Problem Books in Mathematics

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The IMO Compendium

A Collection of Problems Suggested for the International Mathematical Olympiads: 1959–2004

With 200 Figures

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Preface

The International Mathematical Olympiad (IMO) is nearing its fiftieth anniversary and has already created a very rich legacy and firmly established itself as the most prestigious mathematical competition in which a high-school student could aspire to participate. Apart from the opportunity to tackle interesting and very challenging mathematical problems, the IMO represents a great opportunity for high-school students to see how they measure up against students from the rest of the world. Perhaps even more importantly, it is an opportunity to make friends and socialize with students who have similar interests, possibly even to become acquainted with their future colleagues on this first leg of their journey into the world of professional and scientific mathematics. Above all, however pleasing or disappointing the final score may be, preparing for an IMO and participating in one is an adventure that will undoubtedly linger in one's memory for the rest of one's life. It is to the high-school-aged aspiring mathematician and IMO participant that we devote this entire book.

The goal of this book is to include all problems ever shortlisted for the IMOs in a single volume. Up to this point, only scattered manuscripts traded among different teams have been available, and a number of manuscripts were lost for many years or unavailable to many.

In this book, all manuscripts have been collected into a single compendium of mathematics problems of the kind that usually appear on the IMOs. Therefore, we believe that this book will be the definitive and authoritative source for high-school students preparing for the IMO, and we suspect that it will be of particular benefit in countries lacking adequate preparation literature. A high-school student could spend an enjoyable year going through the numerous problems and novel ideas presented in the solutions and emerge ready to tackle even the most difficult problems on an IMO. In addition, the skill acquired in the process of successfully attacking difficult mathematics problems will prove to be invaluable in a serious and prosperous career in mathematics.

However, we must caution our aspiring IMO participant on the use of this book. Any book of problems, no matter how large, quickly depletes itself if the reader merely glances at a problem and then five minutes later, having determined that the problem seems unsolvable, glances at the solution.

The authors therefore propose the following plan for working through the book. Each problem is to be attempted at least half an hour before the reader looks at the solution. The reader is strongly encouraged to keep trying to solve the problem without looking at the solution as long as he or she is coming up with fresh ideas and possibilities for solving the problem. Only after all venues seem to have been exhausted is the reader to look at the solution, and then only in order to study it in close detail, carefully noting any previously unseen ideas or methods used. To condense the subject matter of this already very large book, most solutions have been streamlined, omitting obvious derivations and algebraic manipulations. Thus, reading the solutions requires a certain mathematical maturity, and in any case, the solutions, especially in geometry, are intended to be followed through with pencil and paper, the reader filling in all the omitted details. We highly recommend that the reader mark such unsolved problems and return to them in a few months to see whether they can be solved this time without looking at the solutions. We believe this to be the most efficient and systematic way (as with any book of problems) to raise one's level of skill and mathematical maturity.

We now leave our reader with final words of encouragement to persist in this journey even when the difficulties seem insurmountable and a sincere wish to the reader for all mathematical success one can hope to aspire to.

Belgrade,
October 2004

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For the most current information regarding The IMO Compendium you are invited to go to our website: `www.imo.org.yu`. At this site you can also find, for several of the years, scanned versions of available original shortlist and longlist problems, which should give an illustration of the original state the IMO materials we used were in.

We are aware that this book may still contain errors. If you find any, please notify us at `imo@matf.bg.ac.yu`. A full list of discovered errors can be found at our website. If you have any questions, comments, or suggestions regarding both our book and our website, please do not hesitate to write to us at the above email address. We would be more than happy to hear from you.

Acknowledgements

The making of this book would have never been possible without the help of numerous individuals, whom we wish to thank.

First and foremost, obtaining manuscripts containing suggestions for IMOs was vital in order for us to provide the most complete listing of problems possible. We obtained manuscripts for many of the years from the former and current IMO team leaders of Yugoslavia / Serbia and Montenegro, who carefully preserved these valuable papers throughout the years. Special thanks are due to Prof. Vladimir Mićić, for some of the oldest manuscripts, and to Prof. Zoran Kadelburg. We also thank Prof. Djordje Dugošija and Prof. Pavle Mladenović. In collecting shortlisted and longlisted problems we were also assisted by Prof. Ioan Tomescu from Romania and Hà Duy Hưng from Vietnam.

A lot of work was invested in cleaning up our giant manuscript of errors. Special thanks in this respect go to David Kramer, our copy-editor, and to Prof. Titu Andreescu and his group for checking, in great detail, the validity of the solutions in this manuscript, and for their proposed corrections and alternative solutions to several problems. We also thank Prof. Abderrahim Ouardini from France for sending us the list of countries of origin for the shortlisted problems of 1998, Prof. Dorin Andrica for helping us compile the list of books for reference, and Prof. Ljubomir Čukić for proofreading part of the manuscript and helping us correct several errors.

We would also like to express our thanks to all anonymous authors of the IMO problems. It is a pity that authors' names are not registered together with their proposed problems. Without them, the IMO would obviously not be what it is today. In many cases, the original solutions of the authors were used, and we duly acknowledge this immense contribution to our book, though once again, we regret that we cannot do this individually. In the same vein, we also thank all the students participating in the IMOs, since we have also included some of their original solutions in this book.

The illustrations of geometry problems were done in WinGCLC, a program created by Prof. Predrag Janičić. This program is specifically designed for creating geometric pictures of unparalleled complexity quickly and efficiently. Even though it is still in its testing phase, its capabilities and utility are already remarkable and worthy of highest compliment.

Finally, we would like to thank our families for all their love and support during the making of this book.

Contents

1

Introduction

1.1 The International Mathematical Olympiad

The International Mathematical Olympiad (IMO) is the most important and prestigious mathematical competition for high-school students. It has played a significant role in generating wide interest in mathematics among high school students, as well as identifying talent.

In the beginning, the IMO was a much smaller competition than it is today. In 1959, the following seven countries gathered to compete in the first IMO: Bulgaria, Czechoslovakia, German Democratic Republic, Hungary, Poland, Romania, and the Soviet Union. Since then, the competition has been held annually. Gradually, other Eastern-block countries, countries from Western Europe, and ultimately numerous countries from around the world and every continent joined in. (The only year in which the IMO was not held was 1980, when for financial reasons no one stepped in to host it. Today this is hardly a problem, and hosts are lined up several years in advance.) In the 45th IMO, held in Athens, no fewer than 85 countries took part.

