{n(a_1)}}(a_1) \cap B_{\varepsilon_{n(a_2)}} \neq \emptyset.$$

By Lemma 1.3 we may assume that $B_{\varepsilon_{n(a_1)}}(a_1) \subseteq B_{\varepsilon_{n(a_2)}}(a_2)$. But then Lemma 1.2.ii. implies that $B_{\varepsilon_{n(a_2)}}(a_1) = B_{\varepsilon_{n(a_2)}}(a_2)$ and hence $B_{\varepsilon_{n(a_1)}}(a_1) \subseteq B_{\varepsilon_{n(a_2)}}(a_1)$. Due to the minimality of $n(a_1)$ we must have $n(a_1) \leq n(a_2)$, resp. $\varepsilon_{n(a_1)} \geq \varepsilon_{n(a_2)}$. It follows that $B_{\varepsilon_{n(a_1)}}(a_1) = B_{\varepsilon_{n(a_2)}}(a_1) = B_{\varepsilon_{n(a_2)}}(a_2)$. □

As usual the metric space X is called *complete* if every Cauchy sequence in X is convergent.

Lemma 1.5. *A sequence $(x_n)_{n\in\mathbb{N}}$ in X is a Cauchy sequence if and only if $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.*

For a subset $A \subseteq X$ we call

$$d(A) := \sup\{d(x,y) : x, y \in A\}$$

the *diameter* of A.

Lemma 1.6. *Let $B \subseteq X$ be a ball with $\varepsilon := d(B) > 0$ and pick any point $a \in B$; we then have $B = B_\varepsilon^-(a)$ or $B = B_\varepsilon(a)$.*

Proof. The inclusion $B \subseteq B_\varepsilon(a)$ is obvious. By Lemma 1.2.ii. the ball B is of the form $B = B_\delta^-(a)$ or $B = B_\delta(a)$. The strict triangle inequality then implies $\varepsilon = d(B) \leq \delta$. If $\varepsilon = \delta$ there is nothing further to prove. If $\varepsilon < \delta$ we have $B \subseteq B_\varepsilon(a) \subseteq B_\delta^-(a) \subseteq B$ and hence $B = B_\varepsilon(a)$. □

Let us consider a descending sequence of balls

$$B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$$

in X. If X is complete and if $\lim_{n\to\infty} d(B_n) = 0$ then we claim that

$$\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset.$$

If we pick points $x_n \in B_n$ then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Put $x := \lim_{n\to\infty} x_n$. Since each B_n is closed we must have $x \in B_n$ and therefore $x \in \bigcap_n B_n$.

Without the condition on the diameters the intersection $\bigcap_n B_n$ can be empty (compare the exercise further below). This motivates the following definition.

Definition. *The ultrametric space (X, d) is called spherically complete if any descending sequence of balls $B_1 \supseteq B_2 \supseteq \cdots$ in X has a nonempty intersection.*

Lemma 1.7. i. *If X is spherically complete then it is complete.*

ii. *Suppose that X is complete; if 0 is the only accumulation point of the set $d(X \times X) \subseteq \mathbb{R}_+$ of values of the metric d then X is spherically complete.*

Proof. i. Let $(x_n)_{n\in\mathbb{N}}$ be any Cauchy sequence in X. We may assume that this sequence does not become constant after finitely many steps. Then the

$$\varepsilon_n := \max\{d(x_m, x_{m+1}) : m \geq n\}$$

are strictly positive real numbers satisfying $\varepsilon_n \geq \varepsilon_{n+1}$ and $\varepsilon_n \geq d(x_n, x_{n+1})$. Using Lemma 1.2.ii. we obtain $B_{\varepsilon_n}(x_n) = B_{\varepsilon_n}(x_{n+1}) \supseteq B_{\varepsilon_{n+1}}(x_{n+1})$. By assumption the intersection $\bigcap_n B_{\varepsilon_n}(x_n)$ must contain a point x. We have $d(x, x_n) \leq \varepsilon_n$ for any $n \in \mathbb{N}$. Since the sequence $(\varepsilon_n)_n$ converges to zero this implies that $x = \lim_{x\to\infty} x_n$.

