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Carmelo Clavero · José Luis Gracia
Francisco J. Lisbona Editors

# BAIL 2010 - Boundary and Interior Layers, Computational and Asymptotic Methods



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# BAIL 2010 - Boundary and Interior Layers, Computational and Asymptotic Methods

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# Preface

These proceedings contain selected papers associated with the lectures presented at BAIL 2010 (Boundary and Interior Layers – Computational and Asymptotic Methods). This conference was held from 5 to 9 July 2010 at the University of Zaragoza, Spain. The 64 participants came from many different countries, namely: Argentina, China, France, Germany, India, Ireland, Italy, Russia, South Korea, Spain, Sweden, the UK, and the USA. The BAIL series of conferences are the result of an initiative by Professor John Miller, who organized the first three in Dublin in 1980, 1982, and 1984. Subsequent conferences were then held in Novosibirsk (1986), Shanghai (1988), Copper Mountain, Colorado (1992), Beijing (1994), Perth (2002), Toulouse (2004), Göttingen (2006), and Limerick (2008). The next BAIL Conference will be in Pohang, South Korea, in 2012.

Totally 61 lectures were presented at the BAIL 2010, of which 5 were plenary lectures, 17 were given at the mini-symposia Finite Element Methods Using Layer-Adapted Grids and Robust Methods for Time-Dependent Singularly Perturbed Problems, and 39 were contributions on other subjects. The main objective of the BAIL conferences is to bring together researchers interested in boundary and interior layers. This includes mathematicians and engineers who work on their theoretical and numerical aspects, and also those researchers concerned with their application to a variety of areas such as fluid dynamics, semiconductors, control theory, chemical reactions, and porous media.

The lectures presented at the conference showed the diversity of investigations related to these topics. The proceedings provide a unique overview of research into various aspects of singularly perturbed problems and in particular the efficient resolution of boundary and interior layers using numerical methods. They also include examples of applications of this class of problems.

All papers in the proceedings were subjected to a standardized refereeing process. We would like to thank the authors for their cooperation in the publication of their work in this volume of LNCSE and also the anonymous referees for their work and dedication, without which it would have been impossible to produce this publication.

Finally, we wish to thank the sponsors of the conference: the Spanish Government's project MTM2009-07637-E, the Government of Aragón, the University of Zaragoza, and the Instituto Universitario de Matemáticas y Aplicaciones. Our

thanks also go to the members of the Scientific Committee, the organizers of the mini-symposia, all the attendees for their participation in the conference, and the research group Numerical Methods for Partial Differential and Integral Equations for its work in handling all organizational tasks.

January 2010

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# Modeling Acoustic Streaming On A Vibrating Particle

**Rajai S. Alassar**

**Abstract** In this study, we present the details of a Legendre series truncation method where the stream function and vorticity are expanded in terms of associated Legendre functions to calculate the secondary currents induced by a vibrating spherical particle. The time-dependent differential equations which result from the expansions are solved using a Crank-Nicolson numerical scheme.

## 1 Introduction

The phenomenon of secondary currents produced by the vibration of a particle in a fluid has been observed for a long time. A good review on the subject can be found in Kotas et al. [1], Lighthill [2], and Riley [3,4]. The importance of this phenomenon is currently gaining momentum due to the hypothesis of Yoda et al. [5]. Current models of hearing state that a fish directionalizes sound via direct stimulation of macular hair cells by acoustic particle velocity (Shellart and de Munck [6], Rogers et al. [7]). Yoda et al. [5] hypothesize, instead, that the fish ear is an "auditory retina," where macular hair cells are stimulated by acoustically-induced flow velocities (i.e. secondary currents). The densely packed hair cells visualize the flow patterns due to the acoustically induced flow in the complex three-dimensional geometry between the otolith and the macula, much like a tuft visualization. The complex geometry of fish otoliths may help to distinguish flow patterns for sound from different directions. By converting acoustic signals into spatial patterns sampled with extremely high spatial resolution by the macular hair cells, directionalizing sound becomes a pattern recognition problem, not unlike the visual patterns imaged by the retina.

