VECTOR OPTIMIZATION

Andreas Löhne

Vector Optimization with Infimum and Supremum

Springer

Vector Optimization

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Vector Optimization

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To Jana and Pascal

Preface

Infimum and supremum are indispensable concepts in optimization. Nevertheless their role in vector optimization has been rather marginal. This seems to be due the fact that their existence in partially ordered vector spaces is connected with restrictive assumptions. The key to an approach to vector optimization based on infimum and supremum is to consider set-valued objective functions and to extend the partial ordering of the original objective space to a suitable subspace of the power set. In this new space the infimum and supremum exist under the usual assumptions.

These ideas lead to a novel exposition of vector optimization. The reader is not only required to familiarize with several new concepts, but also a change of philosophy is suggested to those being acquainted with the classical approaches. The goal of this monograph is to cover the most important concepts and results on vector optimization and to convey the ideas, which can be used to derive corresponding variants of all the remaining results and concepts. This selection ranges from the general theory including solution concepts and duality theory, through to algorithms for the linear case.

Researchers and graduate-level students working in the field of vector optimization belong to the intended audience. In view of many facts and notions that are recalled, the book is also addressed to those who are not familiar with classical approaches to vector optimization. However, it should be taken into account that a fundamental motivation of vector optimization and applications are beyond the scope of this book.

Some basic knowledge in (scalar) optimization, convex analysis and general topology is necessary to understand the first part, which deals with general and convex problems. The second part is a self-contained exposition of the linear case. Infimum and supremum are not visible but present in the background. The connections to the first part are explained at several places, but they are not necessary to understand the results for the linear case. Some knowledge on (scalar) linear programming is required.

The results in this book arose from several research papers that have been published over the last five years. The results and ideas of this exposition are contributed by Andreas Hamel, Frank Heyde and Christiane Tammer concerning the first part as well as Frank Heyde, Christiane Tammer and Matthias Ehrgott concerning the second part. A first summary, extension and consolidation of these results has been given in the author's habilitation thesis, which appeared in 2010. This book is an extension. It contains one new chapter with extended variants of algorithms and more detailed explanations.

I thank all persons who supported me to write this book. In particular, I'm greatly indebted to Matthias Ehrgott, Gabriele Eichfelder, Andreas Hamel, Frank Heyde, Johannes Jahn and Christiane Tammer for their valuable comments, important corrections and all their advice that entailed a considerable increase of quality.

Halle (Saale), *Andreas Löhne*
November 2010

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Introduction

From a mathematical point of view, vector optimization is the theory of optimization problems with a vector-valued objective function. Instead of the extended real numbers $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$, one considers an extended partially ordered vector space as the image space of the objective map. One of the main difficulties is the lack of a suitable infimum and supremum. For many instances of extended partially ordered vector spaces, even in finite dimensions, an infimum does not exist at all. But even if the infimum in the sense of a greatest lower bound exists, it is usually not related to the typical optimality notions which are motivated by applications in multiobjective optimization.

The idea of multiobjective optimization is to present a decision maker all or at least a representative selection of minimal or efficient vectors. The decision maker's job is to choose one of these vectors. An infimum in an extended partially ordered vector space, if it exists, is of course a vector. But the requirement from an applicational point of view is to evaluate a set of efficient points in order to present them to the decision maker.

The infimum of a fixed subset is generally changing when the partially ordered set, say the universal set, is extended to a larger partially ordered set. The reason is that more candidates for greatest lower bounds are available in a larger set. This basic idea is applied to vector optimization as we create a suitable notion of infimum by embedding the extended partially ordered vector space into a larger partially ordered set, in fact, into a subset of the power set. This allows us to develop a theory of vector optimization which is based on infimum and supremum. This leads to new insights and a high degree of analogy to the scalar optimization theory.