The format of the competition quickly became stable and unchanging. Each country may send up to six contestants and each contestant competes individually (without any help or collaboration). The country also sends a team leader, who participates in problem selection and is thus isolated from the rest of the team until the end of the competition, and a deputy leader, who looks after the contestants.

The IMO competition lasts two days. On each day students are given four and a half hours to solve three problems, for a total of six problems. The first problem is usually the easiest on each day and the last problem the hardest, though there have been many notable exceptions. ((IMO96-5) is one of the most difficult problems from all the Olympiads, having been fully solved by only six students out of several hundred!) Each problem is worth 7 points, making 42 points the maximum possible score. The number of points obtained by a contestant on each problem is the result of intense negotiations and, ultimately, agreement among the problem coordinators, assigned by the host country, and the team leader and deputy, who defend the interests of their contestants. This system ensures a relatively objective grade that is seldom off by more than two or three points.

Though countries naturally compare each other's scores, only individual prizes, namely medals and honorable mentions, are awarded on the IMO. Fewer than one twelfth of participants are awarded the gold medal, fewer than one fourth are awarded the gold or silver medal, and fewer than one half are awarded the gold, silver or bronze medal. Among the students not awarded a medal, those who score 7 points on at least one problem are awarded an honorable mention. This system of determining awards works rather well. It ensures, on the one hand, strict criteria and appropriate recognition for each level of performance, giving every contestant something to strive for. On the other hand, it also ensures a good degree of generosity that does not greatly depend on the variable difficulty of the problems proposed.

According to the statistics, the hardest Olympiad was that in 1971, followed by those in 1996, 1993, and 1999. The Olympiad in which the winning team received the lowest score was that in 1977, followed by those in 1960 and 1999.

The selection of the problems consists of several steps. Participant countries send their proposals, which are supposed to be novel, to the IMO organizers. The organizing country does not propose problems. From the received proposals (the *longlisted* problems), the problem committee selects a shorter list (the *shortlisted* problems), which is presented to the IMO jury, consisting of all the team leaders. From the short-listed problems the jury chooses six problems for the IMO.

Apart from its mathematical and competitive side, the IMO is also a very large social event. After their work is done, the students have three days to enjoy events and excursions organized by the host country, as well as to interact and socialize with IMO participants from around the world. All this makes for a truly memorable experience.

1.2 The IMO Compendium

Olympiad problems have been published in many books [65]. However, the remaining shortlisted and longlisted problems have not been systematically collected and published, and therefore many of them are unknown to mathematicians interested in this subject. Some partial collections of shortlisted and longlisted problems can be found in the references, though usually only for one year. References [1], [30], [41], [60] contain problems from multiple years. In total, these books cover roughly 50% of the problems found in this book.

The goal of this book is to present, in a single volume, our comprehensive collection of problems proposed for the IMO. It consists of all problems selected for the IMO competitions, shortlisted problems from the 10th IMO and from the 12th through 44th IMOs, and longlisted problems from nineteen IMOs. We do not have shortlisted problems from the 9th and the 11th IMOs, and we could not discover whether competition problems at those two IMOs were selected from the longlisted problems or whether there existed shortlisted problems that have not been preserved. Since IMO organizers usually do not distribute longlisted problems to the representatives of participant countries, our collection is incomplete. The practice of distributing these longlists effectively ended in 1989. A selection of problems from the first eight IMOs has been taken from [60].

The book is organized as follows. For each year, the problems that were given on the IMO contest are presented, along with the longlisted and/or shortlisted problems, if applicable. We present solutions to all shortlisted problems. The problems appearing on the IMOs are solved among the other shortlisted problems. The longlisted problems have not been provided with solutions, except for the two IMOs held in Yugoslavia (for patriotic reasons), since that would have made the book unreasonably long. This book has thus the added benefit for professors and team coaches of being a suitable book from which to assign problems. For each problem, we indicate the country that proposed it with a three-letter code. A complete list of country codes and the corresponding countries is given in the appendix. In all shortlists, we also indicate which problems were selected for the contest. We occasionally make references in our solutions to other problems in a straightforward way. After indicating with LL, SL, or IMO whether the problem is from a longlist, shortlist, or contest, we indicate the year of the IMO and then the number of the problem. For example, (SL89-15) refers to the fifteenth problem of the shortlist of 1989.

We also present a rough list of all formulas and theorems not obviously derivable that were called upon in our proofs. Since we were largely concerned with only the theorems used in proving the problems of this book, we believe that the list is a good compilation of the most useful theorems for IMO problem solving.

The gathering of such a large collection of problems into a book required a massive amount of editing. We reformulated the problems whose original formulations were not precise or clear. We translated the problems that were not in English. Some of the solutions are taken from the author of the problem or other sources, while others are original solutions of the authors of this book. Many of the non-original solutions were significantly edited before being included. We do not make any guarantee that the problems in this book fully correspond to the actual shortlisted or longlisted problems. However, we believe this book to be the closest possible approximation to such a list.

2

Basic Concepts and Facts

The following is a list of the most basic concepts and theorems frequently used in this book. We encourage the reader to become familiar with them and perhaps read up on them further in other literature.

2.1 Algebra

2.1.1 Polynomials

Theorem 2.1. *The quadratic equation* $ax^2 + bx + c = 0$ *(*$a, b, c \in \mathbb{R}$*,* $a \neq 0$*) has solutions*

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The discriminant D *of the quadratic equation is defined as* $D = b^2 - 4ac$*. For* $D < 0$ *the solutions are complex and conjugate to each other, for* $D = 0$ *the solutions degenerate to one real solution, and for* $D > 0$ *the equation has two distinct real solutions.*

Definition 2.2. *Binomial coefficients* $\binom{n}{k}$, $n, k \in \mathbb{N}_0$, $k \leq n$, are defined as

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

They satisfy $\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}$ for $i > 0$ and also $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$, $\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n} = 0$, $\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}$.

Theorem 2.3 ((Newton's) binomial formula). *For* $x, y \in \mathbb{C}$ *and* $n \in \mathbb{N}$*,*

$$(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

Theorem 2.4 (Bézout's theorem). *A polynomial* $P(x)$ *is divisible by the binomial* $x - a$ $(a \in \mathbb{C})$ *if and only if* $P(a) = 0$.