In this paper, the secondary currents caused by the harmonic oscillation of an infinite body of fluid past a spherical particle are calculated by a semi analytical method. The stream function and vorticity are first expanded in terms of associated

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Legendre functions and the resulting time-dependent differential equations are then solved using a Crank-Nicolson numerical scheme. Although no intention is made here to describe the mechanism of fish hearing, the study offers an initial numerical exploration into the relevance of the acoustically-induced flow to directionalization of sound and characterizing the steady streaming region (practically the region that would be sampled by the hair cells next to the sphere which is considered as a simplified geometry of the fish otolith). It is important to mention here that a study on the physics of steady streaming has been conducted by the present author, [8]. The present paper, however, is different in that it presents the mathematics behind the semi-analytical technique used. It shows how some interesting integrals of special functions developed by the author are incorporated and made use of in the context of steady streaming.

We consider a solid spherical particle of diameter $2a$ suspended in an unbounded oscillating incompressible stream, Fig. 1. The unsteady but uniform free-stream exhibits a sinusoidal oscillatory motion. The fluid motion is governed by the conservation principles of momentum and mass which can be expressed by the following equations:

$$\rho\left[\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w} \bullet \nabla)\mathbf{w}\right] = -\nabla p + \mathbf{F} + \mu\nabla^2\mathbf{w} \tag{1}$$

$$\nabla \bullet \mathbf{w} = 0 \tag{2}$$

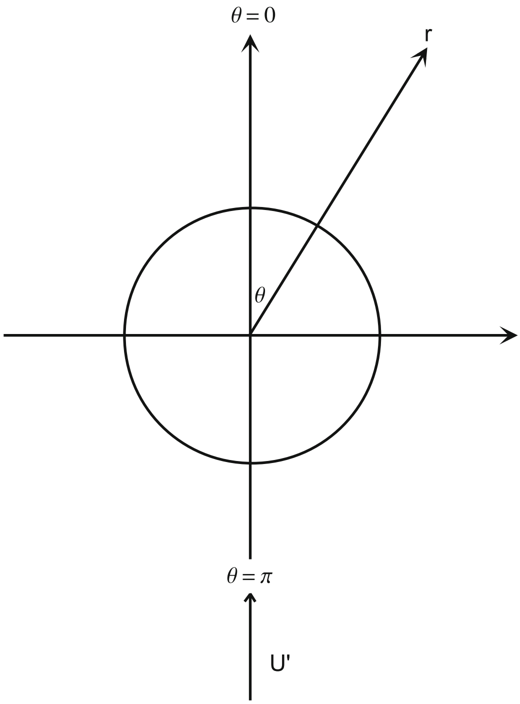


**Fig. 1** Sphere in oscillating stream

where $\rho$ is the fluid density, $t$ is time, **w** is the velocity vector, $p$ is the pressure in the fluid, **F** is the body force vector, and $\mu$ is the dynamic viscosity.

## 2 Method of Solution

First, we recast the equations governing the flow process (1–2) in spherical coordinates. The equations governing, in spherical coordinates, can be written in terms of the dimensionless vorticity ($\zeta$) and the dimensionless stream function ($\psi$) as:

$$e^{3\xi}\sin\theta\,\zeta + \frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\theta^2} - \frac{\partial\psi}{\partial\xi} - \cot\theta\frac{\partial\psi}{\partial\theta} = 0 \tag{3}$$

$$\begin{aligned} e^{2\xi}\frac{\partial\zeta}{\partial \mathrm{t}} + \frac{e^{-\xi}}{\sin\theta}\left[\frac{\partial\psi}{\partial\theta}\left(\frac{\partial\zeta}{\partial\xi} - \zeta\right) - \frac{\partial\psi}{\partial\xi}\left(\frac{\partial\zeta}{\partial\theta} - \cot\theta\,\zeta\right)\right] \\ = \frac{2}{Re}\left[\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial^2\zeta}{\partial\theta^2} + \frac{\partial\zeta}{\partial\xi} + \cot\theta\frac{\partial\zeta}{\partial\theta} - \frac{\zeta}{\sin^2\theta}\right] \end{aligned} \tag{4}$$

where $Re = 2aU_o/\nu$ is the Reynolds number, $U_o$ is the amplitude of the free-stream velocity, and $\nu$ is the coefficient of kinematic viscosity. The logarithmic transformation $\xi = \ln(r/a)$ is used, where $r$ is the dimensional radial distance. The variables $\psi$, $\zeta$, and $t*$(the star is dropped in 3–4) in the governing equations are defined in terms of the usual dimensional quantities $\psi'$, $\zeta'$, and $t$ as: $\psi = \psi'/U_o a^2$, $\zeta = \zeta' a/U_o$, and $t* = U_o\, t/a$.

The oscillations of the free-stream velocity are given in the form $U = U'/U_o = \cos(S\ t)$ where $U'$ is the dimensional free-stream velocity, and $S = a\omega/U_o$ is the Strouhal number with $\omega$ being the frequency of oscillations.