Vector optimization has its origin in economics, in particular, in welfare theory and utility theory. The foundations are connected with the names Vilfredo Pareto (1848-1923) and Francis Ysidro Edgeworth (1845-1926). Independently vector optimization also arose from game theory which was initiated by Émile Borel (1871-1956), Maurice René Fréchet (1878-1973) and

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John von Neumann (1903-1957). For more details the reader is referred to the survey paper by Stadler (1979). From a theoretical perspective the foundations of vector optimization were laid by Georg Cantor (1845-1918) by his famous intersection theorem; by Felix Hausdorff (1868-1942), who showed the existence of utility functions in the context of partially ordered sets; and by Max Zorn (1906-1993), who gave conditions for the existence of maximal elements without using a utility function (see Göpfert *et al.*, 2009). What is today considered to be vector optimization, multiobjective optimization or multicriteria optimization has its origin in the 1950s. The notion of *efficient points* was introduced (compare Stadler, 1979) by Koopmans (1951, Definition 4.2): *"A possible point in the commodity is called efficient whenever an increase in the one of its coordinates (the net output of one good) can be achieved only at the cost of a decrease in some other coordinate (the net output of another good)"*. Kuhn and Tucker (1951) introduced (compare Stadler, 1979) the term *vector maximum problem*. Today there exists a number of textbooks on vector optimization, among them (Sawaragi *et al.*, 1985; Jahn, 1986, 2004; Luc, 1988; Göpfert and Nehse, 1990; Ehrgott, 2000, 2005; Göpfert *et al.*, 2003; Chen *et al.*, 2005; Eichfelder, 2008; Boţ *et al.*, 2009). There are several thousands of research papers on this subject.

This monograph differs from the literature as it is based on the complete lattice $(\mathcal{I}, \preccurlyeq)$ of self-infimal subsets of the original objective space $(Y, \leq)$ of a given extended vector-valued objective function. Starting with a vector optimization problem

$$\text{minimize } f : X \to \overline{Y} \text{ with respect to } \leq \text{ over } S \subseteq X, \tag{V}$$

we assign to f an $\mathcal{I}$-valued objective function

$$\bar{f} : X \to \mathcal{I}, \qquad \bar{f}(x) := \operatorname{Inf}\{f(x)\}$$

and consider the related problem

$$\text{minimize } \bar{f} : X \to \mathcal{I} \text{ with respect to } \preccurlyeq \text{ over } S \subseteq X. \tag{$\mathcal{V}$}$$

There is a close connection between the values of f and $\bar{f}$; that is, for all $x^1, x^2 \in X$ we have

$$f(x^1) \leq f(x^2) \qquad \Longleftrightarrow \qquad \bar{f}(x^1) \preccurlyeq \bar{f}(x^2).$$

Since the objective space $\mathcal{I}$ in Problem $(\mathcal{V})$ is a complete lattice, the latter correspondence allows us to develop the theory of vector optimization based on infimum and supremum.

This approach was firstly pointed out in (Löhne and Tammer, 2007; Heyde *et al.*, 2009a), but it is based on a couple of pre-investigations, such as (Hamel *et al.*, 2004; Hamel, 2005; Löhne, 2005a,b). It turned out that it is possible to formulate and prove vectorial duality theorems very similar to the corresponding scalar results if the vectorial image space is replaced by the complete lattice $\mathcal{I}$. But, the space $\mathcal{I}$ of self-infimal sets does not only provide a complete lattice; the infimum with respect to this complete lattice is also closely related to the standard solution concepts in vector optimization. Even though infimal sets were used before (see e.g. Nieuwenhuis, 1980; Sawaragi *et al.*, 1985; Tanino, 1988, 1992; Song, 1997, 1998), in particular in duality theory, the deeper context was not pointed out: the complete lattice $\mathcal{I}$.

An approach to vector optimization based on infimum and supremum leads to the question how to integrate conventional solution concepts into the theory. It turned out that there is no standard way to say what is a solution to a vector optimization problem from a mathematical point of view. On the one hand this is concerned with the question whether a solution is a set of vectors or just a single vector (see also the introduction to Chapter 2). On the other hand there are different types of efficient vectors depending on different possible interpretations of "less than" when the ordering relation is more complex than the one in $\mathbb{R}$.