Theorem 2.5 (The rational root theorem). *If* $x = p/q$ *is a rational zero of a polynomial* $P(x) = a_n x^n + \cdots + a_0$ *with integer coefficients and* $(p, q) = 1$, *then* $p \mid a_0$ *and* $q \mid a_n$.

Theorem 2.6 (The fundamental theorem of algebra). *Every nonconstant polynomial with coefficients in* $\mathbb{C}$ *has a complex root.*

Theorem 2.7 (*Eisenstein's criterion (extended)*). *Let* $P(x) = a_n x^n + \cdots + a_1 x + a_0$ *be a polynomial with integer coefficients. If there exist a prime* p *and an integer* $k \in \{0, 1, \ldots, n-1\}$ *such that* $p \mid a_0, a_1, \ldots, a_k$, $p \nmid a_{k+1}$, *and* $p^2 \nmid a_0$, *then there exists an irreducible factor* $Q(x)$ *of* $P(x)$ *whose degree is at least* k. *In particular, if* p *can be chosen such that* $k = n - 1$, *then* $P(x)$ *is irreducible.*

Definition 2.8. *Symmetric polynomials* in $x_1, \ldots, x_n$ are polynomials that do not change on permuting the variables $x_1, \ldots, x_n$. *Elementary symmetric polynomials* are $\sigma_k(x_1, \ldots, x_n) = \sum x_{i_1} \cdots x_{i_k}$ (the sum is over all k-element subsets $\{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$).

Theorem 2.9. *Every symmetric polynomial in* $x_1, \ldots, x_n$ *can be expressed as a polynomial in the elementary symmetric polynomials* $\sigma_1, \ldots, \sigma_n$.

Theorem 2.10 (Vieta's formulas). *Let* $\alpha_1, \ldots, \alpha_n$ *and* $c_1, \ldots, c_n$ *be complex numbers such that*

$$(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n .$$

Then $c_k = (-1)^k \sigma_k(\alpha_1, \ldots, \alpha_n)$ *for* $k = 1, 2, \ldots, n$.

Theorem 2.11 (Newton's formulas on symmetric polynomials). *Let* $\sigma_k = \sigma_k(x_1, \ldots, x_n)$ *and let* $s_k = x_1^k + x_2^k + \cdots + x_n^k$, *where* $x_1, \ldots, x_n$ *are arbitrary complex numbers. Then*

$$k\sigma_k = s_1 \sigma_{k-1} - s_2 \sigma_{k-2} + \cdots + (-1)^k s_{k-1} \sigma_1 + (-1)^{k-1} s_k .$$

2.1.2 Recurrence Relations

Definition 2.12. A *recurrence relation* is a relation that determines the elements of a sequence x_n, $n \in \mathbb{N}_0$, as a function of previous elements. A recurrence relation of the form

$$(\forall n \geq k) \qquad x_n + a_1 x_{n-1} + \cdots + a_k x_{n-k} = 0$$

for constants $a_1, \ldots, a_k$ is called a *linear homogeneous recurrence relation of order* k. We define the *characteristic polynomial* of the relation as $P(x) = x^k + a_1 x^{k-1} + \cdots + a_k$.

Theorem 2.13. *Using the notation introduced in the above definition, let $P(x)$ factorize as $P(x) = (x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}\cdots(x-\alpha_r)^{k_r}$, where $\alpha_1, \dots, \alpha_r$ are distinct complex numbers and $k_1, \dots, k_r$ are positive integers. The general solution of this recurrence relation is in this case given by*

$$x_n = p_1(n)\alpha_1^n + p_2(n)\alpha_2^n + \cdots + p_r(n)\alpha_r^n,$$

where p_i is a polynomial of degree less than k_i. In particular, if $P(x)$ has k distinct roots, then all p_i are constant.

If $x_0, \dots, x_{k-1}$ are set, then the coefficients of the polynomials are uniquely determined.

2.1.3 Inequalities

Theorem 2.14. *The quadratic function is always positive; i.e., $(\forall x \in \mathbb{R})\ x^2 \geq 0$. By substituting different expressions for x, many of the inequalities below are obtained.*

Theorem 2.15 (Bernoulli's inequalities).

1. *If $n \geq 1$ is an integer and $x > -1$ a real number then $(1+x)^n \geq 1 + nx$.*
2. *If $a > 1$ or $a < 0$ then for $x > -1$ the following inequality holds: $(1+x)^\alpha \geq 1 + \alpha x$.*
3. *If $a \in (0,1)$ then for $x > -1$ the following inequality holds: $(1+x)^\alpha \leq 1 + \alpha x$.*

Theorem 2.16 (The mean inequalities). *For positive real numbers x_1, x_2, $\dots, x_n$ it follows that $QM \geq AM \geq GM \geq HM$, where*

$$QM = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}, \qquad AM = \frac{x_1 + \cdots + x_n}{n},$$
$$GM = \sqrt[n]{x_1 \cdots x_n}, \qquad HM = \frac{n}{1/x_1 + \cdots + 1/x_n}.$$

Each of these inequalities becomes an equality if and only if $x_1 = x_2 = \cdots = x_n$. The numbers QM, AM, GM, and HM are respectively called the quadratic mean, *the* arithmetic mean, *the* geometric mean, *and the* harmonic mean *of $x_1, x_2, \dots, x_n$.*

Theorem 2.17 (The general mean inequality). *Let $x_1, \dots, x_n$ be positive real numbers. For each $p \in \mathbb{R}$ we define the* mean of order p *of $x_1, \dots, x_n$ by $M_p = \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{1/p}$ for $p \neq 0$, and $M_q = \lim_{p\to q} M_p$ for $q \in \{\pm\infty, 0\}$. In particular, $\max x_i$, QM, AM, GM, HM, and $\min x_i$ are M_∞, M_2, M_1, M_0, M_{-1}, and $M_{-\infty}$ respectively. Then*

$$M_p \leq M_q \quad \text{whenever} \quad p \leq q.$$

Theorem 2.18 (Cauchy–Schwarz inequality). *Let a_i, b_i, $i = 1, 2, \ldots, n$, be real numbers. Then*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Equality occurs if and only if there exists $c \in \mathbb{R}$ such that $b_i = ca_i$ for $i = 1, \ldots, n$.

