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Hyperfinite Dirichlet Forms and Stochastic Processes

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In memory of the dear father, Mr. Decai Li (1926–2003), of the second author.

Preface

The theory of stochastic processes has developed rapidly in the past decades. Martingale theory and the study of smooth diffusion processes as solutions of stochastic differential equations have been extended in several directions, such as the study of infinite dimensional diffusion processes, the study of diffusion processes with non-smooth unbounded coefficients, diffusion processes on manifolds and on singular spaces. The interplay between stochastic analysis and mathematical physics has been one of the most important and exciting research areas.

One of the best techniques to deal with the problems of these areas is Dirichlet space theory. In the original framework of this theory, the state space is a locally compact separable metric space, e.g., $\mathbb{R}^d$, or a d-dimensional manifold. This theory has given us a nice understanding about the property of diffusion processes with non-smooth unbounded coefficients. Moreover, it has been fruitfully applied to mathematical physics. This framework has been generalized to state spaces which are more general topological spaces or some infinite dimensional vector spaces or manifolds. Several key problems, such as the closability of quadratic forms and the construction of strong Markov processes associated with quasi-regular Dirichlet forms, have been solved. The study of infinite dimensional stochastic analysis as well as the study of processes on singular structures (like fractals, trees, or general metric spaces) has enriched and extended the Dirichlet space theory. In the meantime, a new framework has been introduced into Dirichlet space theory by the development of nonstandard probabilistic analysis [25, 166]. As is well-known, nonstandard analysis is an alternative setting for analysis (and, indeed, all areas of mathematics), namely by enriching the set of real numbers by infinitesimal and infinite elements. It has its origin in seminal work by Schmieden, Laugwitz [325] and most notably Robinson [310]. By now, several textbooks and surveys exist on this theory and its applications (see, e.g. [25, 63, 125, 217]). Nonstandard analysis gives a novel approach to the theory of stochastic processes. In particular, it has led to hyperfinite symmetric Dirichlet space theory. Besides being interesting by itself, it has also many applications. In the first part of the book, we extend the research to the nonsymmetric case, and remove some restrictive conditions in the previous treatment of the subject (Chap. 5 of [25]). In addition, we shall apply the theory to present a new approach to infinite dimensional stochastic analysis.

In writing this book we have two main aims: (1) to give a presentation of research on nonsymmetric hyperfinite Dirichlet space theory and its applications in (standard) finite and infinite dimensional stochastic analysis, Chaps. 1–4; (2) to find nonstandard representations for a special class of (finite dimensional) Feller processes and their infinitesimal generators, viz. stochastically continuous processes with stationary and independent increments (i.e., *Lévy processes*), Chap. 5. Chapter 6 is a complement to illustrate the usefulness of the hyperfinite probability spaces. The first part (Chaps. 1–4) is based on Chap. 5 of Albeverio et al. [25] and the further in depth research of Sergio and Ruzong; the second part (Chaps. 5–6) is based on results obtained recently by Tom Lindstrøm and their extensions by Sergio and Frederik.

As mentioned earlier, the interplay between stochastic analysis and mathematical physics has been one of the most important and exciting themes of research in the last decades. This is already a sufficient rationale for the research of the first part of the present book. The motivation for including the second part, Chap. 5, into this book is that many of the issues discussed in the more general framework of the first part, such as existence of standard parts of hyperfinite Markov chains, become much less technical to resolve for hyperfinite Lévy processes. Furthermore, the more restrictive setting of the second part also allows one to obtain finer results on the relation between Lévy processes and their hyperfinite analogues, one example being a hyperfinite version of the Lévy–Khintchine formula.

The contents of this book are arranged as follows: In Chap. 1, we introduce the framework of hyperfinite Dirichlet forms. We develop the potential theory of hyperfinite Dirichlet forms in Chap. 2. In Chap. 3, we consider standard representations of hyperfinite Markov chains under certain conditions, and translate the conditions on hyperfinite Markov chains into the language of hyperfinite Dirichlet forms. As an interesting and important application in classical stochastic analysis, we construct tight dual strong Markov processes associated with quasi-regular Dirichlet forms by using the language of hyperfinite Dirichlet forms in Chap. 4. The results show that hyperfinite Dirichlet space theory is a powerful tool to study classical problems. In the first sections of Chap. 5, the notion of a hyperfinite Lévy process is introduced and its relation to hyperfinite random walks as well as to standard Lévy processes is investigated. These results can be used to show that the jump part of any Lévy process is essentially a hyperfinite convolution of Poisson processes. Finally, Chap. 6 is an epilogue, providing a rigorous motivation for the study of hyperfinite Loeb path spaces as generic probability spaces.

The entire book is based on nonstandard analysis. For the reader's convenience, we present some basic notions of nonstandard analysis, such as internal sets and saturation, linear spaces, Loeb measure spaces, structure of $^*\mathbb{R}$ and topology in the appendix. Because of its monographical character centered around the hyperfinite approach, the book has by no means the goal of including all aspects of recent developments in the theory of stochastic processes and its connections with Dirichlet forms theory or the theory of Lévy processes. For this, we rather refer to surveys and proceedings like Albeverio [2], Barndorff-Nielsen et al. [73], and Ma et al. [275], respectively.

The germ of this book goes back to the year 1989 when the second author, Ruzong Fan, worked on the construction of symmetric Markov processes associated with Dirichlet forms at Peking University, Beijing ([165] and Chap. 4). At that time, Ruzong was unaware that Sergio's group was working on the same project using standard methods [41]. The second author, Dr. Zhiming Ma, of [41] did privately inquire Ruzong about the progress of Ruzong's research in 1989 at the Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing. In response to Dr. Ma's request of a private meeting, Ruzong presented his work to Dr. Ma in a classroom with Dr. Ma as the only audience. Dr. Ma, however, did not mention his ongoing work with Sergio in any way. Thus, Ruzong was totally unaware of Sergio's research. In the spring of 1990, Ruzong first realized this when he saw a manuscript of Albeverio and Ma [41] in Beijing with a surprise. These events notwithstanding, Ruzong continued to work on a "symmetric version" of Chaps. 1–4 using non-standard language when he was at Peking University till 1991 and when he visited the Humboldt-University, Berlin, between 1991 and 1992. Under Sergio's supervision and encouragement, Ruzong extended the project to the current "nonsymmetric version" from 1992 to 1994 at Ruhr-University, Bochum. In 2006, Frederik kindly joined the project with a contribution on hyperfinite Lévy processes (Chap. 5) and the Epilogue (Chap. 6). In the summer of 2006, the three authors gathered at the University of Bonn to finalize this monograph. We gratefully acknowledge the manifold support of various institutions in the long process of work on this project.

In the run-up to its completion, Sergio and Frederik were supported partially by the collaborative research center SFB 611 of the German Research Foundation (DFG), Germany; in addition, Ruzong's visit to Bonn was partially funded through a research fellowship from the Alexander von Humboldt Foundation, Germany.

Over the course of his career, Ruzong has received a lot of generous support from Sergio. As a Ph.D candidate in Beijing around 1987–1988, Ruzong was greatly fascinated by Sergio and Raphael Høegh-Krohn's novel work on infinite dimensional stochastic analysis, in which Ruzong finished his Ph.D thesis. Unfortunately, Ruzong got no chance to meet Raphael Høegh-Krohn; right before Ruzong went to Europe, he was shocked to learn that Raphael

Høegh-Krohn died of a heart attack. In a relatively isolated environment, Ruzong mostly worked on himself by reading numerous papers and books of Sergio and Raphael Høegh-Krohn; and many times, Ruzong had to spend a few days on a single equation or lemma to guess and to understand it. Whilst it seemed like a helpless or hopeless situation for Ruzong at that time, Ruzong eventually came to the forefront of research in areas of infinite dimensional stochastic analysis: he studied the hard and central questions regarding Beurling–Deny formulae, representation of martingale additive functionals and absolute continuity of symmetric diffusion processes on Banach spaces, potential theory of symmetric hyperfinite Dirichlet forms, and construction of the symmetric strong Markov processes associated with quasi-regular Dirichlet forms by using the non-standard analysis language. This direction of research was initiated by Sergio, although Ruzong was unaware that Sergio's group already worked on the construction of Markov processes using the language of standard stochastic analysis.