It might be worth noting that the solution concept proposed in this work involves two different types of minimality notions: weakly minimal and minimal vectors. This could shed a new light on the role of weakly efficient solutions in vector optimization. Jahn (2004, p. 110) writes that *"the concept of weak minimality is of theoretical interest, and it is not an appropriate notion for applied problems."* This is in accordance with the fact that weak minimality is essential to construct the complete lattice $\mathcal{I}$, but our solution concept itself is based on minimality. One can say that the theoretical benefits of weak minimality and the application-oriented properties of minimality are involved in one concept.

This monograph is organized as follows. Part I is devoted to the general ideas and to convex problems. In Chapter 1 we introduce the complete lattice $\mathcal{I}$, which is the basis of this exposition. We also provide several concepts and facts from the literature as far as they are needed in this book. Chapter 2 is devoted to solution concepts and Chapter 3 is concerned with duality. Part II deals with linear problems. Even though the connections to the first part are often discussed, the second part is a self-contained exposition of the linear theory. In Chapter 4 we focus on solution concepts and duality. The concepts from Part I are adapted and special features of the linear duality theory are shown. Chapter 5 is devoted to algorithms to solve linear problems. Each chapter begins with a specific introduction and ends with several notes on the literature; in particular, the origin of the results is discussed.

This book offers a systematic introduction and a summary of recent developments in the theory of vector optimization with infimum and supremum. It is based on the cited papers, but the theory is presented in a more general setting and with several extensions. This book aims to be a self-contained summary and an extension of recent results.

Part I
General and Convex Problems

Chapter 1
A complete lattice for vector optimization

Extended real-valued objective functions are characteristic for scalar optimization problems. The space of extended real numbers $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ enjoys several properties which are quite important for optimization:

(i) $\mathbb{R}$ is a vector space, but $\overline{\mathbb{R}}$ is not. The linear operations can be partially extended to $\overline{\mathbb{R}}$.
(ii) The linear operations on $\mathbb{R}$ are continuous, i.e., the topology is compatible with the linear structure.
(iii) $\overline{\mathbb{R}}$ is totally ordered by the usual ordering $\leq$. The ordering on $\mathbb{R}$ is compatible with the linear operations.
(iv) $\overline{\mathbb{R}}$ is a complete lattice, i.e., every subset has an infimum and a supremum.

In vector optimization we have to replace $\mathbb{R}$ and $\overline{\mathbb{R}}$ by a more general space. Certain properties can be maintained, others must be abandoned. Underlying a partially ordered topological vector space Y and its extension $\overline{Y} := Y \cup \{\pm\infty\} := Y \cup \{-\infty, +\infty\}$, we obtain all the mentioned properties up to the following two exceptions: First, Y is not totally but partially ordered only. Secondly, a complete lattice is obtained by $\overline{Y}$ only in special cases. This depends on the choice of Y and the choice of the partial ordering. But even in the special cases where $\overline{Y}$ is a complete lattice (e.g. $Y = \mathbb{R}^q$ equipped with the "natural" componentwise ordering), the infimum is different to the typical vectorial minimality notions, which arise from applications. This is illustrated in Figure 1.1.

As a consequence, infimum and supremum (at least in the sense of greatest lower and least upper bounds) do not occur in the standard literature on vector optimization. Some authors, among them Nieuwenhuis (1980); Tanino (1988, 1992), used a generalization of the infimum in $\overline{\mathbb{R}}$. Although the same notion is also involved into this work, the new idea is that we provide an appropriate complete lattice. To this end we work with a subset of the power set of a given partially ordered topological vector space. The construction and the properties of this complete lattice are the subject of this chapter. We