Theorem 2.19 (Hölder's inequality). *Let a_i, b_i, $i = 1, 2, \ldots, n$, be nonnegative real numbers, and let p, q be positive real numbers such that $1/p + 1/q = 1$. Then*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}.$$

Equality occurs if and only if there exists $c \in \mathbb{R}$ such that $b_i = ca_i$ for $i = 1, \ldots, n$. The Cauchy–Schwarz inequality is a special case of Hölder's inequality for $p = q = 2$.

Theorem 2.20 (Minkowski's inequality). *Let a_i, b_i $(i = 1, 2, \ldots, n)$ be nonnegative real numbers and p any real number not smaller than 1. Then*

$$\left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^p\right)^{1/p}.$$

For $p > 1$ equality occurs if and only if there exists $c \in \mathbb{R}$ such that $b_i = ca_i$ for $i = 1, \ldots, n$. For $p = 1$ equality occurs in all cases.

Theorem 2.21 (Chebyshev's inequality). *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be real numbers. Then*

$$n \sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq n \sum_{i=1}^{n} a_i b_{n+1-i}.$$

The two inequalities become equalities at the same time when $a_1 = a_2 = \cdots = a_n$ or $b_1 = b_2 = \cdots = b_n$.

Definition 2.22. A real function f defined on an interval I is *convex* if $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$. for all $x, y \in I$ and all $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. A function f is said to be *concave* if the opposite inequality holds, i.e., if $-f$ is convex.

Theorem 2.23. *If f is continuous on an interval I, then f is convex on that interval if and only if*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad \textit{for all } x, y \in I.$$

Theorem 2.24. *If f is differentiable, then it is convex if and only if the derivative f' is nondecreasing. Similarly, differentiable function f is concave if and only if f' is nonincreasing.*

Theorem 2.25 (Jensen's inequality). *If $f : I \to \mathbb{R}$ is a convex function, then the inequality*

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$$

holds for all $\alpha_i \geq 0$, $\alpha_1 + \cdots + \alpha_n = 1$, and $x_i \in I$. For a concave function the opposite inequality holds.

Theorem 2.26 (Muirhead's inequality). *Given $x_1, x_2, \ldots, x_n \in \mathbb{R}^+$ and an n-tuple $\mathbf{a} = (a_1, \cdots, a_n)$ of positive real numbers, we define*

$$T_{\mathbf{a}}(x_1, \ldots, x_n) = \sum y_1^{a_1} \ldots y_n^{a_n},$$

the sum being taken over all permutations $y_1, \ldots, y_n$ of $x_1, \ldots, x_n$. We say that an n-tuple $\mathbf{a}$ majorizes *an n-tuple $\mathbf{b}$ if $a_1 + \cdots + a_n = b_1 + \cdots + b_n$ and $a_1 + \cdots + a_k \geq b_1 + \cdots + b_k$ for each $k = 1, \ldots, n-1$. If a nonincreasing n-tuple $\mathbf{a}$ majorizes a nonincreasing n-tuple $\mathbf{b}$, then the following inequality holds:*

$$T_{\mathbf{a}}(x_1, \ldots, x_n) \geq T_{\mathbf{b}}(x_1, \ldots, x_n).$$

Equality occurs if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 2.27 (Schur's inequality). *Using the notation introduced for Muirhead's inequality,*

$$T_{\lambda+2\mu,0,0}(x_1, x_2, x_3) + T_{\lambda,\mu,\mu}(x_1, x_2, x_3) \geq 2T_{\lambda+\mu,\mu,0}(x_1, x_2, x_3),$$

where $\lambda, \mu \in \mathbb{R}^+$. Equality occurs if and only if $x_1 = x_2 = x_3$ or $x_1 = x_2$, $x_3 = 0$ (and in analogous cases).

2.1.4 Groups and Fields

Definition 2.28. A *group* is a nonempty set G equipped with an operation $*$ satisfying the following conditions:

(i) $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$.

(ii) There exists a (unique) *additive identity* $e \in G$ such that $e * a = a * e = a$ for all $a \in G$.

(iii) For each $a \in G$ there exists a (unique) *additive inverse* $a^{-1} = b \in G$ such that $a * b = b * a = e$.

If $n \in \mathbb{Z}$, we define a^n as $a * a * \cdots * a$ (n times) if $n \geq 0$, and as $(a^{-1})^{-n}$ otherwise.

Definition 2.29. A group $\mathcal{G} = (G, *)$ is *commutative* or *abelian* if $a * b = b * a$ for all $a, b \in G$.

Definition 2.30. A set A *generates* a group $(G, *)$ if every element of G can be obtained using powers of the elements of A and the operation $*$. In other words, if A is the generator of a group G then every element $g \in G$ can be written as $a_1^{i_1} * \cdots * a_n^{i_n}$, where $a_j \in A$ and $i_j \in \mathbb{Z}$ for every $j = 1, 2, \ldots, n$.

Definition 2.31. The *order* of $a \in G$ is the smallest $n \in \mathbb{N}$ such that $a^n = e$, if it exists. The *order* of a group is the number of its elements, if it is finite. Each element of a finite group has a finite order.

Theorem 2.32 (Lagrange's theorem). *In a finite group, the order of an element divides the order of the group.*

Definition 2.33. A *ring* is a nonempty set R equipped with two operations $+$ and $\cdot$ such that $(R, +)$ is an abelian group and for any $a, b, c \in R$,

(i) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

(ii) $(a + b) \cdot c = a \cdot c + b \cdot c$ and $c \cdot (a + b) = c \cdot a + c \cdot b$.

A ring is *commutative* if $a \cdot b = b \cdot a$ for any $a, b \in R$ and *with identity* if there exists a *multiplicative identity* $i \in R$ such that $i \cdot a = a \cdot i = a$ for all $a \in R$.

Definition 2.34. A *field* is a commutative ring with identity in which every element a other than the additive identity has a *multiplicative inverse* a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = i$.

Theorem 2.35. *The following are common examples of groups, rings, and fields:*

Groups: $(\mathbb{Z}_n, +)$, $(\mathbb{Z}_p \setminus \{0\}, \cdot)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{R} \setminus \{0\}, \cdot)$.