The boundary conditions to be satisfied are the no slip and impermeability conditions on the surface of the sphere and the free-stream conditions away from it. These can be written as:

$$\psi = \frac{\partial\psi}{\partial\theta} = \frac{\partial\psi}{\partial\xi} = 0 \qquad at \quad \xi = 0 \tag{5}$$

$$\left.\begin{array}{l} \frac{\partial\psi}{\partial\xi} \to e^{2\xi}\sin^2\theta \quad \cos(S\,t) \quad ,\ and \quad \frac{\partial\psi}{\partial\theta} \to e^{2\xi}\sin\theta\ \cos\theta\ \ \cos(S\,t) \\ or,\ \psi \to \frac{e^{2\xi}}{2}\sin^2\theta \quad \cos(S\,t) \\ \zeta \to 0 \qquad\qquad , \end{array}\right\} as\ \xi \to \infty \tag{6}$$

In order to solve the governing equations subject to the boundary conditions, we adopt a series truncation method based on expanding $\psi$ and $\zeta$ using Associated Legendre polynomials, Alassar et al. [9], as:

$$\left\{\begin{array}{c}\psi \\ \zeta\end{array}\right\} = \left\{\begin{array}{c}\sum\limits_{n=1}^{\infty} f_n(\xi,t)\int\limits_z^1 P_n(\gamma)\,d\gamma \\ \sum\limits_{n=1}^{\infty} g_n(\xi,t)\ P_n^1(z)\end{array}\right\} \tag{7}$$

where $P_n(z)$ *and* $P_n^1(z)$ are the Legendre and first associated Legendre polynomials of order n respectively, and $z = \cos\theta$. The integrals needed to undergo the transformation of the differential equations onto the modes of the series (7) can be obtained using an approach similar to that reported by Mavromatis and Alassar [10].

The Legendre function $P_n(x)$, as known to physicists, usually arises in studies of systems with three dimensional spherical symmetry. They satisfy the differential equation $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$, and the orthogonality relation $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0 \;\; for\ n \neq m$. The first associated Legendre function $P_n^1(x)$ is a special case of the more general associated Legendre functions (not necessarily polynomials) $P_n^m(x)$ which are obtained from derivatives of the Legendre polynomials according to $P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m\,P_n(x)}{dz^m}$. Notice that $P_n^m(x)$ reduce to $P_n(x)$ for $m = 0$.

Substituting from (7) into (3–4) and integrating over z from $-1$ to 1, the following expressions can be obtained by manipulation of the Legendre functions,

$$\frac{\partial^2 f_n}{\partial \xi^2} - (n + 1/2)^2 f_n = n(n+1)\, e^{5/2\xi}\, g_n \tag{8}$$

$$e^{2\xi}\frac{\partial g_n}{\partial t} = \frac{2}{Re}\left[\frac{\partial^2 g_n}{\partial \xi^2} + \frac{\partial g_n}{\partial \xi} - n(n+1)\, g_n\right] + S_n \tag{9}$$

where,

$$S_n = -e^{-\xi/2}\left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\alpha_{ij}^n\, f_i(\frac{\partial g_j}{\partial \xi} - g_j) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\beta_{ij}^n\, g_j(\frac{\partial f_i}{\partial \xi} + \frac{1}{2}f_i)\right] \tag{10}$$

$$\alpha_{ij}^n = -(2n+1)\sqrt{\frac{j(j+1)}{n(n+1)}}\quad \begin{pmatrix} n & i & j \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} n & i & j \\ 0 & 0 & 0 \end{pmatrix} \tag{11}$$

$$\beta_{ij}^n = (2n+1)\sqrt{\frac{j(j^2-1)(j+2)}{n(n+1)\,i(i+1)}}\quad \begin{pmatrix} n & i & j \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} n & i & j \\ 0 & 0 & 0 \end{pmatrix} \tag{12}$$

and $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ are the 3-j symbols.

The power of this technique is evident through the fact that the series expansions resulted in the elimination of the independent variable ($\theta$). The governing equations are now written in the form of a set of differential equations with the dependent variables being the coefficients ($f_n$ , $g_n$) of the series. The resulting equations represent two sets of differential equations, with every set containing infinite number of equations, as compared to the original two partial differential equations. However, we will solve only few of these equations and yet obtain a highly accurate solution.