ii. Let $B_1 \supseteq B_2 \supseteq \cdots$ be any decreasing sequence of balls in X. Obviously we have $d(B_1) \geq d(B_2) \geq \cdots$. By our above discussion we only need to consider the case that $\inf_n d(B_n) > 0$. Our assumption on accumulation points implies that $d(B_n) \in D(X \times X)$ for any $n \in \mathbb{N}$ and then in fact that the sequence $(d(B_n))_n$ must become constant after finitely many steps. Hence there exists an $m \in \mathbb{N}$ such that $0 < \varepsilon := d(B_m) = d(B_{m+1}) = \cdots$. By Lemma 1.6 we have, for any $n \geq m$ and any $a \in B_n$, that

$$B_n = B_\varepsilon^-(a) \qquad \text{or} \qquad B_n = B_\varepsilon(a).$$

Moreover, which of the two equations holds is independent of the choice of a by Lemma 1.2.ii. *Case* 1: We have $B_n = B_\varepsilon(a)$ for any $n \geq m$ and any $a \in B_n$. It immediately follows that $B_n = B_m$ for any $n \geq m$ and hence that $\bigcap_n B_n = B_m$. *Case* 2: There is an $\ell \geq m$ such that $B_\ell = B_\varepsilon^-(a)$ for any $a \in B_\ell$. For any $n \geq \ell$ and any $a \in B_n \subseteq B_\ell$ we then obtain $B_\varepsilon^-(a) = B_\ell \supseteq B_n \supseteq B_\varepsilon^-(a)$ so that $B_\ell = B_n$ and hence $\bigcap_n B_n = B_\ell$. $\square$

Exercise. *Suppose that X is complete, and let $B_1 \supset B_2 \supset \cdots$ be a decreasing sequence of balls in X such that $d(B_1) > d(B_2) > \cdots$ and $\inf_n d(B_n) > 0$. Then the subspace $Y := X \setminus (\bigcap_n B_n)$ is complete but not spherically complete.*

Lemma 1.8. *Suppose that X is spherically complete; for any family $(B_i)_{i\in I}$ of closed balls in X such that $B_i \cap B_j \neq \emptyset$ for any $i, j \in I$ we then have $\bigcap_{i\in I} B_i \neq \emptyset$.*

Proof. We choose a sequence $(i_n)_{n\in\mathbb{N}}$ of indices in I such that:

– $d(B_{i_1}) \geq d(B_{i_2}) \geq \cdots \geq d(B_{i_n}) \geq \cdots$,

– for any $i \in I$ there is an $n \in \mathbb{N}$ with $d(B_i) \geq d(B_{i_n})$.

The proof of Lemma 1.6 shows that $B_i = B_{d(B_i)}(a)$ for any $a \in B_i$. Our assumption on the family $(B_i)_i$ therefore implies that:

– $B_{i_1} \supseteq B_{i_2} \supseteq \cdots \supseteq B_{i_n} \supseteq \cdots$,

– for any $i \in I$ there is an $n \in \mathbb{N}$ with $B_i \supseteq B_{i_n}$.

We see that

$$\bigcap_{i \in I} B_i = \bigcap_{n \in \mathbb{N}} B_{i_n} \neq \emptyset.$$

□

2 Nonarchimedean Fields

Let K be any field.

Definition. *A nonarchimedean absolute value on K is a function*

$$| \ | : K \longrightarrow \mathbb{R}$$

which satisfies:

(i) $|a| \geq 0$,

(ii) $|a| = 0$ *if and only if* $a = 0$,

(iii) $|ab| = |a| \cdot |b|$,

(iv) $|a + b| \leq \max(|a|, |b|)$.