Rings: $(\mathbb{Z}_n, +, \cdot)$, $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Z}[x], +, \cdot)$, $(\mathbb{R}[x], +, \cdot)$.

Fields: $(\mathbb{Z}_p, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{Q}(\sqrt{2}), +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$.

2.2 Analysis

Definition 2.36. A sequence $\{a_n\}_{n=1}^{\infty}$ has a *limit* $a = \lim_{n\to\infty} a_n$ (also denoted by $a_n \to a$) if

$$(\forall \varepsilon > 0)(\exists n_\varepsilon \in \mathbb{N})(\forall n \geq n_\varepsilon)\ |a_n - a| < \varepsilon.$$

A function $f : (a, b) \to \mathbb{R}$ has a limit $y = \lim_{x\to c} f(x)$ if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in (a, b))\ 0 < |x - c| < \delta \Rightarrow |f(x) - y| < \varepsilon.$$

Definition 2.37. A sequence x_n *converges* to $x \in \mathbb{R}$ if $\lim_{n\to\infty} x_n = x$. A series $\sum_{n=1}^{\infty} x_n$ converges to $s \in \mathbb{R}$ if and only if $\lim_{m\to\infty} \sum_{n=1}^{m} x_n = s$. A sequence or series that does not converge is said to *diverge*.

Theorem 2.38. *A sequence a_n is convergent if it is monotonic and bounded.*

Definition 2.39. A function f is *continuous* on $[a,b]$ if for every $x_0 \in [a,b]$, $\lim_{x\to x_0} f(x) = f(x_0)$.

Definition 2.40. A function $f : (a,b) \to \mathbb{R}$ is *differentiable* at a point $x_0 \in (a,b)$ if the following limit exists:

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

A function is differentiable on (a,b) if it is differentiable at every $x_0 \in (a,b)$. The function f' is called the *derivative* of f. We similarly define the second derivative f'' as the derivative of f', and so on.

Theorem 2.41. *A differentiable function is also continuous. If f and g are differentiable, then fg, $\alpha f + \beta g$ ($\alpha, \beta \in \mathbb{R}$), $f \circ g$, $1/f$ (if $f \neq 0$), f^{-1} (if well-defined) are also differentiable. It holds that $(\alpha f + \beta g)' = \alpha f' + \beta g'$, $(fg)' = f'g + fg'$, $(f \circ g)' = (f' \circ g) \cdot g'$, $(1/f)' = -f'/f^2$, $(f/g)' = (f'g - fg')/g^2$, $(f^{-1})' = 1/(f' \circ f^{-1})$.*

Theorem 2.42. *The following are derivatives of some elementary functions (a denotes a real constant): $(x^a)' = ax^{a-1}$, $(\ln x)' = 1/x$, $(a^x)' = a^x \ln a$, $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$.*

Theorem 2.43 (Fermat's theorem). *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. The function f attains its maximum and minimum in this interval. If $x_0 \in (a,b)$ is an extremum (i.e., a maximum or minimum), then $f'(x_0) = 0$.*

Theorem 2.44 (Rolle's theorem). *Let $f(x)$ be a continuously differentiable function defined on $[a,b]$, where $a, b \in \mathbb{R}$, $a < b$, and $f(a) = f(b) = 0$. Then there exists $c \in [a,b]$ such that $f'(c) = 0$.*

Definition 2.45. Differentiable functions $f_1, f_2, \ldots, f_k$ defined on an open subset D of $\mathbb{R}^n$ are *independent* if there is no nonzero differentiable function $F : \mathbb{R}^k \to \mathbb{R}$ such that $F(f_1, \ldots, f_k)$ is identically zero on some open subset of D.

Theorem 2.46. *Functions $f_1, \ldots, f_k : D \to \mathbb{R}$ are independent if and only if the $k \times n$ matrix $[\partial f_i/\partial x_j]_{i,j}$ is of rank k, i.e. when its k rows are linearly independent at some point.*

Theorem 2.47 (Lagrange multipliers). *Let D be an open subset of $\mathbb{R}^n$ and $f, f_1, f_2, \ldots, f_k : D \to \mathbb{R}$ independent differentiable functions. Assume* *that a point a in D is an extremum of the function f within the set of points in D such that $f_1 = f_2 = \cdots = f_n = 0$. Then there exist real numbers $\lambda_1, \dots, \lambda_k$ (so-called Lagrange multipliers) such that a is a stationary point of the function $F = f + \lambda_1 f_1 + \cdots + \lambda_k f_k$, i.e., such that all partial derivatives of F at a are zero.*

Definition 2.48. Let f be a real function defined on $[a, b]$ and let $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ and $\xi_k \in [x_{k-1}, x_k]$. The sum $S = \sum_{k=1}^{n}(x_k - x_{k-1})f(\xi_k)$ is called a *Darboux sum*. If $I = \lim_{\delta \to 0} S$ exists (where $\delta = \max_k(x_k - x_{k-1})$), we say that f is *integrable* and I its *integral*. Every continuous function is integrable on a finite interval.

2.3 Geometry

2.3.1 Triangle Geometry

Definition 2.49. The *orthocenter* of a triangle is the common point of its three altitudes.

Definition 2.50. The *circumcenter* of a triangle is the center of its circumscribed circle (i.e. *circumcircle*). It is the common point of the perpendicular bisectors of the sides of the triangle.

Definition 2.51. The *incenter* of a triangle is the center of its inscribed circle (i.e. *incircle*). It is the common point of the internal bisectors of its angles.

Definition 2.52. The *centroid* of a triangle (*median point*) is the common point of its medians.

Theorem 2.53. *The orthocenter, circumcenter, incenter and centroid are well-defined (and unique) for every non-degenerate triangle.*

Theorem 2.54 (Euler's line). *The orthocenter H, centroid G, and circumcircle O of an arbitrary triangle lie on a line (Euler's line) and satisfy $\overrightarrow{HG} = 2\overrightarrow{GO}$.*

Theorem 2.55 (The nine-point circle). *The feet of the altitudes from A, B, C and the midpoints of AB, BC, CA, AH, BH, CH lie on a circle (The nine-point circle).*

Theorem 2.56 (Feuerbach's theorem). *The nine-point circle of a triangle is tangent to the incircle and all three excircles of the triangle.*

Theorem 2.57. *Given a triangle $\triangle ABC$, let $\triangle ABC'$, $\triangle AB'C$, and $\triangle A'BC$ be equilateral triangles constructed outwards. Then AA', BB', CC' intersect in one point, called* Torricelli's point.