In the process of obtaining (8–9), one encounters integrals such as $\int_{-1}^{1} P_n^k(z)\, P_m^k(z)\, dz$, $\int_{-1}^{1} P_n^1(z)\; P_i^1(z)\; P_j(z)\; dz$, $\int_{-1}^{1} P_n^1(z)\; P_i^1(z)\; P_j^2(z)\; dz$, and others. These integrals make it possible to eliminate the angular direction $\theta$. They are of the general form:

$$\int_{-1}^{1} P_{j_1}^{m_1}(z)\; P_{j_2}^{m_2}(z)\; P_{j_3}^{m_3}(z)\; dz \tag{13}$$

These integrals are very essential and can be obtained from the following relation:

$$\begin{aligned}\int_{-1}^{1} P_{j_1}^{m_1}(z)\; P_{j_2}^{m_2}(z)\; P_{j_3}^{m_3}(z)\; dz = &\sqrt{\frac{(j_2+m_2)!(j_1+m_1)!}{(j_2-m_2)!(j_1-m_1)!}}\\ &\times\sum_n\Bigg[(-1)^{m_1+m_2}(2n+1)\begin{pmatrix} j_1 & j_2 & n\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} j_1 & j_2 & n\\ m_1 & m_2 & -m_1-m_2\end{pmatrix}\\ &\times\sqrt{\frac{(n-m_1-m_2)!}{(n+m_1+m_2)!}}\quad\int_{-1}^{1} P_{j_3}^{m_3}(z)\, P_n^{m_2+m_1}(z)\, dz\Bigg]\end{aligned} \tag{14}$$

where $|j_1 - j_2| \le n \le j_1 + j_2$ , and

$$\begin{aligned}\int_{-1}^{1} P_{j_1}^{m_1}(z)\, P_{j_2}^{m_2}(z)\, dz = &\frac{(-1)^{m_2}\pi}{2^{2(|m_2-m_1|)+1}\,\Gamma(\frac{1}{2}+\frac{|m_2-m_1|}{2})\,\Gamma(\frac{3}{2}+\frac{|m_2-m_1|}{2})}\sqrt{\frac{(j_1+m_1)!(j_2+m_2)!}{(j_1-m_1)!(j_2-m_2)!}}\\ &\times\sum_k(-1)^{-m_1+m_2}(2k+1)\begin{pmatrix} j_1 & j_2 & k\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} j_1 & j_2 & k\\ -m_1 & m_2 & m_1-m_2\end{pmatrix}\\ &\times(1+(-1)^{k+|m_2-m_1|})\sqrt{\frac{(k+|m_2-m_1|)!}{(k-|m_2-m_1|)!}}\\ &\times{}_3F_2\left[\frac{|m_2-m_1|+k+1}{2}, \frac{|m_2-m_1|-k}{2}, \frac{|m_2-m_1|}{2}+1; |m_2-m_1|\;+1, \frac{3+|m_2-m_1|}{2}; 1\right]\end{aligned} \tag{15}$$

where, $|j_1 - j_2| \le k \le j_1 + j_2$ , $\Gamma$ is the Gamma function, and ${}_3F_2$ is the generalized hypergeometric function. A detailed discussion on these integrals can be found in Mavromatis and Alassar [10] who showed that the hypergeometric function in (15) is always a finite series, and indeed is also Saalschutzian, i.e.

$$\begin{aligned}&{}_3F_2\left[\frac{|m_2-m_1|+k+1}{2}, \frac{|m_2-m_1|-k}{2}, \frac{|m_2-m_1|}{2}+1; |m_2-m_1|\;+1, \frac{3+|m_2-m_1|}{2}; 1\right] =\\ &\frac{\Gamma(1/2)\Gamma(k/2)\Gamma(|m_2-m_1|+1)\Gamma(-k/2-1/2)}{\Gamma((|m_2-m_1|-k)/2+1/2)\,\Gamma(|m_2-m_1|/2)\,\Gamma((|m_2-m_1|+k)/2+1)\,\Gamma(-|m_2-m_1|/2-1/2)}\end{aligned} \tag{16}$$

The 3-$j$ symbols $\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}$ are transformation coefficients that appear in the problem of adding angular momenta. They represent the probability amplitude that three angular momenta $j_1$, $j_2$, and $j_3$ with projections $m_1$, $m_2$, and $m_3$ are coupled to yield zero angular momentum. They are related to the famous Clebsch-Gordan coefficients (C). These symbols, however, possess simpler symmetry properties. The relation between the 3-$j$ symbols and the Clebsch-Gordan coefficients is given by:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_3+m_3+2j_1} \frac{1}{\sqrt{2j_3+1}} C^{j_3 m_3}_{j_1 - m_1\, j_2 - m_2} \tag{17}$$