Exercise. i. $|n \cdot 1| \leq 1$ *for any* $n \in \mathbb{Z}$.

ii. $| \ | : K^\times \longrightarrow \mathbb{R}_+^\times$ *is a homomorphism of groups*; *in particular*, $|1| = |-1| = 1$.

iii. *K is an ultrametric space with respect to the metric* $d(a, b) := |b - a|$; *in particular, we have* $|a + b| = \max(|a|, |b|)$ *whenever* $|a| \neq |b|$.

iv. *Addition and multiplication on the ultrametric space K are continuous maps.*

Definition. *A nonarchimedean field $(K, | \ |)$ is a field K equipped with a nonarchimedean absolute value $| \ |$ such that*:

(i) $| \ |$ *is non-trivial, i. e., there is an* $a \in K$ *with* $|a| \neq 0, 1$,

(ii) *K is complete with respect to the metric* $d(a, b) := |b - a|$.

The most important class of examples is constructed as follows. We fix a prime number p. Then

$$|a|_p := p^{-r} \quad \text{if } a = p^r \tfrac{m}{n} \text{ with } r, m, n \in \mathbb{Z} \text{ and } p \nmid mn$$

is a nonarchimedean absolute value on the field $\mathbb{Q}$ of rational numbers. The corresponding completion $\mathbb{Q}_p$ is called the *field of p-adic numbers*. Of course, it is nonarchimedean as well. We note that $|\mathbb{Q}_p|_p = p^{\mathbb{Z}} \cup \{0\}$. Hence $\mathbb{Q}_p$ is spherically complete by Lemma 1.7.ii. On the other hand we see that in the ultrametric space $\mathbb{Q}_p$ we can have $B_\varepsilon(a) = B_\delta(a)$ even if $\varepsilon \neq \delta$. To have more examples we state without proof (compare [Se1] Chap. II §§1–2) the following fact. Let $K/\mathbb{Q}_p$ be any finite extension of fields. Then

$$|a| := \sqrt[{[K:\mathbb{Q}_p]}]{|\mathrm{Norm}_{K/\mathbb{Q}_p}(a)|_p}$$

is the unique extension of $|\ |_p$ to a nonarchimedean absolute value on K. The corresponding ultrametric space K is complete and spherically complete and, in fact, locally compact.

In the following we fix a nonarchimedean field $(K, |\ |)$. By the strict triangle inequality the closed unit ball

$$o_K := B_1(0)$$

is a subring of K, called the *ring of integers* in K, and the open unit ball

$$\mathfrak{m}_K := B_1^-(0)$$

is an ideal in o_K. Because of $o_K^\times = o_K \setminus \mathfrak{m}_K$ this ideal $\mathfrak{m}_K$ is the only maximal ideal of o_K. The field $o_K/\mathfrak{m}_K$ is called the *residue class field* of K.

Exercise 2.1. i. *If the residue class field $o_K/\mathfrak{m}_K$ has characteristic zero then K has characteristic zero as well and we have $|a| = 1$ for any nonzero $a \in \mathbb{Q} \subseteq K$.*

ii. *If K has characteristic zero but $o_K/\mathfrak{m}_K$ has characteristic $p > 0$ then we have*

$$|a| = |a|_p^{-\frac{\log |p|}{\log p}} \qquad \textit{for any } a \in \mathbb{Q} \subseteq K;$$

in particular, K contains $\mathbb{Q}_p$.

A nonarchimedean field K as in the second part of Exercise 2.1 is called a *p-adic field*.

Lemma 2.2. *If K is p-adic then we have*

$$|n| \geq |n!| \geq |p|^{\frac{n-1}{p-1}} \qquad \text{for any } n \in \mathbb{N}.$$

Proof. We may obviously assume that $K = \mathbb{Q}_p$. Then the reader should do this as an exercise but also may consult [B-LL] Chap. II §8.1 Lemma 1. □

Now let V be any K-vector space.

Definition. *A (nonarchimedean) norm on V is a function $\| \; \| : V \longrightarrow \mathbb{R}$ such that for any $v, w \in V$ and any $a \in K$ we have:*

(i) $\|av\| = |a| \cdot \|v\|$,

(ii) $\|v + w\| \leq \max(\|v\|, \|w\|)$,

(iii) *if* $\|v\| = 0$ *then* $v = 0$.