Definition 2.58. Let ABC be a triangle, P a point, and X, Y, Z respectively the feet of the perpendiculars from P to BC, AC, AB. Triangle XYZ is called the *pedal triangle* of $\triangle ABC$ corresponding to point P.

Theorem 2.59 (Simson's line). *The pedal triangle XYZ is degenerate, i.e., X, Y, Z are collinear, if and only if P lies on the circumcircle of ABC. Points X, Y, Z are in this case said to lie on* Simson's line*.*

Theorem 2.60 (Carnot's theorem). *The perpendiculars from X, Y, Z to BC, CA, AB respectively are concurrent if and only if*

$$BX^2 - XC^2 + CY^2 - YA^2 + AZ^2 - ZB^2 = 0.$$

Theorem 2.61 (Desargues's theorem). *Let $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles. The lines A_1A_2, B_1B_2, C_1C_2 are concurrent or mutually parallel if and only if the points $A = B_1C_2 \cap B_2C_1$, $B = C_1A_2 \cap A_1C_2$, and $C = A_1B_2 \cap A_2B_1$ are collinear.*

2.3.2 Vectors in Geometry

Definition 2.62. For any two vectors $\overrightarrow{a}$, $\overrightarrow{b}$ in space, we define the *scalar product* (also known as *dot product*) of $\overrightarrow{a}$ and $\overrightarrow{b}$ as $\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}|\cos\varphi$, and the *vector product* as $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{p}$, where $\varphi = \angle(\overrightarrow{a}, \overrightarrow{b})$ and $\overrightarrow{p}$ is the vector with $|\overrightarrow{p}| = |\overrightarrow{a}||\overrightarrow{b}||\sin\varphi|$ perpendicular to the plane determined by $\overrightarrow{a}$ and $\overrightarrow{b}$ such that the triple of vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{p}$ is positively oriented (note that if $\overrightarrow{a}$ and $\overrightarrow{b}$ are collinear, then $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{0}$). These products are both linear with respect to both factors. The scalar product is commutative, while the vector product is anticommutative, i.e. $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}$. We also define the *mixed vector product* of three vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ as $[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}] = (\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{c}$.
Remark. Scalar product of vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is often denoted by $\langle \overrightarrow{a}, \overrightarrow{b} \rangle$.

Theorem 2.63 (Thales' theorem). *Let lines AA' and BB' intersect in a point O, $A' \neq O \neq B'$. Then $AB \parallel A'B' \Leftrightarrow \frac{\overrightarrow{OA}}{\overrightarrow{OA'}} = \frac{\overrightarrow{OB}}{\overrightarrow{OB'}}$. (Here $\frac{\overrightarrow{a}}{\overrightarrow{b}}$ denotes the ratio of two nonzero collinear vectors).*

Theorem 2.64 (Ceva's theorem). *Let ABC be a triangle and X, Y, Z be points on lines BC, CA, AB respectively, distinct from A, B, C. Then the lines AX, BY, CZ are concurrent if and only if*

$$\frac{\overrightarrow{BX}}{\overrightarrow{XC}} \cdot \frac{\overrightarrow{CY}}{\overrightarrow{YA}} \cdot \frac{\overrightarrow{AZ}}{\overrightarrow{ZB}} = 1, \text{ or equivalently, } \frac{\sin\angle BAX}{\sin\angle XAC}\frac{\sin\angle CBY}{\sin\angle YBA}\frac{\sin\angle ACZ}{\sin\angle ZCB} = 1$$

(the last expression being called the trigonometric form *of Ceva's theorem).*

Theorem 2.65 (Menelaus's theorem). *Using the notation introduced for Ceva's theorem, points X, Y, Z are collinear if and only if*

$$\frac{\overrightarrow{BX}}{\overrightarrow{XC}} \cdot \frac{\overrightarrow{CY}}{\overrightarrow{YA}} \cdot \frac{\overrightarrow{AZ}}{\overrightarrow{ZB}} = -1.$$

Theorem 2.66 (Stewart's theorem). *If D is an arbitrary point on the line BC, then*

$$AD^2 = \frac{\overrightarrow{DC}}{\overrightarrow{BC}} BD^2 + \frac{\overrightarrow{BD}}{\overrightarrow{BC}} CD^2 - \overrightarrow{BD} \cdot \overrightarrow{DC}.$$

Specifically, if D is the midpoint of BC, then $4AD^2 = 2AB^2 + 2AC^2 - BC^2$.

2.3.3 Barycenters

Definition 2.67. A *mass point* (A, m) is a point A which is assigned a *mass* $m > 0$.

Definition 2.68. The *mass center* (*barycenter*) of the set of mass points (A_i, m_i), $i = 1, 2, \ldots, n$, is the point T such that $\sum_i m_i \overrightarrow{TA_i} = 0$.

Theorem 2.69 (Leibniz's theorem). *Let T be the mass center of the set of mass points $\{(A_i, m_i) \mid i = 1, 2, \ldots, n\}$ of total mass $m = m_1 + \cdots + m_n$, and let X be an arbitrary point. Then*

$$\sum_{i=1}^{n} m_i XA_i^2 = \sum_{i=1}^{n} m_i TA_i^2 + mXT^2.$$

Specifically, if T is the centroid of $\triangle ABC$ and X an arbitrary point, then

$$AX^2 + BX^2 + CX^2 = AT^2 + BT^2 + CT^2 + 3XT^2 .$$

2.3.4 Quadrilaterals

Theorem 2.70. *A quadrilateral $ABCD$ is cyclic (i.e., there exists a circumcircle of $ABCD$) if and only if $\angle ACB = \angle ADB$ and if and only if $\angle ADC + \angle ABC = 180°$.*

Theorem 2.71 (Ptolemy's theorem). *A convex quadrilateral $ABCD$ is cyclic if and only if*

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

For an arbitrary quadrilateral $ABCD$ we have Ptolemy's inequality *(see 2.3.7, Geometric Inequalities).*