In early 1989, Ruzong applied for a fellowship from the Alexander von Humboldt Foundation from Peking University, Beijing; soon after a rejection from the Foundation in the fall 1989, Ruzong received a warm letter from Sergio with encouragement and a kind offer to nominate, as an academic host, Ruzong for the fellowship and by writing a strong letter of recommendation. This is just one anecdote to illustrate how Ruzong has constantly been able to count on Sergio's help via communications by either mail or face-to-face conversations starting from 1989. Between 1992 and 1994, Sergio generously supported Ruzong at Ruhr-University Bochum to complete the main part of Chaps. 1–4 of this monograph, and helped Ruzong to pass the hard period of time in his career.

The story of Ruzong is an example how Sergio has helped many young mathematicians to grow and to mature. Quite probably, Ruzong would have disappeared from academia a long time ago without the support of Sergio. In a true sense, Sergio has been an academic father figure for Ruzong when he desperately needed one. In recent years, after his departure from Sergio's research group, Ruzong has been mainly working on statistical genetics guided by his beloved American mentor, Dr. Kenneth Lange, at the University of Michigan and UCLA. Nevertheless, Ruzong has fond memories and deep appreciation of numerous communications with his European academic father Sergio; and both Ruzong and Frederik are deeply grateful for Sergio's mentoring.

Thus, especially right after Sergio's 70th birthday in 2009 − which also marks the 50th anniversary of his remarkable scientific career −, Ruzong and Frederik are sure that they will be joined by many other young mathematicians in thanking Sergio for his wonderful role in our professional and personal development and in wishing him all the best for the rest of his life: Not just continued productivity, but most of all good health, happiness, joy, and peace.

We owe a huge debt of gratitude to our families: In the summer of 2006, Dr. Li Zhu (Ruzong's wife) kindly took care of two young children when her husband was visiting Bonn. Their adorable daughter, Olivia Wenlu Fan, was with the second author in Germany for the "hot and interesting" summer of Bonn, where she liked everything except German milk. Frederik thanks his wife, Angélique Herzberg, for her love and manifold support with the words of Proverbs 31,10–12: "A wife of noble character [. . .] is worth far more than rubies. Her husband [. . .] lacks nothing of value. She brings him good [. . .] all the days of her life." We are all very grateful to our families for their love and understanding during the entire process of writing this book.

Finally, we would like to thank Dr. Catriona Byrne as well as Susanne Denskus and Ute McCrory of Springer Verlag for their kind, unfainting editorial assistance in the long process of publishing this work.

Contents

Chapter 1
Hyperfinite Dirichlet Forms

The interplay between methods from functional analysis and the theory of stochastic processes is one of the most important and exciting aspects of mathematical physics today. It is a highly technical and sophisticated theory based on decades of research in both areas. Numerous papers have been written on the standard theory of Dirichlet forms. Apart from the articles and monographs cited below, other notable contributions to the area include: Albeverio and Bernabei [5], Albeverio, Kondratiev, and Röckner [32], Albeverio and Kondratiev [33], Albeverio and Ma [39], Albeverio, Rüdiger, and Wu [54], Bliedtner [94], Bouleau [98], Bouleau and Hirsch [99], Chen et al. [112], Chen, Ma, and Röckner [116], Eberle [149], Exner [154], Fabes, Fukushima, Gross, Kenig, Röckner, and Stroock [155], Fitzsimmons and Kuwae [172], Fukushima [177, 179, 180], Fukushima and Tanaka [185], Fukushima and Ying [188, 189], Gesztesy et al. [191, 192], Grothaus et al. [198], Hesse et al. [208], Jacob [218–220], Jacob and Moroz [221], Jacob and Schilling [222], Jost et al. [225], Kassmann [232], Kim et al. [240], Kumagai and Sturm [248], Le Jan [258], Liskevich and Röckner [265], Ma and Röckner [272, 273], Ma et al. [274], Mosco [283], Okura [292], Oshima [294, 295], D.W. Robinson [312], Röckner and Wang [317], Röckner and Zhang [319], Schmuland and Sun [329], Shiozawa and Takeda [331], da Silva et al. [332], Stannat [336, 338], Stroock [340], Sturm [343], Takeda [346, 347], Wu [363], and Yosida [364].

In this monograph, we present the theory of Dirichlet forms from a unified vantage point, using nonstandard analysis, thus viewing the continuum of the time line as a discrete lattice of infinitesimal spacing. This approach is close in spirit to the discrete classical formulation of Dirichlet space theory in A. Beurling and J. Deny's seminal article [87].

The discrete setup in this monograph permits to study the diffusion and the jump part by essentially the same methods. This setting being independent of special topological properties of the state space, it is also considerably less technical than other approaches. Thus, the theory has found its natural setting and no longer depends on choosing particular topological spaces; in particular, it is valid for both finite and infinite dimensional spaces.

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Whilst Albeverio et al. [25], Chap. 5, only discussed symmetric hyperfinite Dirichlet forms and related Markov chains (refer to [165, 166] also), we shall extend the theory to the nonsymmetric case. We shall try to follow as much as possible the path suggested by the work on the symmetric case.

An important sub-class of Markov process are Feller processes with stationary and independent increments (*Lévy processes*), and in recent years, these processes have attracted a lot of interest, including from nonstandard analysts. Initiated by T. Lindstrøm [263], a number of articles have been devoted to the investigation of *hyperfinite Lévy processes*. Chapter 5 of this monograph is a detailed exposition of Lindstrøm's theory [263] and its subsequent continuation by Albeverio and Herzberg [14]. The book ends with an expository summary (without proofs) of the model theory of stochastic processes as developed by H.J. Keisler and his coauthors, who formulated and proved the "universality" of hyperfinite adapted probability spaces in a rigorous manner, and a short description of recent fundamental results about the definability of nonstandard universes.

Meanwhile, our purpose in the first chapter is to develop a general theory of hyperfinite quadratic forms. We shall set the scene in Sect. 1.1. Sections 1.2 and 1.3 will study the domains of symmetric parts, the standard parts and resolvents. We shall discuss the property of weak coercive quadratic forms in Sects. 1.4 and 1.7. In Sect. 1.5, we shall study Markov forms and begin the analysis of associated Markov chains and get the basic Beurling–Deny formula. We discuss the hyperfinite lifting theory of standard Dirichlet forms in Sect. 1.6.

1.1 Hyperfinite Quadratic Forms

We shall develop a hyperfinite theory of nonnegative quadratic forms on infinite dimensional spaces. It is well-known that in the Hilbert space case the theory of closed forms of this kind is equivalent to the theory of nonnegative operators. In fact, there is a natural correspondence between forms $E(\cdot,\cdot)$ and operators A given by $E(u,u) = \langle Au, u\rangle$, where $\langle\cdot,\cdot\rangle$ is the scalar product in the Hilbert space. We have chosen to present the theory in terms of forms and not operators for two reasons: partly because forms are real-valued, and this makes it simpler to take standard parts, but also because in most of our applications, the form is what is naturally given.