Moreover, V is called normed if it is equipped with a norm.

Exercise. i. $\|v\| \geq 0$ *for any* $v \in V$ *and* $\|0\| = 0$.

ii. *V is an ultrametric space with respect to the metric $d(v, w) := \|w - v\|$; in particular, we have $\|v + w\| = \max(\|v\|, \|w\|)$ whenever $\|v\| \neq \|w\|$.*

iii. *Addition $V \times V \xrightarrow{+} V$ and scalar multiplication $K \times V \longrightarrow V$ are continuous.*

Lemma 2.3. *Let $(V_1, \| \; \|_1)$ and $(V_2, \| \; \|_2)$ let two normed K-vector spaces; a linear map $f : V_1 \longrightarrow V_2$ is continuous if and only if there is a constant $c > 0$ such that*

$$\|f(v)\|_2 \leq c \cdot \|v\|_1 \quad \text{for any } v \in V_1.$$

Proof. We suppose first that such a constant $c > 0$ exists. Consider any sequence $(v_n)_{n \in \mathbb{N}}$ in V_1 which converges to some $v \in V_1$. Then $(\|v_n - v\|_1)_n$ and hence $(\|f(v_n) - f(v)\|_2)_n = (\|f(v_n - v)\|_2)_n$ are zero sequences. It follows that the sequence $(f(v_n))_n$ converges to $f(v)$ in V_2. This means that f is continuous.

Now we assume vice versa that f is continuous. We find a $0 < \varepsilon < 1$ such that

$$B_\varepsilon(0) \subseteq f^{-1}(B_1(0)).$$

Since $| \; |$ is non-trivial we may assume that $\varepsilon = |a|$ for some $a \in K$. In other words

$$\|v\|_1 \leq |a| \quad \text{implies} \quad \|f(v)\|_2 \leq 1$$

for any $v \in V_1$. Let now $0 \neq v \in V_1$ be an arbitrary nonzero vector. We find an $m \in \mathbb{Z}$ such that

$$|a|^{m+2} < \|v\|_1 \leq |a|^{m+1}.$$

Setting $c := |a|^{-2}$ we obtain

$$\|f(v)\|_2 = |a|^m \cdot \|f(a^{-m}v)\|_2 \leq |a|^m < c \cdot \|v\|_1.$$

□

Definition. *The normed K-vector space $(V, \| \ \|)$ is called a K-Banach space if V is complete with respect to the metric $d(v, w) := \|w - v\|$.*

Examples. 1) *K^n with the norm $\|(a_1, \ldots, a_n)\| := \max_{1 \leq i \leq n} |a_i|$ is a K-Banach space.*

2) *Let I be a fixed but arbitrary index set. A family $(a_i)_{i \in I}$ of elements in K is called bounded if there is a $c > 0$ such that $|a_i| \leq c$ for any $i \in I$. The set*

$$\ell^\infty(I) := \text{ set of all bounded families } (a_i)_{i \in I} \text{ in } K$$

with componentwise addition and scalar multiplication and with the norm

$$\|(a_i)_i\|_\infty := \sup_{i \in I} |a_i|$$

is a K-Banach space.

3) *With I as above let*

$$c_0(I) := \{(a_i)_{i \in I} \in \ell^\infty(I) : \text{for any } \varepsilon > 0 \text{ we have } |a_i| \geq \varepsilon \text{ for at most finitely many } i \in I\}.$$

It is a closed vector subspace of $\ell^\infty(I)$ and hence a K-Banach space in its own right. Moreover, for $(a_i)_i \in c_0(I)$ we have

$$\|(a_i)_i\|_\infty = \max_{i \in I} |a_i|.$$

For example, $c_0(\mathbb{N})$ is the Banach space of all zero sequences in K.