Theorem 2.72 (Casey's theorem). *Let k_1, k_2, k_3, k_4 be four circles that all touch a given circle k. Let t_{ij} be the length of a segment determined by an external common tangent of circles k_i and k_j ($i, j \in \{1, 2, 3, 4\}$) if both k_i and k_j touch k internally, or both touch k externally. Otherwise, t_{ij} is set to be the internal common tangent. Then one of the products $t_{12}t_{34}$, $t_{13}t_{24}$, and $t_{14}t_{23}$ is the sum of the other two.*

Some of the circles k_1, k_2, k_3, k_4 may be degenerate, i.e. of 0 radius and thus reduced to being points. In particular, for three points A, B, C on a circle k and a circle k' touching k at a point on the arc of AC not containing B, we have $AC \cdot b = AB \cdot c + a \cdot BC$, where a, b, and c are the lengths of the tangent segments from points A, B, and C to k'. Ptolemy's theorem *is a special case of Casey's theorem when all four circles are degenerate.*

Theorem 2.73. *A convex quadrilateral $ABCD$ is tangent (i.e., there exists an incircle of $ABCD$) if and only if*

$$AB + CD = BC + DA.$$

Theorem 2.74. *For arbitrary points A, B, C, D in space, $AC \perp BD$ if and only if*

$$AB^2 + CD^2 = BC^2 + DA^2.$$

Theorem 2.75 (Newton's theorem). *Let $ABCD$ be a quadrilateral, $AD \cap BC = E$, and $AB \cap DC = F$ (such points A, B, C, D, E, F form a* complete quadrilateral*). Then the midpoints of AC, BD, and EF are collinear. If $ABCD$ is tangent, then the incenter also lies on this line.*

Theorem 2.76 (Brocard's theorem). *Let $ABCD$ be a quadrilateral inscribed in a circle with center O, and let $P = AB \cap CD$, $Q = AD \cap BC$, $R = AC \cap BD$. Then O is the orthocenter of $\triangle PQR$.*

2.3.5 Circle Geometry

Theorem 2.77 (Pascal's theorem). *If $A_1, A_2, A_3, B_1, B_2, B_3$ are distinct points on a conic γ (e.g., circle), then points $X_1 = A_2B_3 \cap A_3B_2$, $X_2 = A_1B_3 \cap A_3B_1$, and $X_3 = A_1B_2 \cap A_2B_1$ are collinear. The special result when γ consists of two lines is called* Pappus's theorem.

Theorem 2.78 (Brianchon's theorem). *Let $ABCDEF$ be an arbitrary convex hexagon circumscribed about a conic (e.g., circle). Then AD, BE and CF meet in a point.*

Theorem 2.79 (The butterfly theorem). *Let AB be a segment of circle k and C its midpoint. Let p and q be two different lines through C that, respectively, intersect k on one side of AB in P and Q and on the other in P' and Q'. Let E and F respectively be the intersections of PQ' and $P'Q$ with AB. Then it follows that $CE = CF$.*

Definition 2.80. The *power* of a point X with respect to a circle $k(O,r)$ is defined by $\mathcal{P}(X) = OX^2 - r^2$. For an arbitrary line l through X that intersects k at A and B ($A = B$ when l is a tangent), it follows that $\mathcal{P}(X) = \overrightarrow{XA} \cdot \overrightarrow{XB}$.

Definition 2.81. The *radical axis* of two circles is the locus of points that have equal powers with respect to both circles. The radical axis of circles $k_1(O_1,r_1)$ and $k_2(O_2,r_2)$ is a line perpendicular to O_1O_2. The radical axes of three distinct circles are concurrent or mutually parallel. If concurrent, the intersection of the three axes is called the *radical center*.

Definition 2.82. The *pole* of a line $l \not\ni O$ with respect to a circle $k(O,r)$ is a point A on the other side of l from O such that $OA \perp l$ and $d(O,l) \cdot OA = r^2$. In particular, if l intersects k in two points, its pole will be the intersection of the tangents to k at these two points.

Definition 2.83. The *polar* of the point A from the previous definition is the line l. In particular, if A is a point outside k and AM, AN are tangents to k ($M, N \in k$), then MN is the polar of A.
Poles and polares are generally defined in a similar way with respect to arbitrary non-degenerate conics.

Theorem 2.84. *If A belongs to a polar of B, then B belongs to a polar of A.*

2.3.6 Inversion

Definition 2.85. An *inversion* of the plane π around the circle $k(O,r)$ (which belongs to π), is a transformation of the set $\pi \backslash \{O\}$ onto itself such that every point P is transformed into a point P' on $(OP$ such that $OP \cdot OP' = r^2$. In the following statements we implicitly assume exclusion of O.

Theorem 2.86. *The fixed points of the inversion are on the circle k. The inside of k is transformed into the outside and vice versa.*

Theorem 2.87. *If A, B transform into A', B' after an inversion, then $\angle OAB = \angle OB'A'$, and also $ABB'A'$ is cyclic and perpendicular to k. A circle perpendicular to k transforms into itself. Inversion preserves angles between continuous curves (which includes lines and circles).*

Theorem 2.88. *An inversion transforms lines not containing O into circles containing O, lines containing O into themselves, circles not containing O into circles not containing O, circles containing O into lines not containing O.*

2.3.7 Geometric Inequalities

Theorem 2.89 (The triangle inequality). *For any three points A, B, C in a plane $AB + BC \geq AC$. Equality occurs when A, B, C are collinear and $\mathcal{B}(A, B, C)$.*

Theorem 2.90 (Ptolemy's inequality). *For any four points A, B, C, D,*

$$AC \cdot BD \le AB \cdot CD + AD \cdot BC.$$

Theorem 2.91 (The parallelogram inequality). *For any four points A, B, C, D,*

$$AB^2 + BC^2 + CD^2 + DA^2 \ge AC^2 + BD^2.$$

Equality occurs if and only if ABCD is a parallelogram.