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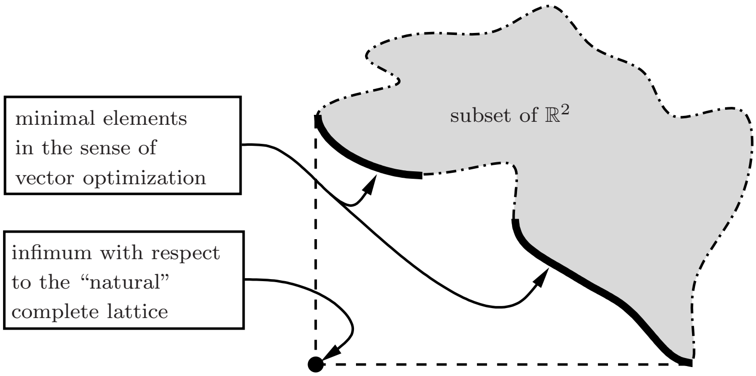

Fig. 1.1 $\overline{\mathbb{R}^2}$ equipped with the natural ordering provides a complete lattice. But, the infimum can be far away from the minimal elements.

recall in this chapter several standard notions and results but we also present the basics of a set-valued approach to vector optimization.

1.1 Partially ordered sets and complete lattices

This section is a short summary of several concepts and results related to ordered sets as they are required for this exposition.

Definition 1.1. Let Z be a nonempty set. A relation $R \subseteq Z \times Z$ is called a *partial ordering* on Z if the following properties are satisfied:

(i) R is reflexive: $\forall z \in Z : (z,z) \in R$,
(ii) R is transitive: $[(z^1,z^2) \in R \wedge (z^2,z^3) \in R] \implies (z^1,z^3) \in R$,
(iii) R is antisymmetric: $[(z^1,z^2) \in R \wedge (z^2,z^1) \in R] \implies z^1 = z^2$.

Instead of $(z^1,z^2) \in R$, we write $z^1 \leq_R z^2$.

The index R is usually omitted or replaced (for instance, if the ordering is generated by a cone C, we write $z^1 \leq_C z^2$ whenever $z^2 - z^1 \in C$) and we just say that $\leq$ is a partial ordering. A nonempty set Z equipped with a partial ordering on Z is called a *partially ordered set*. It is denoted by $(Z,\leq)$. The following convention is used throughout: If $(Z,\leq)$ is a partially ordered set and $A \subseteq Z$, we speak about a subset of the partially ordered set $(Z,\leq)$.

Definition 1.2. Let $(Z,\leq)$ be a partially ordered set and let $A \subseteq Z$. An element $l \in Z$ is called a *lower bound* of A if $l \leq z$ for all $z \in A$. An upper bound is defined analogously.

Next we define an *infimum* and a *supremum* for a subset A of a partially ordered set $(Z, \leq)$.

Definition 1.3. Let $(Z, \leq)$ be a partially ordered set and let $A \subseteq Z$. An element $k \in Z$ is called a *greatest lower bound* or *infimum* of $A \subseteq Z$ if k is a lower bound of A and for every other lower bound l of A we have $l \leq k$. We use the notation $k = \inf A$ for the infimum of A, if it exists.

The *least upper bound* or *supremum* is defined analogously and is denoted by $\sup A$. The lower (upper) bound of Z, if it exists, is called *least (greatest)* element.

Proposition 1.4. *Let $(Z, \leq)$ be a partially ordered set and let $A \subseteq Z$. If the infimum of A exists, then it is uniquely defined.*

Proof. Let both k and l be infima of A. Then, l and k are lower bounds of A. The definition of the infimum yields $l \leq k$ and $k \leq l$. As $\leq$ is antisymmetric, we get $l = k$. □

Definition 1.5. A partially ordered set $(Z, \leq)$ is called a *complete lattice* if the infimum and supremum exist for every subset $A \subseteq Z$.

Note that a one-sided condition is already sufficient to characterize a complete lattice.