Remark. *Any K-Banach space $(V, \| \ \|)$ over a finite extension $K/\mathbb{Q}_p$ which satisfies $\|V\| \subseteq |K|$ is isometric to some K-Banach space $(c_0(I), \| \ \|_\infty)$; moreover, all such I have the same cardinality.*

Proof. Compare [NFA] Remark 10.2 and Lemma 10.3. □

Let V and W be two normed K-vector spaces. From now on we denote, unless this causes confusion, all occurring norms indiscriminately by $\| \ \|$. It is clear that

$$\mathcal{L}(V,W) := \{f \in \mathrm{Hom}_K(V,W) : f \text{ is continuous}\}$$

is a vector subspace of $\mathrm{Hom}_K(V,W)$. By Lemma 2.3 the *operator norm*

$$\|f\| := \sup\left\{\frac{\|f(v)\|}{\|v\|} : v \in V, v \neq 0\right\} = \sup\left\{\frac{\|f(v)\|}{\|v\|} : v \in V, 0 < \|v\| \leq 1\right\}$$

is well defined for any $f \in \mathcal{L}(V,W)$.

Lemma 2.4. *$\mathcal{L}(V,W)$ with the operator norm is a normed K-vector space.*

Proof. This is left to the reader as an exercise. □

Proposition 2.5. *If W is a K-Banach space then so, too, is $\mathcal{L}(V,W)$.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}(V,W)$. Then, on the one hand, $(\|f_n\|)_n$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges, of course. On the other hand, because of

$$\|f_{n+1}(v) - f_n(v)\| = \|(f_{n+1} - f_n)(v)\| \leq \|f_{n+1} - f_n\| \cdot \|v\|$$

we obtain, for any $v \in V$, the Cauchy sequence $(f_n(v))_n$ in W. By assumption the limit $f(v) := \lim_{n\to\infty} f_n(v)$ exists in W. Obviously we have

$$f(av) = af(v) \quad \text{for any } a \in K.$$

For $v, v' \in V$ we compute

$$\begin{aligned} f(v) + f(v') &= \lim_{n\to\infty} f_n(v) + \lim_{n\to\infty} f_n(v') = \lim_{n\to\infty}(f_n(v) + f_n(v')) \\ &= \lim_{n\to\infty} f_n(v + v') = f(v + v'). \end{aligned}$$

This means that $v \longmapsto f(v)$ is a K-linear map which we denote by f. Since

$$\|f(v)\| = \lim_{n\to\infty} \|f_n(v)\| \leq (\lim_{n\to\infty} \|f_n\|) \cdot \|v\|$$

it follows from Lemma 2.3 that f is continuous. Finally the inequality

$$\begin{aligned}\|f-f_n\| &= \sup\left\{\frac{\|(f-f_n)(v)\|}{\|v\|} : v \neq 0\right\} \\ &= \sup\left\{\frac{\lim_{m\to\infty}\|f_m(v)-f_n(v)\|}{\|v\|} : v \neq 0\right\} \\ &\leq \lim_{m\to\infty}\|f_m-f_n\| \leq \sup_{m\geq n}\|f_{m+1}-f_m\|\end{aligned}$$

shows that f indeed is the limit of the sequence $(f_n)_n$ in $\mathcal{L}(V,W)$. □

In particular,

$$V' := \mathcal{L}(V,K)$$

always is a K-Banach space. It is called the *dual space* to V.

Lemma 2.6. *Let I be an index set; for any $j \in I$ let $1_j \in c_0(I)$ denote the family $(a_i)_{i\in I}$ with $a_i = 0$ for $i \neq j$ and $a_j = 1$; then*

$$\begin{aligned} c_0(I)' &\xrightarrow{\cong} \ell^\infty(I) \\ \ell &\longmapsto (\ell(1_i))_{i\in I}\end{aligned}$$

is an isometric linear isomorphism.