Theorem 2.92. *For a given triangle* $\triangle ABC$ *the point* X *for which* $AX + BX + CX$ *is minimal is Toricelli's point when all angles of* $\triangle ABC$ *are less than or equal to* $120°$*, and is the vertex of the obtuse angle otherwise. The point* X_2 *for which* $AX_2^2 + BX_2^2 + CX_2^2$ *is minimal is the centroid (see Leibniz's theorem).*

Theorem 2.93 (The Erdős–Mordell inequality). *Let* P *be a point in the interior of* $\triangle ABC$ *and* X, Y, Z *projections of* P *onto* BC, AC, AB*, respectively. Then*

$$PA + PB + PC \ge 2(PX + PY + PZ).$$

Equality holds if and only if $\triangle ABC$ *is equilateral and* P *is its center.*

2.3.8 Trigonometry

Definition 2.94. The *trigonometric circle* is the unit circle centered at the origin O of a coordinate plane. Let A be the point $(1, 0)$ and $P(x, y)$ be a point on the trigonometric circle such that $\measuredangle AOP = \alpha$. We define $\sin\alpha = y$, $\cos\alpha = x$, $\tan\alpha = y/x$, and $\cot\alpha = x/y$.

Theorem 2.95. *The functions* sin *and* cos *are periodic with period* 2π*. The functions* tan *and* cot *are periodic with period* π*. The following simple identities hold:* $\sin^2 x + \cos^2 x = 1$, $\sin 0 = \sin\pi = 0$, $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\sin(\pi/2) = 1$, $\sin(\pi/4) = 1/\sqrt{2}$, $\sin(\pi/6) = 1/2$, $\cos x = \sin(\pi/2 - x)$*. From these identities other identities can be easily derived.*

Theorem 2.96. *Additive formulas for trigonometric functions:*

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta, \quad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta,$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}, \quad \cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\alpha \pm \cot\beta}.$$

Theorem 2.97. *Formulas for trigonometric functions of* $2x$ *and* $3x$*:*

$$\sin 2x = 2\sin x\cos x, \quad \sin 3x = 3\sin x - 4\sin^3 x,$$
$$\cos 2x = 2\cos^2 x - 1, \quad \cos 3x = 4\cos^3 x - 3\cos x,$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}, \quad \tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.$$

Theorem 2.98. *For any* $x \in \mathbb{R}$, $\sin x = \frac{2t}{1+t^2}$ *and* $\cos x = \frac{1-t^2}{1+t^2}$, *where* $t = \tan \frac{x}{2}$.

Theorem 2.99. *Transformations from product to sum:*

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta),$$
$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta),$$
$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta).$$

Theorem 2.100. *The angles* α, β, γ *of a triangle satisfy*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1,$$
$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma.$$

Theorem 2.101 (De Moivre's formula). *If* $i^2 = -1$, *then*

$$(\cos x + i\sin x)^n = \cos nx + i\sin nx.$$

2.3.9 Formulas in Geometry

Theorem 2.102 (Heron's formula). *The area of a triangle ABC with sides* a, b, c *and semiperimeter* s *is given by*

$$S = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4}\sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}.$$

Theorem 2.103 (The law of sines). *The sides* a, b, c *and angles* α, β, γ *of a triangle* ABC *satisfy*

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where R *is the circumradius of* $\triangle ABC$.

Theorem 2.104 (The law of cosines). *The sides and angles of* $\triangle ABC$ *satisfy*

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Theorem 2.105. *The circumradius* R *and inradius* r *of a triangle* ABC *satisfy* $R = \frac{abc}{4S}$ *and* $r = \frac{2S}{a+b+c} = R(\cos\alpha + \cos\beta + \cos\gamma - 1)$. *If* x, y, z *denote the distances of the circumcenter in an acute triangle to the sides, then* $x + y + z = R + r$.

Theorem 2.106 (Euler's formula). *If* O *and* I *are the circumcenter and incenter of* $\triangle ABC$, *then* $OI^2 = R(R - 2r)$, *where* R *and* r *are respectively the circumradius and the inradius of* $\triangle ABC$. *Consequently,* $R \geq 2r$.

Theorem 2.107. *The area S of a quadrilateral $ABCD$ with sides a, b, c, d, semiperimeter p, and angles α, γ at vertices A, C respectively is given by*

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\frac{\alpha+\gamma}{2}}\ .$$

If $ABCD$ is a cyclic quadrilateral, the above formula reduces to

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}.$$

Theorem 2.108 (Euler's theorem for pedal triangles). *Let X, Y, Z be the feet of the perpendiculars from a point P to the sides of a triangle ABC. Let O denote the circumcenter and R the circumradius of $\triangle ABC$. Then*

$$S_{XYZ} = \frac{1}{4}\left|1 - \frac{OP^2}{R^2}\right| S_{ABC}\ .$$

Moreover, $S_{XYZ} = 0$ if and only if P lies on the circumcircle of $\triangle ABC$ (see Simson's line*).*

Theorem 2.109. *If*
overrightarrowa $= (a_1, a_2, a_3)$, $\overrightarrow{b} = (b_1, b_2, b_3)$, $\overrightarrow{c} = (c_1, c_2, c_3)$ are three vectors in coordinate space, then

$$\overrightarrow{a}\cdot\overrightarrow{b} = a_1b_1 + a_2b_2 + a_3b_3, \quad \overrightarrow{a}\times\overrightarrow{b} = (a_1b_2 - a_2b_1, a_2b_3 - a_3b_2, a_3b_1 - a_1b_3),$$

$$[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}] = \left|\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\right|.$$

Theorem 2.110. *The area of a triangle ABC and the volume of a tetrahedron $ABCD$ are equal to $|\overrightarrow{AB}\times\overrightarrow{AC}|$ and $\left|\left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right]\right|$, respectively.*

Theorem 2.111 (Cavalieri's principle). *If the sections of two solids by the same plane always have equal area, then the volumes of the two solids are equal.*

2.4 Number Theory

2.4.1 Divisibility and Congruences

Definition 2.112. *The greatest common divisor* $(a, b) = \gcd(a, b)$ of $a, b \in \mathbb{N}$ is the largest positive integer that divides both a and b. Positive integers a and b are *coprime* or *relatively prime* if $(a, b) = 1$. *The least common multiple* $[a, b] = \operatorname{lcm}(a, b)$ of $a, b \in \mathbb{N}$ is the smallest positive integer that is divisible by both a and b. It holds that $[a, b](a, b) = ab$. The above concepts are easily generalized to more than two numbers; i.e., we also define $(a_1, a_2, \ldots, a_n)$ and $[a_1, a_2, \ldots, a_n]$.