Many representations of the 3-j symbols are available. They may be represented by the square 3×3 array of the Regge R-symbol, by algebraic sums, or in terms of the generalized hypergeometric function of unit argument (${}_3F_2$). The following formula should give a flavor of the many representations available:

$$C^{c\gamma}_{a\alpha b\beta} = \delta_{\gamma,\alpha+\beta} \frac{\Delta(abc)}{(a+b-c)!(-b+c+\alpha)!(-a+c-\beta)!} \left[\frac{(a+\alpha)!(b-\beta)!(c+\gamma)!(c-\gamma)!(2c+1)!}{(a-\alpha)!(b+\beta)!}\right]^{\frac{1}{2}}$$
$$\times {}_3F_2 \left[ \begin{matrix} -a-b+c, -a+\alpha, -b-\beta \\ -a+c-\beta+1, -b+c+\alpha+1 \end{matrix} \middle| 1 \right] \tag{18}$$

where,

$$\Delta(abc) = \left[\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!}\right]^{\frac{1}{2}} \tag{19}$$

For detailed discussion, representations, properties, and tabulated values, the reader is referred to Varshalovich et al. [11, pp. 235–411]. The 3-j symbols can also be obtained through the famous software MATHEMATICA.

The boundary conditions (5–6) are transferred on to the modes of the series (7) by utilizing the same process by which the differential equations are treated with. The boundary conditions can now be written as:

$$f_n(0,t) = \frac{\partial f_n}{\partial \xi}(0,t) = 0 \tag{20}$$

$$f_n(\xi,t) \to e^{3/2\xi} \cos(St)\ \delta_{n1}\,, \quad \frac{\partial f_n(\xi,t)}{\partial \xi} \to \frac{3}{2}\, e^{3/2\xi} \cos(St)\ \delta_{n1}\ \ as \quad \xi \to \infty \tag{21}$$

$$g_n(\xi,t) \to 0 \qquad as \quad \xi \to \infty \tag{22}$$

where $\delta_{ij}$ is the Kronecker delta.

Finally, an integral condition based on (8) to be satisfied by the functions $g_n$ can be obtained after making use of the boundary conditions (20–22) as:

$$\int_0^\infty e^{(2-n)\,\xi} g_n\, d\xi = \frac{3}{2} \cos(St)\, \delta_{n1} \tag{23}$$

The solutions of the functions $\psi$ and $\zeta$ are advanced in time by first solving (9) using a Crank-Nicolson finite-difference scheme similar to that used by Dennis et al. [12].

Since the problem is solved numerically the conditions at $\infty$ are applied at $\xi = \xi_m$ where $\xi_m$ defines the distance away from the sphere at which $\zeta$ has negligible value. Equation (9), when written in difference form using the Crank-Nicolson finite difference scheme and applied at every mesh point in the range from $\xi = 0\ to\ \xi = \xi_m$, will result in a set of algebraic equations that forms a tridiagonal matrix problem which is solved for each value of n between 1 and N iteratively. N designates the number of terms taken in the series defined in (7). The boundary conditions $g_n(0,t)$ which are needed to complete the integration procedure are obtained by writing the integral condition defined in (19) as a numerical quadrature formula which then relates the boundary value to values of the corresponding function at internal points of the computational domain. This gives the extra condition needed to determine the boundary values for $g_n$ and thus the formulation of the solution of (9) is complete.

A straightforward finite-difference solution for (8) results in an unstable solution especially for large n. Therefore, the solution of these equations is obtained using a step-by-step integration scheme modified from that used by Badr et al. [13]. The method is based on splitting (8) into two first order differential equations one of which is integrated by a stable method in the direction of increasing $\xi$ while the other is integrated in the backward direction from $\xi = \infty$ to $\xi = 0$. The method is well explained by Badr et al. [13] and can be easily modified to suit our problem and need not be discussed further.

The whole iterative numerical scheme can be summarized as follows:

At time $t$, the known solution at time $(t - \Delta t)$ is used as a starting solution. The tridiagonal system resulting from (9) with the most recently available information is solved to obtain the functions $g_n(\xi,t)$. Secondly, we apply the integral condition (19) to obtain a better approximation for $g_n(0,t)$. Then, (8) is solved using the stable step-by-step numerical procedure mentioned above to obtain $f_n(\xi,t)$. The procedure is then repeated until convergence is reached. The condition set for convergence is $\left|g_n^{m+1}(\xi) - g_n^m(\xi)\right| < 10^{-10}$ where m denotes the iteration number. Time is then incremented and the whole process is repeated.