Let H be an internal, hyperfinite dimensional linear space[1] equipped with an inner product $\langle\cdot,\cdot\rangle$ generating a norm $||\cdot||$. Let $^*\mathbb{R}$ be the nonstandard

[1] The notions of hyperfinite dimensional linear space are given in Albeverio et al. [25].

real line[2]. We call a map $\mathcal{E} : H \times H \longrightarrow {}^*\mathbb{R}$ *nonnegative quadratic form* if and only if for all $\alpha \in {}^*\mathbb{R}, u, v, w \in H$,

$$\begin{aligned}
\mathcal{E}(u,u) &\geq 0, \\
\mathcal{E}(\alpha u, v) &= \alpha\mathcal{E}(u,v), \\
\mathcal{E}(u, \alpha v) &= \alpha\mathcal{E}(u,v), \\
\mathcal{E}(u+v, w) &= \mathcal{E}(u,w) + \mathcal{E}(v,w), \\
\mathcal{E}(w, u+v) &= \mathcal{E}(w,u) + \mathcal{E}(w,v).
\end{aligned}$$

Since $\mathcal{E}(\cdot,\cdot)$ is a nonnegative quadratic form on the hyperfinite dimensional space H, elementary linear algebra tells us that there is a unique nonnegative definite operator $A : H \longrightarrow H$ such that

$$\mathcal{E}(u,v) = \langle Au, v\rangle \quad \text{for all} \quad u, v \in H. \tag{1.1.1}$$

To see this, let ${}^*\mathbb{N}_0$ be the nonstandard integers[3]. Let $\{e_i \mid 1 \leq i \leq N\}$ be an orthonormal basis of $(H, \langle\cdot,\cdot\rangle)$ for an $N \in {}^*\mathbb{N}$. We put $Ae_i = \sum_{j=1}^N \mathcal{E}(e_i, e_j)e_j$. Then (1.1.1) follows immediately. Hence, A is given by the matrix $A = (\mathcal{E}(e_i, e_j))_{1\leq i,j\leq N}$, i.e.,

$$A = \begin{pmatrix} \mathcal{E}(e_1,e_1) & \mathcal{E}(e_1,e_2) & \dots & \mathcal{E}(e_1,e_N) \\ \mathcal{E}(e_2,e_1) & \mathcal{E}(e_2,e_2) & \dots & \mathcal{E}(e_2,e_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{E}(e_N,e_1) & \mathcal{E}(e_N,e_2) & \dots & \mathcal{E}(e_N,e_N) \end{pmatrix}. \tag{1.1.2}$$

Moreover, $\langle Au, u\rangle \geq 0$ for all $u \in H$. This means that A is a hyperfinite dimensional matrix (not necessarily symmetric). Let $\hat{A}$ be the *adjoint operator* of A, that is,

$$\mathcal{E}(u,v) = \langle u, \hat{A}v\rangle \quad \text{for all} \quad u, v \in H.$$

By (1.1.2), we have that $\hat{A}$ is the transpose of A. If $||A||$ and $||\hat{A}||$ are the operator norms of A and $\hat{A}$, respectively, we have $||A|| = ||\hat{A}||$. We fix an infinitesimal[4] Δt such that

[2] ${}^*\mathbb{R}$ is the standard notation for the nonstandard real line, refer to Appendix, Albeverio et al. [25], Cutland [125], Davis [135], Hurd [216], Hurd and Loeb [217], Lindstrøm [262], Stroyan and Bayod [341], and Stroyan and Luxemburg [342].

[3] ${}^*\mathbb{N}_0$ is the standard notation for the nonstandard integers, refer to Appendix, Albeverio et al. [25], Cutland [125], Davis [135], Hurd [216], Hurd and Loeb [217], Lindstrøm [262], Stroyan and Bayod [341], and Stroyan and Luxemburg [342].

[4] In the sense of nonstandard analysis, refer to Appendix, Albeverio et al. [25], Keisler [237, 238], Stroyan and Bayod [341], and Stroyan and Luxemburg [342].

$$0 < \Delta t \leq \frac{1}{||A||} = \frac{1}{||\hat{A}||}. \tag{1.1.3}$$

Let us define new operators $Q^{\Delta t}$ and $\hat{Q}^{\Delta t}$ by

$$Q^{\Delta t} = I - \Delta t A,$$
$$\hat{Q}^{\Delta t} = I - \Delta t \hat{A}.$$

The relation (1.1.3) implies that the operators $Q^{\Delta t}$ and $\hat{Q}^{\Delta t}$ are nonnegative. Because A is nonnegative, the operator norms of $Q^{\Delta t}$ and $\hat{Q}^{\Delta t}$ are less than or equal to one. Similarly, we define the *nonnegative quadratic co-form* $\hat{\mathcal{E}}(\cdot,\cdot)$ of $\mathcal{E}(\cdot,\cdot)$ by

$$\hat{\mathcal{E}}(u,v) = \mathcal{E}(v,u) \text{ for all } u,v \in H.$$

Introduce a nonstandard time line T by

$$T = \{k\Delta t \mid k \in {}^*\mathbb{N}_0\}.$$

For each element $t = k\Delta t$ in T, define Q^t and $\hat{Q}^t$ to be the operators

$$Q^t = (Q^{\Delta t})^k,$$
$$\hat{Q}^t = (\hat{Q}^{\Delta t})^k.$$

The families $\{Q^t\}_{t\in T}$ and $\{\hat{Q}^t\}_{t\in T}$ are obviously semigroups. We shall call $\{Q^t\}_{t\in T}$ the *semigroup* and $\{\hat{Q}^t\}_{t\in T}$ the *co-semigroup* associated with $\mathcal{E}(\cdot,\cdot)$ and Δt, respectively. Whenever we refer to $\mathcal{E}(\cdot,\cdot)$, $\hat{\mathcal{E}}(\cdot,\cdot)$, $A, \hat{A}, T, Q^t$ and $\hat{Q}^t$ in the rest of this book, we shall assume that they are linked by above relations.

In applications, the primary objects will often be the semigroup $\{Q^t\}_{t\in T}$ and co-semigroup $\{\hat{Q}^t\}_{t\in T}$. We can then define A and $\hat{A}$ (and hence $\mathcal{E}(\cdot,\cdot)$) by

$$A = \frac{1}{\Delta t}\left(I - Q^{\Delta t}\right),$$
$$\hat{A} = \frac{1}{\Delta t}\left(I - \hat{Q}^{\Delta t}\right).$$

The operator A is called the *infinitesimal generator* of $\mathcal{E}(\cdot,\cdot)$, and $\hat{A}$ is called the *infinitesimal co-generator* of $\mathcal{E}(\cdot,\cdot)$. For each $t \in T$, we may define approximations $A^{(t)}$ of A and $\hat{A}^{(t)}$ of $\hat{A}$ by

$$A^{(t)} = \frac{1}{t}\left(I - Q^t\right),$$
$$\hat{A}^{(t)} = \frac{1}{t}\left(I - \hat{Q}^t\right). \tag{1.1.4}$$

From $A^{(t)}$ and $\hat{A}^{(t)}$, we get the forms

$$\begin{aligned}\mathcal{E}^{(t)}(u,v) &= \langle A^{(t)}u, v\rangle \\ &= \langle u, \hat{A}^{(t)}v\rangle, \end{aligned} \tag{1.1.5}$$

and

$$\begin{aligned}\hat{\mathcal{E}}^{(t)}(u,v) &= \mathcal{E}^{(t)}(v,u) \\ &= \langle \hat{A}^{(t)}u, v\rangle \\ &= \langle A^{(t)}v, u\rangle.\end{aligned}$$

We define the *symmetric part* $\overline{\mathcal{E}}(\cdot,\cdot)$ and *anti-symmetric part* $\mathring{\mathcal{E}}(\cdot,\cdot)$ of $\mathcal{E}(\cdot,\cdot)$ by

$$\begin{aligned}\overline{\mathcal{E}}(u,v) &= \frac{1}{2}\Big(\mathcal{E}(u,v) + \mathcal{E}(v,u)\Big), \\ \mathring{\mathcal{E}}(u,v) &= \frac{1}{2}\Big(\mathcal{E}(u,v) - \mathcal{E}(v,u)\Big).\end{aligned}$$

For $\alpha \in {}^*\mathbb{R}, \alpha \geq 0$, we set

$$\overline{\mathcal{E}}_\alpha(u,v) = \overline{\mathcal{E}}(u,v) + \alpha\langle u, v\rangle.$$