Proof. We give the proof only in the case $I = \mathbb{N}$. The general case follows the same line but requires the technical concept of summability (cf. [NFA] end of §3). Let us denote the map in question by ι. Because of

$$|\ell(1_i)| \leq \|\ell\| \cdot \|1_i\|_\infty = \|\ell\|$$

it is well defined and satisfies

$$\|\iota(\ell)\|_\infty \leq \|\ell\| \qquad \text{for any } \ell \in c_0(\mathbb{N})'.$$

For trivial reasons ι is a linear map. Consider now an arbitrary nonzero vector $v = (a_i)_i \in c_0(\mathbb{N})$. In the Banach space $c_0(\mathbb{N})$ we then have the convergent series expansion

$$v = \sum_{i\in\mathbb{N}} a_i \cdot 1_i.$$

Applying any $\ell \in c_0(\mathbb{N})'$ by continuity leads to

$$\ell(v) = \sum_{i\in\mathbb{N}} a_i \ell(1_i).$$

We obtain

$$\frac{|\ell(v)|}{\|v\|_\infty} \leq \frac{\sup_i |a_i||\ell(1_i)|}{\sup_i |a_i|} \leq \sup_i |\ell(1_i)| = \|\iota(\ell)\|_\infty .$$

It follows that

$$\|\ell\| \leq \|\iota(\ell)\|_\infty$$

and together with the previous inequality that ι in fact is an isometry and in particular is injective. For surjectivity let $(c_i)_i \in \ell^\infty(\mathbb{N})$ be any vector and put $\varepsilon := \|(c_i)_i\|_\infty$. We consider the linear form

$$\begin{aligned} \ell : c_0(\mathbb{N}) &\longrightarrow K \\ (a_i)_i &\longmapsto \sum_i a_i c_i \end{aligned}$$

(note that the defining sum is convergent). Using Lemma 2.3 together with the inequality

$$|\ell((a_i)_i)| = \left| \sum_i a_i c_i \right| \leq \sup_i |a_i||c_i| \leq \sup_i |a_i| \cdot \sup_i |c_i| = \varepsilon \cdot \|(a_i)_i\|_\infty$$

we see that ℓ is continuous. It remains to observe that

$$\iota(\ell) = (\ell(1_i))_i = (c_i)_i .$$

□

3 Convergent Series

From now on throughout the book $(K, | \ |)$ is a fixed nonarchimedean field.

For the convenience of the reader we collect in this section the most basic facts about convergent series in Banach spaces (some of which we have used already in the proof of Lemma 2.6).

Let $(V, \| \ \|)$ be a K-Banach space.

Lemma 3.1. *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in V; we then have:*

i. *The series $\sum_{n=1}^\infty v_n$ is convergent if and only if $\lim_{n\to\infty} v_n = 0$;*

ii. *if the limit $v := \lim_{n\to\infty} v_n$ exists in V and is nonzero then $\|v_n\| = \|v\|$ for all but finitely many $n \in \mathbb{N}$;*

iii. *let* $\sigma : \mathbb{N} \to \mathbb{N}$ *be any bijection and suppose that the series* $v = \sum_{n=1}^{\infty} v_n$ *is convergent in* V; *then the series* $\sum_{n=1}^{\infty} v_{\sigma(n)}$ *is convergent as well with the same limit* v.

Proof. i. This is immediate from Lemma 1.5. ii. If $v \neq 0$ then $\|v\| \neq 0$ and hence $\|v_n - v\| < \|v\|$ for any sufficiently big $n \in \mathbb{N}$. For these n Lemma 1.1 then implies that

$$\|v_n\| = \|(v_n - v) + v\| = \max(\|v_n - v\|, \|v\|) = \|v\|.$$

iii. We fix an $\varepsilon > 0$ and choose an $m \in \mathbb{N}$ such that

$$\left\| v - \sum_{n=1}^{s} v_n \right\| < \varepsilon \qquad \text{for any } s \geq m.$$