Following the start of fluid motion, very small time steps were used since the time variation of vorticity is quite fast. As time increases, the time step was gradually increased. Smaller time steps were used for higher Strouhal numbers. The number of points in the $\xi$ direction used is 201 with a space step of 0.025. This makes $\xi_m = 5$ which sets the outer boundary at a physical distance of approximately 148 times the radius of the sphere. This is necessary to ensure that the conditions at infinity are appropriately incorporated in the numerical solution. The effect of $\xi_m$ on the flow field near the sphere was examined by comparing the results when using different values of $\xi_m$. The effect of the step size on the flow field near the sphere was also examined by comparing the results when using different values. No significant changes in the values of the drag or the surface vorticity were detected by reducing the step size further than the given value. As there is no intrinsic way to determine them, the total number of terms taken in the series was found by numerical experiments. The number of terms taken in the series starts with only 3 terms. One more term is added when the last term in the series exceeds $10^{-6}$. The total number of terms is dependent on Reynolds and Strouhal numbers. More terms are needed for high Reynolds and low Strouhal numbers.

One last modification is taken here through defining a dimensionless time $\tau$ which is related to the previously defined dimensionless time $t$ by

$$\tau = St/2\pi \tag{24}$$

Scaling time by the Strouhal number is appropriate in dealing with relatively high-frequency flows. Consequently, each cycle has a period of unity with 400 divisions and $\Delta\tau = 0.0025$.

The accuracy of the method of solution was verified by Alassar et al. [9] through comparisons with the forced and mixed convection cases available in the literature such as Wong et al. [14], Sayegh and Gauvin [15], Dennis and Walker [16], and others. The comparisons were satisfactory.

Figure 2 shows the secondary currents calculated by the present method for the cases Re $= 5, 50$, and 200 with S $= \pi/4$ and a photo from experiments by Kotas et al. [1].

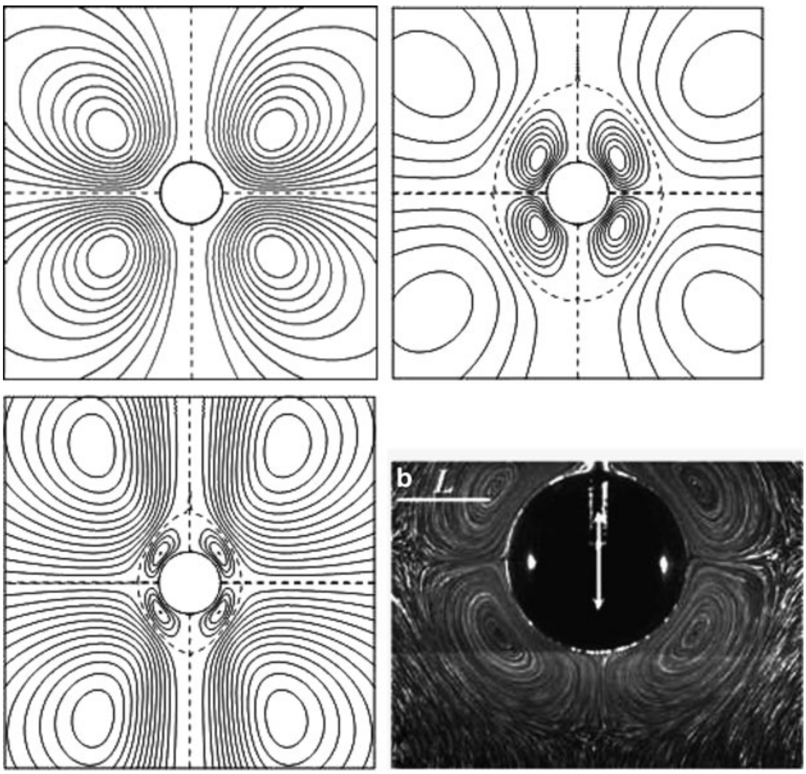


**Fig. 2** Secondary currents for the cases Re = 5, 50, and 200 with S = $\pi/4200$, and a photo from experiments by Kotas et al. [1]