Each of these forms generates a norm (possibly a semi-norm in the case $\alpha = 0$):

$$\begin{aligned}|u|_\alpha &= \sqrt{\overline{\mathcal{E}}_\alpha(u,u)} \\ &= \sqrt{\mathcal{E}_\alpha(u,u)}.\end{aligned}$$

We recall that the original Hilbert space norm on H is denoted by $||\cdot||$. Similarly, we set for $\alpha \in {}^*\mathbb{R}, \alpha \geq 0$,

$$\begin{aligned}\mathcal{E}_\alpha(u,v) &= \mathcal{E}(u,v) + \alpha\langle u,v\rangle, \\ \hat{\mathcal{E}}_\alpha(u,v) &= \hat{\mathcal{E}}(u,v) + \alpha\langle u,v\rangle.\end{aligned}$$

Let $\overline{A}$ and $\{\overline{Q}^t\}$ be the generator and semigroup of $\overline{\mathcal{E}}(\cdot,\cdot)$, respectively. Then

$$\overline{A} = \frac{1}{2}\left(A + \hat{A}\right), \quad \overline{Q}^{\Delta t} = \frac{1}{2}\left(Q^{\Delta t} + \hat{Q}^{\Delta t}\right) \text{ and } \overline{Q}^{k\Delta t} = (\overline{Q}^{\Delta t})^k, \forall k \in {}^*\mathbb{N}.$$

Since $\overline{A}$ and $\overline{Q}^t$ are nonnegative, self-adjoint operators, they have unique nonnegative square roots, which we denote by $\overline{A}^{\frac{1}{2}}$ and $\overline{Q}^{\frac{t}{2}}$, respectively.

In the same manner as (1.1.4) and (1.1.5), we can define approximations $\overline{A}^{(t)}$ of A and $\overline{\mathcal{E}}^{(t)}(\cdot,\cdot)$ of $\overline{\mathcal{E}}(\cdot,\cdot)$ by

$$\overline{A}^{(t)} = \frac{1}{t}\left(I - \overline{Q}^t\right), \quad \overline{\mathcal{E}}^{(t)}(u,v) = \langle \overline{A}^{(t)} u, v\rangle, \quad t \in T.$$

If a nonnegative quadratic form $\mathcal{E}(\cdot,\cdot) : H \times H \longrightarrow {}^*\mathbb{R}$ satisfies

$$\mathcal{E}(u,v) = \mathcal{E}(v,u) \text{ for all } u, v \in H,$$

i.e., $\overline{\mathcal{E}}(u,v) = \mathcal{E}(u,v)$, we shall call it a *nonnegative symmetric quadratic form.* It is easy to see that a nonnegative quadratic form $\mathcal{E}(u,v)$ is symmetric if and only if $A = \hat{A}$ or $Q^t = \hat{Q}^t, \forall t \in T$.

In this book, we shall deal with nonnegative quadratic forms $\mathcal{E}(\cdot,\cdot)$ and the related theory. For the framework, potential theory and applications of nonnegative symmetric quadratic form, we refer the reader to Albeverio et al. [25], Chap. 5, Sect. 5.1 and Fan [165, 166]. We shall utilize the known results of symmetric forms in our study, and extend them to the nonsymmetric case. In particular, we need the notion of the symmetric part $\overline{\mathcal{E}}(\cdot,\cdot)$ of $\mathcal{E}(\cdot,\cdot)$, and the related notations. In Sect. 1.2, we shall define the domain $\mathcal{D}(\overline{\mathcal{E}})$ of the symmetric part $\overline{\mathcal{E}}(\cdot,\cdot)$ by using the semigroup $\{\overline{Q}^t \mid t \in T\}$. We shall introduce the resolvent $\{\overline{G}_\alpha \mid \alpha \in {}^*(-\infty, 0)\}$ of $\overline{\mathcal{E}}(\cdot,\cdot)$ in Sect. 1.3, and characterize the domain $\mathcal{D}(\overline{\mathcal{E}})$ by this resolvent. In Sect. 1.4, we shall define the domain $\mathcal{D}(\mathcal{E})$ of $\mathcal{E}(\cdot,\cdot)$ by its resolvent $\{G_\alpha \mid \alpha \in {}^*(-\infty, 0)\}$; under the hyperfinite weak sector condition, we shall show that $\mathcal{D}(\mathcal{E}) = \mathcal{D}(\overline{\mathcal{E}})$. In Sect. 1.5, we shall introduce hyperfinite Dirichlet forms and related Markov chains. For standard coercive forms, we shall construct their nonstandard representation in Sect. 1.6.

1.2 Domain of the Symmetric Part

In this section, we shall define the domain $\mathcal{D}(\overline{\mathcal{E}})$ of the symmetric part $\overline{\mathcal{E}}(\cdot,\cdot)$ for a hyperfinite nonnegative quadratic form $\mathcal{E}(\cdot,\cdot)$. Before giving a strict definition (Definition 1.2.1), we shall mention an intuitive description. At first, let $\mathrm{Fin}(H)$ be the set of all elements in H with finite norm. By defining $x \approx y$ if $||x - y|| \approx 0$, we know from Proposition A.5.2 in the Appendix that the space[5]

$$^\circ H = \mathrm{Fin}(H)/\approx$$

[5] $\approx$ stands for differing by an infinitesimal, in the sense of nonstandard analysis, refer to Albeverio et al. [25], Cutland [125], Davis [135], Hurd [216], Hurd and Loeb [217], and Lindstrøm [262].

is a Hilbert space with respect to the inner product $({}^{\circ}x, {}^{\circ}y) = \mathrm{st}(\langle x, y\rangle)$, where ${}^{\circ}x$ denotes the equivalence class of x and $\mathrm{st} : {}^{*}\mathbb{R} \longrightarrow \mathbb{R}$ is the mapping of standard part[6]. We call $({}^{\circ}H, (\cdot,\cdot))$ the *hull* of $(H, \langle\cdot,\cdot\rangle)$.

Consider the standard part $\overline{E}(\cdot,\cdot)$ of the nonnegative symmetric quadratic form $\overline{\mathcal{E}}(\cdot,\cdot)$. If $\overline{\mathcal{E}}(\cdot,\cdot)$ is *S-bounded*, i.e., there exists a constant $K \in \mathbb{R}_+$ such that

$$|\overline{\mathcal{E}}(u,v)| \le K||u||||v|| \qquad \text{for all } u, v \in H,$$

we can simply define $\overline{E}(\cdot,\cdot)$ by

$$\overline{E}({}^{\circ}u, {}^{\circ}v) = {}^{\circ}\overline{\mathcal{E}}(u,v).$$

If $\overline{\mathcal{E}}(\cdot,\cdot)$ is not S-bounded, we shall meet two difficulties. We no longer have that $\overline{\mathcal{E}}(u,v) \approx \overline{\mathcal{E}}(\tilde{u},\tilde{v})$ whenever $u \approx \tilde{u}$ and $v \approx \tilde{v}$, and there may be elements $v \in \mathrm{Fin}(H)$ such that $\overline{\mathcal{E}}(\tilde{v},\tilde{v})$ is infinite for all $\tilde{v} \approx v$. The latter problem should not surprise us. It is an immediate consequence of the fact that unbounded forms on Hilbert spaces cannot be defined everywhere. We shall solve it by simply letting $\overline{E}({}^{\circ}u, {}^{\circ}v)$ be undefined when $\overline{\mathcal{E}}(\tilde{v},\tilde{v})$ is infinite for all $\tilde{v} \in {}^{\circ}v$. The most natural solution to the first problem may be to define

$$\overline{E}({}^{\circ}u, {}^{\circ}u) = \inf\{{}^{\circ}\overline{\mathcal{E}}(v,v) \mid v \in {}^{\circ}u\}, \tag{1.2.1}$$

and then extend $\overline{E}(\cdot,\cdot)$ to be a bilinear form by the usual trick

$$\overline{E}({}^{\circ}u, {}^{\circ}v) = \frac{1}{2}\left\{\overline{E}({}^{\circ}u + {}^{\circ}v, {}^{\circ}u + {}^{\circ}v) - \overline{E}({}^{\circ}u, {}^{\circ}u) - \overline{E}({}^{\circ}v, {}^{\circ}v)\right\}.$$