Then also

$$\|v_s\| = \left\| \left(v - \sum_{n=1}^{s-1} v_n \right) - \left(v - \sum_{n=1}^{s} v_n \right) \right\| \leq \max\left(\left\| v - \sum_{n=1}^{s-1} v_n \right\|, \left\| v - \sum_{n=1}^{s} v_n \right\| \right) < \varepsilon$$

for any $s > m$. Setting $\ell := \max\{\sigma^{-1}(n) : n \leq m\} \geq m$ we have

$$\{\sigma^{-1}(1), \ldots, \sigma^{-1}(m)\} \subseteq \{1, \ldots, \ell\}$$

and hence, for any $s \geq \ell$,

$$\{\sigma(1), \ldots, \sigma(s)\} = \{1, \ldots, m\} \cup \{n_1, \ldots, n_{s-m}\}$$

with appropriate natural numbers $n_i > m$. We conclude that

$$\begin{aligned} \left\| v - \sum_{n=1}^{s} v_{\sigma(n)} \right\| &= \left\| \left(v - \sum_{n=1}^{m} v_n \right) - v_{n_1} - \cdots - v_{n_{s-m}} \right\| \\ &\leq \max\left(\left\| v - \sum_{n=1}^{m} v_n \right\|, \|v_{n_1}\|, \ldots, \|v_{n_{s-m}}\| \right) \\ &< \varepsilon \end{aligned}$$

for any $s \geq \ell$. □

The following identities between convergent series are obvious:

– $\sum_{n=1}^{\infty} a v_n = a \cdot \sum_{n=1}^{\infty} v_n$ for any $a \in K$.

– $(\sum_{n=1}^{\infty} v_n) + (\sum_{n=1}^{\infty} w_n) = \sum_{n=1}^{\infty}(v_n + w_n)$.

Lemma 3.2. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} v_n$ be convergent series in K and V, respectively; then the series $\sum_{n=1}^{\infty} w_n$ with $w_n := \sum_{\ell+m=n} a_\ell v_m$ is convergent, and*

$$\sum_{n=1}^{\infty} w_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} v_n\right).$$

Proof. Let $A := \sup_n |a_n|$ and $C := \sup_n \|v_n\|$. The other cases being trivial we will assume that $A, C > 0$. For any given $\varepsilon > 0$ we find an $N \in \mathbb{N}$ such that

$$|a_n| < \frac{\varepsilon}{C} \quad \text{and} \quad \|v_n\| < \frac{\varepsilon}{A} \quad \text{for any } n \geq N.$$

Then

$$\|w_n\| \leq \max_{\ell+m=n} |a_\ell| \cdot \|v_m\| \leq \max\left(C \cdot \max_{\ell \geq N} |a_\ell|, A \cdot \max_{m \geq N} \|v_m\|\right) < \varepsilon$$

for any $n \geq 2N$. By Lemma 3.1.i. the series $\sum_{n=1}^{\infty} w_n$ therefore is convergent. To establish the asserted identity we note that its left hand side is the limit of the sequence

$$W_s := \sum_{n=1}^{s} w_n = \sum_{\ell+m \leq s} a_\ell v_m$$

whereas its right hand side is the limit of the sequence

$$W_s' := \left(\sum_{n=1}^{s} a_n\right)\left(\sum_{n=1}^{s} v_n\right) = \sum_{\ell,m \leq s} a_\ell v_m.$$

It therefore suffices to show that the differences $W_s - W_s'$ converge to zero. But we have

$$\begin{aligned} \|W_s - W_s'\| &= \left\| \sum_{\substack{\ell,m \leq s \\ \ell+m > s}} a_\ell v_m \right\| \leq \max_{\substack{\ell,m \leq s \\ \ell+m > s}} |a_\ell| \cdot \|v_m\| \\ &\leq \max\left(C \cdot \max_{\ell > \frac{s}{2}} |a_\ell|, A \cdot \max_{m > \frac{s}{2}} \|v_m\|\right). \end{aligned}$$

□

Analogous assertions hold true for series $\sum_{n_1,\ldots,n_r=1}^{\infty} v_{n_1,\ldots,n_r}$ indexed by multi-indices in $\mathbb{N} \times \cdots \times \mathbb{N}$. But we point out the following additional fact.