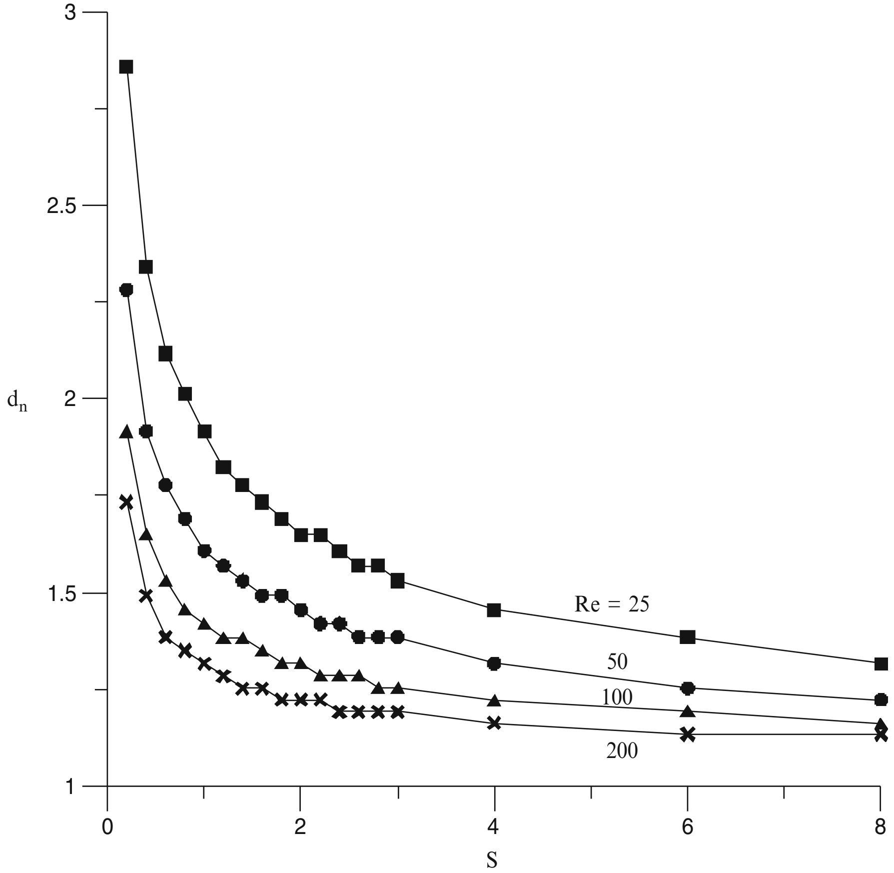


**Fig. 3** Variation of $d_n$ with S

An important characteristic length is the distance from the center of the sphere to the center of the near (inner) recirculation region $d_n$. Figure 3 shows the variation of $d_n$ with Strouhal and Reynolds numbers. As S increases, the distance from the center of the sphere to the center of the inner rotating region (stagnation point) becomes smaller for all Re cases. Obviously, $d_n$ is smaller for higher Reynolds numbers.

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# Performance of Stabilized Higher-Order Methods for Nonstationary Convection-Diffusion-Reaction Equations

**Markus Bause**

**Abstract** We study the performance properties of a class of stabilized higher-order finite element approximations of convection-diffusion-reaction models with nonlinear reaction mechanisms. Streamline upwind Petrov-Galerkin (SUPG) stabilization together with anisotropic shock-capturing as an additional stabilization in crosswind-direction is used. We show that these techniques reduce spurious oscillations in crosswind-direction and increase the accuracy of simulations.

## 1 Introduction

Time-dependent convection-diffusion-reaction equations

$$\partial_t u + \boldsymbol{b} \cdot \nabla u - \nabla \cdot (a \nabla u) + r(u) = f \tag{1}$$

are often studied in various technical and environmental applications. Here, $u = u(\boldsymbol{x}, t)$ denotes the unknown where $\boldsymbol{x} \in \Omega \subset \mathbb{R}^d$, with $d \geq 2$, and $t \in (0, T)$ for some $T > 0$. Further, $a \in L^\infty(0, T; W^{1,\infty}(\Omega))$ is the diffusion coefficient, $\boldsymbol{b} \in L^\infty(0, T; \boldsymbol{W}^{1,\infty}(\Omega))$ is the velocity field, $r \in C^1(\mathbb{R}_0^+)$ is the parametrization of the reaction rate and $f \in L^2(0, T; L^2(\Omega))$ is a prescribed right-hand side term. We suppose that $\nabla \cdot \boldsymbol{b}(\boldsymbol{x}, t) = 0$ and $a(x, t) \geq \alpha > 0$ almost everywhere. Throughout the paper we use standard notation.