The disadvantage of this approach is that it gives us little understanding of how the infimum in (1.2.1) is obtained. For an easier access to the regularity properties of $\overline{\mathcal{E}}(\cdot,\cdot)$ and $\overline{E}(\cdot,\cdot)$, we prefer a more indirect way of attack. Our plan is to define a subset $\mathcal{D}(\overline{\mathcal{E}})$ of Fin(H) – we call it the *domain of* $\overline{\mathcal{E}}(\cdot,\cdot)$ – satisfying

$$\text{if} \quad {}^{\circ}\overline{\mathcal{E}}(u,u) < \infty, \text{there is a} \quad v \in \mathcal{D}(\overline{\mathcal{E}}) \text{ such that } v \approx u, \tag{1.2.2}$$

$$\text{if } u, v \in \mathcal{D}(\overline{\mathcal{E}}) \text{ and } u \approx v, \ \text{ then } {}^{\circ}\overline{\mathcal{E}}(u,u) = {}^{\circ}\overline{\mathcal{E}}(v,v) < \infty. \tag{1.2.3}$$

We then define $\overline{E}(\cdot,\cdot)$ by

$$\overline{E}({}^{\circ}u, {}^{\circ}u) = {}^{\circ}\overline{\mathcal{E}}(v,v), \tag{1.2.4}$$

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[6] Refer to Albeverio et al. [25].

when $v \in \mathcal{D}(\overline{\mathcal{E}}) \cap {}^{\circ}u$. It turns out that the two definitions (1.2.1) and (1.2.4) agree (see Proposition 1.2.4).

If we look at the standard nonsymmetric Dirichlet theory, see Albeverio et al. [9], Kim [241] and Ma and Röckner [270], the domain of a quadratic form is given from the very beginning. After that, the authors such as those of Ma and Röckner [270] introduced the symmetric and anti-symmetric parts (see page 15, [270]). This method makes the domains of the quadratic form and its symmetric part coincide. On the other hand, Albeverio et al. [25] has given us a very nice definition of domain for the symmetric hyperfinite quadratic forms by their semigroups. Therefore, we may define the domain $\mathcal{D}(\overline{\mathcal{E}})$ of $\overline{\mathcal{E}}(\cdot,\cdot)$ via the semigroup of $\{\overline{Q}^t \mid t \in T\}$. In the next section, we shall discuss the property of the resolvent $\{\overline{G}_\alpha \mid \alpha \in {}^*(-\infty,0)\}$ of $\overline{\mathcal{E}}(\cdot,\cdot)$. We can define the domain of $\mathcal{D}(\overline{\mathcal{E}})$ through $\{\overline{G}_\alpha \mid \alpha \in {}^*(-\infty,0)\}$.

Now it is very natural to ask: can we as well define the domain $\mathcal{D}(\mathcal{E})$ of $\mathcal{E}(\cdot,\cdot)$ directly from $\{Q^t \mid t \in T\}$? Here we would mention that it seems not easy to do the job. In Sect. 1.4, we shall define $\mathcal{D}(\mathcal{E})$ by means of the resolvent $\{G_\alpha \mid \alpha < 0\}$ of $\mathcal{E}(\cdot,\cdot)$. Under the hypothesis of weak sector condition, we shall prove $\mathcal{D}(\overline{\mathcal{E}}) = \mathcal{D}(\mathcal{E})$ by showing that the two definitions satisfy (1.2.1). This is similar to the procedure in the standard nonsymmetric Dirichlet space theory, see, e.g., Albeverio et al. [9], Albeverio et al. [47], Albeverio and Ugolini [57], Kim [241], and Ma and Röckner [270].

Notice that even when $\overline{\mathcal{E}}(\cdot,\cdot)$ is not S-bounded, $\overline{\mathcal{E}}^{(t)}(\cdot,\cdot)$ is S-bounded for all non-infinitesimal t. One of the motivations behind our definition of the domain $\mathcal{D}(\overline{\mathcal{E}})$ is that we want to single out the elements where $\overline{\mathcal{E}}(\cdot,\cdot)$ is really approximated by the bounded forms $\overline{\mathcal{E}}^{(t)}(\cdot,\cdot), t \not\approx 0$, i.e., those $u \in H$ such that

$$ {}^{\circ}\overline{\mathcal{E}}(u,u) = \lim_{\substack{t \downarrow 0 \\ t \not\approx 0}} {}^{\circ}\overline{\mathcal{E}}^{(t)}(u,u). \tag{1.2.5} $$

We could have taken this to be our definition of $\mathcal{D}(\overline{\mathcal{E}})$, but for technical and expository reasons we have chosen another one which we shall soon show to be equivalent to (1.2.5) (see Proposition 1.2.2).

Definition 1.2.1. Let $\mathcal{E}(\cdot,\cdot)$ be a nonnegative quadratic form on a hyperfinite dimensional linear space H. The domain $\mathcal{D}(\overline{\mathcal{E}})$ of the symmetric part of $\mathcal{E}(\cdot,\cdot)$ is the set of all $u \in H$ satisfying

(i) ${}^{\circ}\mathcal{E}_1(u,u) = {}^{\circ}\overline{\mathcal{E}}_1(u,u) < \infty$.
(ii) For all $t \approx 0$, $\overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) \approx \overline{\mathcal{E}}(u,u)$.

Let us try to convey the intuition behind this definition. Thinking of $\overline{A}$ as a differential operator, the elements of $\mathcal{D}(\overline{\mathcal{E}})$ are "smooth" functions and

$\overline{Q}^t$ is a "smoothing" operator often given by an integral kernel. If an element u is already smooth, then an infinitesimal amount of smoothing $\overline{Q}^t, t \approx 0$, should not change it noticeably, and hence $\overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) \approx \overline{\mathcal{E}}(u,u)$. We shall give a partial justification of this rather crude image later, when we show that if ${}^\circ\overline{\mathcal{E}}_1(u,u) < \infty$, then the "smoothed" elements $\overline{Q}^t u, t \not\approx 0$, are all in $\mathcal{D}(\overline{\mathcal{E}})$ (Lemma 1.2.3, see also Corollary 1.2.3).

Our first task will be to establish a list of alternative definitions of $\mathcal{D}(\overline{\mathcal{E}})$, among them (1.2.5). We begin with the following simple identity giving the relationship between $\mathcal{E}(\cdot,\cdot)$ and $\mathcal{E}^{(t)}(\cdot,\cdot)$, and also the relationship between $\overline{\mathcal{E}}(\cdot,\cdot)$ and $\overline{\mathcal{E}}^{(t)}(\cdot,\cdot)$:

Lemma 1.2.1. *For all $u \in H$ and $t \in T$, we have*

(i) $\mathcal{E}^{(t)}(u,u) \geq 0$ *and* $\overline{\mathcal{E}}^{(t)}(u,u) \geq 0$,

(ii) $\mathcal{E}^{(t)}(u,u) = \dfrac{\Delta t}{t} \sum_{0 \leq s < t} \mathcal{E}(Q^s u, u) = \dfrac{\Delta t}{t} \sum_{0 \leq s < t} \mathcal{E}(u, \hat{Q}^s u)$,

(iii) $\overline{\mathcal{E}}^{(t)}(u,u) = \dfrac{\Delta t}{t} \sum_{0 \leq s < t} \overline{\mathcal{E}}(\overline{Q}^s u, u) = \dfrac{\Delta t}{t} \sum_{0 \leq s < t} \overline{\mathcal{E}}(\overline{Q}^{s/2} u, \overline{Q}^{s/2} u)$.

Proof. (i) We have

$$\begin{aligned}
\mathcal{E}^{(t)}(u,u) &= \frac{1}{t}\langle (I - Q^t)u, u\rangle \\
&= \frac{1}{t}\big(\langle u,u\rangle - \langle Q^t u, u\rangle\big) \\
&\geq \frac{1}{t}\big(\langle u,u\rangle - ||Q^t||\langle u,u\rangle\big) \\
&\geq 0,
\end{aligned}$$

since $||Q^t|| \leq ||Q^{\Delta t}||^{\frac{t}{\Delta t}} \leq 1$. It is then easy to see that $\overline{\mathcal{E}}^{(t)}(u,u) \geq 0$.

(ii) By an easy calculation, we have

$$\begin{aligned}
\mathcal{E}^{(t)}(u,u) &= \frac{1}{t}\langle (I - Q^t)u, u\rangle \\
&= \frac{1}{t}\sum_{0 \leq s < t} \langle (Q^s - Q^{s+\Delta t})u, u\rangle \\
&= \frac{\Delta t}{t}\sum_{0 \leq s < t} \mathcal{E}(Q^s u, u) \\
&= \frac{\Delta t}{t}\sum_{0 \leq s < t} \mathcal{E}(u, \hat{Q}^s u).
\end{aligned} \tag{1.2.6}$$

(iii) In the same way as (1.2.6), we can prove that the first equation holds. The second one is due to the symmetry and the semigroup property of $\overline{Q}^s$. □

Among other things, Lemma 1.2.1 tells us that $\mathcal{E}^{(t)}(\cdot,\cdot)$ and $\overline{\mathcal{E}}^{(t)}(\cdot,\cdot)$ are nonnegative.

Lemma 1.2.2. *Let $B, C : H \longrightarrow H$ be nonnegative, symmetric operators commuting with $\overline{A}$ and each other. Then the functions*

$$t \mapsto \langle \overline{Q}^t Bu, Cu\rangle \quad \textit{and} \quad t \mapsto \overline{\mathcal{E}}^{(s)}(\overline{Q}^t Bu, Cu)$$

are nonnegative and decreasing for all $u \in H$ and $s \in T$.

Proof. We first notice that the $\overline{\mathcal{E}}^{(s)}(\cdot,\cdot)$ part follows from the other one since

$$\overline{\mathcal{E}}^{(s)}(\overline{Q}^t Bu, Cu) = \frac{1}{s}\langle \overline{Q}^t (I - \overline{Q}^s)Bu, Cu\rangle,$$

and the operator $B' = (I - \overline{Q}^s)B$ is nonnegative and commutes with $\overline{A}$ and C. If $t > r$, then

$$\begin{aligned}\langle \overline{Q}^r Bu, Cu\rangle - \langle \overline{Q}^t Bu, Cu\rangle &= \langle (I - \overline{Q}^{t-r})\overline{Q}^r Bu, Cu\rangle \\ &= (t-r)\overline{\mathcal{E}}^{(t-r)}(\overline{Q}^{r/2}B^{1/2}C^{1/2}u, \overline{Q}^{r/2}B^{1/2}C^{1/2}u) \\ &\geq 0,\end{aligned}$$

where we used that $\overline{\mathcal{E}}^{(t-r)}(\cdot,\cdot)$ is nonnegative. Hence, $t \longrightarrow \langle \overline{Q}^t Bu, Cu\rangle$ decreases. For the positivity, we observe that

$$\begin{aligned}\langle \overline{Q}^t Bu, Cu\rangle &= \langle \overline{Q}^{t/2}B^{1/2}C^{1/2}u, \overline{Q}^{t/2}B^{1/2}C^{1/2}u\rangle \\ &\geq 0.\end{aligned}$$

□

From Lemma 1.2.2 we may now obtain our main inequalities.

Proposition 1.2.1. *For all $u \in H, t \in T$:*

(i) $0 \leq \overline{\mathcal{E}}(u, u - \overline{Q}^t u) \leq \overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) \leq 2\overline{\mathcal{E}}(u, u - \overline{Q}^t u)$.

(ii) $0 \leq \overline{\mathcal{E}}(\overline{Q}^{\Delta t}u, \overline{Q}^{\Delta t}u) - \overline{\mathcal{E}}(\overline{Q}^{2\Delta t}u, \overline{Q}^{2\Delta t}u) \leq \overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^{\Delta t}u, \overline{Q}^{\Delta t}u)$.

Proof. By trivial algebra, we have

$$\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) = \overline{\mathcal{E}}(u, u - \overline{Q}^t u) + \overline{\mathcal{E}}(\overline{Q}^t u, u - \overline{Q}^t u).$$

Applying Lemma 1.2.2 with $B = I, C = I - \overline{Q}^t$, we see that

$$0 \le \overline{\mathcal{E}}(\overline{Q}^t u, u - \overline{Q}^t u) \le \overline{\mathcal{E}}(u, u - \overline{Q}^t u),$$

and part (i) follows.

(ii) The non-negativity is immediate from (i), and as above we have

$$\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^{\Delta t} u, \overline{Q}^{\Delta t} u) = \overline{\mathcal{E}}(u, u - \overline{Q}^{\Delta t} u) + \overline{\mathcal{E}}(\overline{Q}^{\Delta t} u, u - \overline{Q}^{\Delta t} u).$$

Applying Lemma 1.2.2 to each of the latter two terms, using $B = I, C = I - \overline{Q}^{\Delta t}$ in the first case, and $B = \overline{Q}^{\Delta t}, C = I - \overline{Q}^{\Delta t}$ in the second, we get

$$\begin{aligned}
\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^{\Delta t} u, \overline{Q}^{\Delta t} u) &\ge \overline{\mathcal{E}}(\overline{Q}^{2\Delta t} u, u - \overline{Q}^{\Delta t} u) + \overline{\mathcal{E}}(\overline{Q}^{3\Delta t} u, u - \overline{Q}^{\Delta t} u) \\
&= \overline{\mathcal{E}}(\overline{Q}^{2\Delta t} u, u) - \overline{\mathcal{E}}(\overline{Q}^{2\Delta t} u, \overline{Q}^{\Delta t} u) \\
&\quad + \overline{\mathcal{E}}(\overline{Q}^{3\Delta t} u, u) - \overline{\mathcal{E}}(\overline{Q}^{3\Delta t} u, \overline{Q}^{\Delta t} u) \\
&= \overline{\mathcal{E}}(\overline{Q}^{\Delta t} u, \overline{Q}^{\Delta t} u) - \overline{\mathcal{E}}(\overline{Q}^{2\Delta t} u, \overline{Q}^{2\Delta t} u).
\end{aligned}$$

The proposition is proved. □

The inequalities above are what we need to establish a reasonable characterization of $\mathcal{D}(\overline{\mathcal{E}})$. We first give our promised list of alternative definitions of the domain of $\overline{\mathcal{E}}(\cdot,\cdot)$.

Proposition 1.2.2. *The following statements are equivalent:*

(i) u *is in the domain* $\mathcal{D}(\overline{\mathcal{E}})$ *of* $\mathcal{E}(\cdot,\cdot)$.

(ii) ${}^\circ\mathcal{E}_1(u,u) = {}^\circ\overline{\mathcal{E}}_1(u,u) < \infty$, *and for all* $t \approx 0$, *we have* $\overline{\mathcal{E}}(u, u - \overline{Q}^t u) \approx 0$.

(iii) ${}^\circ\mathcal{E}_1(u,u) < \infty$, *and for all* $t \approx 0$, *we have* $\overline{\mathcal{E}}(u - \overline{Q}^t u, u - \overline{Q}^t u) \approx 0$.

(iv) ${}^\circ\mathcal{E}_1(u,u) < \infty$, *and for all* $t \approx 0$, *we have* $\overline{\mathcal{E}}^{(t)}(u,u) \approx \overline{\mathcal{E}}(u,u)$.

Proof. $(i) \iff (ii)$. Follows immediately from Proposition 1.2.1 (i).