The accurate numerical approximation of (1) is still a challenging task. In applications, the transport equation (1) is often convection- and/or reaction-dominated and characteristic solutions have sharp layers. In these cases standard finite element methods cannot be applied. Stabilized finite element approaches are required. For a review of these techniques we refer to the recent work of John and Schmeyer [3].

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Stabilization methods are well-understood for linear steady convection-diffusion-reaction problems; cf., e.g., [3, 4]. However, there is still a considerable lack in the analysis, design and application of these methods for unsteady nonlinear problems which is addressed here. Rigorous analyses are rare for the unsteady and nonlinear case.

## 2 Discretization Scheme

Equipping (1) with initial and homogeneous Dirichlet boundary conditions and discretizing (1) in time by the $\theta$-scheme, with $\theta \in (0, 1]$, leads to a sequence of stationary boundary value problems: *Find* $\{u^k\}_{k=1}^N$ *such that*

$$\alpha_k u^k + \theta \boldsymbol{b}(t_k) \cdot \nabla u^k - \theta \nabla \cdot (\boldsymbol{a}(t_k) \nabla u^k) + \theta r(u^k) = \tilde{f}^k \quad \text{in } \Omega\,, \tag{2}$$

*with* $\tilde{f}^k = \alpha_k u^{k-1} + \theta f(t_k) + (1-\theta) f(t_{k-1}) - (1-\theta)\boldsymbol{b}(t_{k-1}) \cdot \nabla u^{k-1} + (1-\theta)\nabla \cdot (\boldsymbol{a}(t_{k-1}) \nabla u^{k-1}) - (1-\theta) r(\boldsymbol{u}^{k-1})$, $\alpha_k = 1/(t_k - t_{k-1})$ *and* $u^k = 0$ *on* $\partial\Omega$, $u^0 = u(t_0)$.

In the sequel, we suppose that the solution $u$ of (1) is non-negative and bounded from above, i.e., $0 =: u_0 \leq u \leq u_1$ almost everywhere in $\Omega \times (0, T)$, which is admissible from the sake of physical realism, for instance, if $u$ denotes the concentration of a chemical species. We make the assumption that

$$r \in C^1(\mathbb{R}_0^+)\,, \qquad r(0) = 0\,, \qquad r'(s) \geq r_0 \geq 0 \quad \text{for } s \geq 0\,,\ s \in \mathbb{R}\,. \tag{3}$$

To calculate approximations of $\{u^k\}_{k=1}^N$, a standard $hp$-version of the finite element method is assumed; cf. [1, 4, 7]. For a family of admissible and shape-regular triangulations $\mathcal{T}_h = \{T\}$ of the polyhedral domain $\Omega \subset \mathbb{R}^d$ let

$$V_h^{\boldsymbol{p}} = X_h^{\boldsymbol{p}} \cap H_0^1(\Omega) \quad \text{with} \quad X_h^{\boldsymbol{p}} = \{v \in C(\overline{\Omega}) \mid v_{|T} \circ F_T \in \mathcal{P}_{p_T}(\widehat{T})\ \forall T \in \mathcal{T}_h\}$$

denote the underlying finite element space of piecewise polynomials of local order $p_T$ for all $T \in \mathcal{T}_h$. Here, $\widehat{T}$ is the (open) unit simplex or the (open) unit hypercube in $\mathbb{R}^d$ and $\mathcal{P}_n(\widehat{T})$, with $n \geq 1$, is the set of all polynomials of degree at most $n$ on $\widehat{T}$. We assume that each $T \in \mathcal{T}_h$ is a smooth bijective image of $\widehat{T}$, i.e., $T = F_T(\widehat{T})$. The vector $\boldsymbol{p}$ is defined by $\boldsymbol{p} = \{p_T \mid T \in \mathcal{T}_h\}$. In our analysis the local inverse inequalities

$$\|\nabla w_h\|_{L^2(T)} \leq \mu_{\text{inv}} p_T^2 h_T^{-1} \|w\|_{L^2(T)} \quad \forall w_h \in X_h^{\boldsymbol{p}} \quad \text{on } T \in \mathcal{T}_h \tag{4}$$

are applied. Here, $\mu_{\text{inv}}$ depends on the shape-regularity parameter; cf. [7].

Skipping for brevity the indices in (2), the SUPG-stabilized approximation of (2) is: *Find* $u_h \in V_h^{\boldsymbol{p}}$ *such that*

$$A_s(u_h, v_h) = L_s(v_h) \tag{5}$$