$(ii) \implies (iii)$. We have

$$0 \le \overline{\mathcal{E}}(u - \overline{Q}^t u, u - \overline{Q}^t u) = \overline{\mathcal{E}}(u, u - \overline{Q}^t u) - \overline{\mathcal{E}}(\overline{Q}^t u, u - \overline{Q}^t u),$$

and by Lemma 1.2.2 the term $\overline{\mathcal{E}}(\overline{Q}^t u, u - \overline{Q}^t u)$ is positive.

$(iii) \implies (i)$. We recall that $|u|_0 = \sqrt{\overline{\mathcal{E}}(u,u)}$ is a semi-norm. By Lemma 1.2.2 and the triangle inequality, we have

$$0 \le |u|_0 - |\overline{Q}^t u|_0 \le |u - \overline{Q}^t u|_0.$$

Multiplying both sides by $|u|_0 + |\overline{Q}^t u|_0$, we get

$$0 \le |u|_0^2 - |\overline{Q}^t u|_0^2 \le |u - \overline{Q}^t u|_0(|u|_0 + |\overline{Q}^t u|_0) \le 2|u|_0|u - \overline{Q}^t u|_0.$$

Hence if ${}^\circ\mathcal{E}(u,u) < \infty$ and $\overline{\mathcal{E}}(u - \overline{Q}^t u, u - \overline{Q}^t u) \approx 0$, we have that

$$\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) \approx 0.$$

$(ii) \implies (iv)$. Follows at once from Lemma 1.2.1.

$(iv) \implies (ii)$. Follows from Lemma 1.2.1 and the fact that $s \mapsto \overline{\mathcal{E}}(\overline{Q}^s u, u)$ is decreasing. □

The characterizations of $\mathcal{D}(\overline{\mathcal{E}})$ given in the Proposition 1.2.2 are useful for different purposes. As an illustration, we use Proposition 1.2.2 (iii) to prove that the domain has the right linear structure.

Corollary 1.2.1. *Let $u, v \in \mathcal{D}(\overline{\mathcal{E}})$, and assume that $\alpha \in {}^*\mathbb{R}$ is a nearstandard[7] number. Then αu and $u + v$ are elements of $\mathcal{D}(\overline{\mathcal{E}})$.*

Proof. The αu part is trivial. For $u+v$ we use Proposition 1.2.2 (iii) and the triangle inequality.

$$\begin{aligned} ||(u+v) - \overline{Q}^t(u+v)||_0 &= ||u - \overline{Q}^t u + v - \overline{Q}^t v||_0 \\ &\le ||u - \overline{Q}^t u||_0 + ||v - \overline{Q}^t v||_0. \end{aligned}$$

The latter two terms above are infinitesimals when $t \approx 0$. □

Corollary 1.2.2. *For any infinitesimal $\delta \in T$, we have $\mathcal{D}(\overline{\mathcal{E}}) \subset \mathcal{D}(\overline{\mathcal{E}}^{(\delta)})$.*

Proof. Let $u \in \mathcal{D}(\overline{\mathcal{E}})$. By Proposition 1.2.2 (iv), we know $\overline{\mathcal{E}}^{(k\delta)}(u,u) \approx \overline{\mathcal{E}}(u,u) \approx \overline{\mathcal{E}}^{(\delta)}(u,u)$ for all k such that $k\delta \approx 0$. By Proposition 1.2.2 (iv) again, we get $u \in \mathcal{D}(\overline{\mathcal{E}}^{(\delta)})$. □

The second part of Proposition 1.2.1 informs us that $\overline{Q}^t u$ is more likely to be in $\mathcal{D}(\overline{\mathcal{E}})$ than u is. The next lemma pins this down more precisely.

Lemma 1.2.3. *Assume ${}^\circ\mathcal{E}_1(u,u) < \infty$. Then for all non-infinitesimals t, we have $\overline{Q}^t u \in \mathcal{D}(\overline{\mathcal{E}})$.*

Proof. By Proposition 1.2.1, we have

$${}^\circ\overline{\mathcal{E}}_1(\overline{Q}^t u, \overline{Q}^t u) \le {}^\circ\overline{\mathcal{E}}_1(u,u) < \infty.$$

[7] See Appendix and Albeverio et al. [25] for the concept of nearstandard.

To prove that Definition 1.2.1 (ii) is satisfied, we notice that according to Proposition 1.2.1 (ii), the function

$$t \mapsto \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u)$$

is decreasing and convex, and hence

$$\frac{1}{s}\Big(\overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) - \overline{\mathcal{E}}(\overline{Q}^{t+s} u, \overline{Q}^{t+s} u)\Big) \leq \frac{1}{t}\Big(\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u)\Big)$$

for all $s > 0$. Multiplying through by s, we get

$$0 \leq \overline{\mathcal{E}}(\overline{Q}^t u, \overline{Q}^t u) - \overline{\mathcal{E}}(\overline{Q}^{t+s} u, \overline{Q}^{t+s} u) \leq \frac{s}{t}\Big(\overline{\mathcal{E}}(u,u) - \overline{\mathcal{E}}(\overline{Q}^s u, \overline{Q}^s u)\Big).$$

For $s \approx 0$ and $t \not\approx 0$, the expression on the right is infinitesimal, and the lemma follows. □

We shall now strengthen the lemma above and show that if ${}^\circ\mathcal{E}_1(u,u) < \infty$, then there is an infinitesimal t such that $\overline{Q}^t u \in \mathcal{D}(\overline{\mathcal{E}})$. This is a special case of our next result. First we need to introduce a new definition. A subset F of H is called $\overline{\mathcal{E}}$*-closed* if and only if for all sequences $\{u_n\}_{n\in\mathbb{N}}$ of elements from F such that ${}^\circ|u_n - u_m|_1 \longrightarrow 0$ as $n, m \longrightarrow \infty$, there exists an element u in F such that ${}^\circ|u_n - u|_1 \longrightarrow 0$ as $n \longrightarrow \infty$.

Proposition 1.2.3. *$\mathcal{D}(\overline{\mathcal{E}})$ is $\overline{\mathcal{E}}$-closed. Moreover, if $\{u_n\}_{n\in\mathbb{N}}$ is a $|\cdot|_1$ Cauchy sequence from $\mathcal{D}(\overline{\mathcal{E}})$, and $\{u_n \mid n \in {}^*\mathbb{N}\}$ is an internal extension, then there is a $\gamma \in {}^*\mathbb{N} - \mathbb{N}$ such that $u_\eta \in \mathcal{D}(\overline{\mathcal{E}})$ for all $\eta \leq \gamma$.*

Proof. Let $\{u_n | n \in \mathbb{N}\}$ be a $|\cdot|_1$ Cauchy sequence from $\mathcal{D}(\overline{\mathcal{E}})$, and let $\{u_n | n \in {}^*\mathbb{N}\}$ be an internal extension of it. There is an element $\gamma \in {}^*\mathbb{N} - \mathbb{N}$ such that $|u_n - u_m|_1 \approx 0$ whenever n and m are infinite and less than γ. Let $\eta \in {}^*\mathbb{N} - \mathbb{N}, \eta \leq \gamma$. By the choice of γ, ${}^\circ\overline{\mathcal{E}}_1(u_\eta, u_\eta) < \infty$ and ${}^\circ|u_n - u_\eta|_1 \longrightarrow 0$ as n approaches infinity in $\mathbb{N}$. All that remains is to prove that $u_\eta \in \mathcal{D}(\overline{\mathcal{E}})$.

Assume not, then by Proposition 1.2.2 (iii) there is an $\varepsilon \in \mathbb{R}_+$ and $t \approx 0$ such that

$$|u_\eta - \overline{Q}^t u_\eta|_0 > \varepsilon.$$

Choose $m \in \mathbb{N}$ so large that

$$|u_\eta - u_m|_0 < \frac{\varepsilon}{4}.$$

Then by Proposition 1.2.1 (i), we have

$$|\overline{Q}^t u_\eta - \overline{Q}^t u_m|_0 < \frac{\varepsilon}{4}.$$

Combining the inequalities above, we have

$$\begin{aligned}\varepsilon < |u_\eta - \overline{Q}^t u_\eta|_0 &\le |u_\eta - u_m|_0 + |u_m - \overline{Q}^t u_m|_0 + |\overline{Q}^t u_m - \overline{Q}^t u_\eta|_0 \\ &\le \varepsilon/2 + |u_m - \overline{Q}^t u_m|_0,\end{aligned}$$

but since $u_m \in \mathcal{D}(\overline{\mathcal{E}})$, the last term is infinitesimal by Proposition 1.2.2 (iii). We have the contradiction we wanted. □

Corollary 1.2.3. *If* $^\circ\mathcal{E}_1(u,u) < \infty$, *there is a* $t_0 \approx 0$ *such that* $\overline{Q}^t u \in \mathcal{D}(\overline{\mathcal{E}})$ *for all* $t \ge t_0$.

Proof. First we notice that if $\overline{Q}^{t_0} u \in \mathcal{D}(\overline{\mathcal{E}})$, so is $\overline{Q}^t u$ for all $t > t_0$. Put $u_n = \overline{Q}^{\frac{1}{n}} u$. Then the sequence $\{|u_n|_1\}$ is increasing and bounded by $|u|_1$, and we can apply Proposition 1.2.3 to it. The corollary follows. □

Corollary 1.2.4. *If* $^\circ\mathcal{E}_1(u,u) < \infty$, *there is a* $\delta_u \approx 0$ *such that* $u \in \mathcal{D}(\overline{\mathcal{E}}^{(\delta)})$ *for all infinitesimal* $\delta \ge \delta_u$.

Proof. By Corollary 1.2.3, there is a $t_0 \approx 0$ such that $\overline{Q}^t u \in \mathcal{D}(\overline{\mathcal{E}})$ for all $t \ge t_0$. Let δ_u be an infinitesimal such that $\delta_u > t_0$ and $t_0/\delta_u \approx 0$. For all infinitesimal $\delta \ge \delta_u$, we have $v = \overline{Q}^\delta u \in \mathcal{D}(\overline{\mathcal{E}}) \subset \mathcal{D}(\overline{\mathcal{E}}^{(\delta)})$ by Corollaries 1.2.2 and 1.2.3. For all $k \in {}^*\mathbb{N}$ such that $k\delta \approx 0$, we have the following

$$\begin{aligned}\overline{\mathcal{E}}^{(\delta)}(\overline{Q}^{k\delta} u, \overline{Q}^{k\delta} u) &= \overline{\mathcal{E}}^{(\delta)}(\overline{Q}^{(k-1)\delta} v, \overline{Q}^{(k-1)\delta} v) \\ &\approx \overline{\mathcal{E}}^{(\delta)}(v,v) \\ &= \overline{\mathcal{E}}^{(\delta)}(\overline{Q}^\delta u, \overline{Q}^\delta u).\end{aligned} \tag{1.2.7}$$

By Lemma 1.2.1 (iii), we have

$$\begin{aligned}\overline{\mathcal{E}}^{(\delta)}(\overline{Q}^\delta u, \overline{Q}^\delta u) &= \frac{\Delta t}{\delta} \sum_{0 \le s < \delta} \overline{\mathcal{E}}(\overline{Q}^{s/2}\overline{Q}^\delta u, \overline{Q}^{s/2}\overline{Q}^\delta u) \\ &= \frac{\Delta t}{\delta} \sum_{0 \le s < \delta} \overline{\mathcal{E}}(\overline{Q}^{s/2+\delta-t_0}\overline{Q}^{t_0} u, \overline{Q}^{s/2+\delta-t_0}\overline{Q}^{t_0} u) \\ &\approx \overline{\mathcal{E}}(\overline{Q}^{t_0} u, \overline{Q}^{t_0} u),\end{aligned} \tag{1.2.8}$$

because $\overline{Q}^{t_0} u \in \mathcal{D}(\overline{\mathcal{E}})$. By Lemma 1.2.1 (iii) again, we have

$$\begin{aligned}\overline{\mathcal{E}}^{(\delta)}(u,u) &= \frac{\Delta t}{\delta} \sum_{0 \le s < \delta} \overline{\mathcal{E}}(\overline{Q}^{s/2} u, \overline{Q}^{s/2} u) \\ &= \frac{\Delta t}{\delta}\left[\sum_{0 \le s < 2t_0} \overline{\mathcal{E}}(\overline{Q}^{s/2} u, \overline{Q}^{s/2} u) + \sum_{2t_0 \le s < \delta} \overline{\mathcal{E}}(\overline{Q}^{s/2} u, \overline{Q}^{s/2} u)\right]\end{aligned}$$

$$\approx \frac{\Delta t}{\delta} \sum_{2t_0 \le s < \delta} \overline{\mathcal{E}}(\overline{Q}^{s/2-t_0}\overline{Q}^{t_0}u, \overline{Q}^{s/2-t_0}\overline{Q}^{t_0}u)$$
$$\approx \overline{\mathcal{E}}(\overline{Q}^{t_0}u, \overline{Q}^{t_0}u). \tag{1.2.9}$$

By relations (1.2.7), (1.2.8), and (1.2.9), we know $u \in \mathcal{D}(\overline{\mathcal{E}}^{(\delta)})$. □

Remark 1.2.1. Proposition 1.2.3 is rather surprising since there exist standard forms which are neither closed nor closable. In fact, there are numerous applications where the main difficulty is to show that the form constructed is closed, or at least can be extended to a closed form (see, e.g., [11, 16–24, 26, 27, 36, 48, 49, 94, 98, 99, 103, 151, 175, 176, 178, 225, 232, 236, 247, 251, 259, 278, 301, 318, 345, 359]). If we know that a form comes from a hyperfinite form, this follows immediately from Proposition 1.2.3. In Albeverio et al. [25], Chap. 6, we have got various examples of how useful this observation is. For the time being, we only remark that since we shall soon show that all standard, coercive closed forms can be obtained from hyperfinite forms, the method is quite general (we refer to Sect. 1.6 of this chapter).

Notice that if we can show that whenever ${}^\circ\overline{\mathcal{E}}_1(u,u) < \infty$, then for all $t \approx 0$, $||u - \overline{Q}^t u|| \approx 0$, Corollary 1.2.3 will imply the first part of our program, i.e., (1.2.2) above.

Lemma 1.2.4. *Assume* ${}^\circ\overline{\mathcal{E}}(u,u) < \infty$. *Then for all* $t \approx 0$, *we have*

$$||u - \overline{Q}^t u|| \approx 0.$$

Proof. For $t \approx 0$, we have

$$\begin{aligned} ||u - \overline{Q}^t u||^2 &= \langle u - \overline{Q}^t u, u - \overline{Q}^t u\rangle \\ &= t\overline{\mathcal{E}}^{(t)}(u, u - \overline{Q}^t u) \\ &= t\Big[\overline{\mathcal{E}}^{(t)}(u,u) - \overline{\mathcal{E}}^{(t)}(u, \overline{Q}^t u)\Big] \\ &\le t\overline{\mathcal{E}}(u,u) \approx 0. \end{aligned}$$

□

Let us turn our attention to our second main goal (1.2.3).

Lemma 1.2.5. *If* $u, v \in \mathcal{D}(\overline{\mathcal{E}})$ *and* $u \approx v$, *then*

$$\mathcal{E}(u,u) \approx \mathcal{E}(v,v).$$

Proof. It is obviously enough to show that if $u \in \mathcal{D}(\overline{\mathcal{E}})$ and $u \approx 0$, then $\mathcal{E}(u,u) \approx 0$. But if $u \in \mathcal{D}(\overline{\mathcal{E}})$, we know from Proposition 1.2.2 (iv):

$${}^\circ\overline{\mathcal{E}}(u,u) = \lim_{\substack{t \downarrow 0 \\ t \not\approx 0}} {}^\circ\overline{\mathcal{E}}^{(t)}(u,u). \tag{1.2.10}$$