Proposition 1.6. *A partially ordered set $(Z, \leq)$ is a complete lattice if and only if the infimum exists for every subset $A \subseteq Z$.*

Proof. Let $A \subseteq Z$ be a given set and let $B \subseteq Z$ be the set of all upper bounds of A. By assumption, $p := \inf B$ exists. As p is a lower bound of B, $z \geq p$ holds for every upper bound z of A. Every $z \in A$ is a lower bound of B. By the definition of the infimum we get $p \geq z$ for every $z \in A$. Together we have $p = \sup A$. □

Example 1.7. The extended real numbers $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ equipped with the usual ordering $\leq$ provide a complete lattice.

Example 1.8. Let $\leq$ be the componentwise ordering relation in $\mathbb{R}^q$. If the ordering relation $\leq$ is extended to $Z := \mathbb{R}^q \cup \{\pm\infty\}$ by setting $-\infty \leq z \leq +\infty$ for all $z \in Z$, $(Z, \leq)$ provides a complete lattice. The infimum of a subset $A \subseteq Z$ is

$$\inf A = \begin{cases} \left(\inf\limits_{z \in A} z_1, \ldots, \inf\limits_{z \in A} z_q \right)^T & \text{if } \exists b \in \mathbb{R}^q, \forall z \in A : b \leq z \\ +\infty & \text{if } A = \emptyset \\ -\infty & \text{otherwise.} \end{cases}$$

Example 1.9. Let $Z = \mathbb{R}^3$ and let C be the polyhedral (convex) cone which is spanned by the vectors $(0,0,1)^T,\ (0,1,1)^T,\ (1,0,1)^T,\ (1,1,1)^T$. Then $(Z, \leq_C)$ is not a complete lattice. For instance, there is no supremum of the finite set $\{(0,0,0)^T, (1,0,0)^T\}$.

Note that the previous example is a special case of the following result by Peressini (1967): $\mathbb{R}^n$ *is an Archimedean vector lattice with respect to the order generated by a cone* C *if and only if there are* n *linearly independent vectors* v^i *such that*

$$C := \left\{x \in \mathbb{R}^n \,|\, \forall i = 1, \dots, n : \left\langle x, v^i \right\rangle \geq 0\right\}. \tag{1.1}$$

Note further that, as pointed out by Anderson and Annulis (1973), the word "Archimedean" is inadvertently omitted in (Peressini, 1967). A vector lattice is Archimedean if

$$(\forall n \in \mathbb{N} : 0 \leq nx \leq z) \implies x = 0.$$

The pair $(\mathbb{R}^2, L)$, where

$$L := \left\{x \in \mathbb{R}^2 \,|\, x_1 > 0 \vee [x_1 = 0 \wedge x_2 \geq 0]\right\}$$

is the lexicographic ordering cone provides an example of a vector lattice, which is not Archimedean. As demonstrated in (Anderson and Annulis, 1973), L cannot be expressed as in (1.1).

Example 1.10. Let X be a nonempty set and let $\mathcal{P}(X) = 2^X$ be the power set of X. $(\mathcal{P}(X), \supseteq)$ provides a complete lattice. The infimum and supremum of a nonempty subset $\mathcal{A} \subseteq \mathcal{P}(X)$ are given as

$$\inf \mathcal{A} = \bigcup_{A \in \mathcal{A}} A \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A.$$

Note that $X \in \mathcal{P}(X)$ is the least element and $\emptyset \in \mathcal{P}(X)$ is the greatest element in $(\mathcal{P}(X), \supseteq)$. If $\mathcal{A}$ is empty, we set $\sup \mathcal{A} = X$ and $\inf \mathcal{A} = \emptyset$.

Example 1.11. Let X be a vector space and let $\mathcal{C}(X)$ be the family of all convex subsets of X. $(\mathcal{C}(X), \supseteq)$ provides a complete lattice. The infimum and supremum of a nonempty subset $\mathcal{A} \subseteq \mathcal{C}(X)$ are given as

$$\inf \mathcal{A} = \operatorname{co} \bigcup_{A \in \mathcal{A}} A \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A.$$

If $\mathcal{A}$ is empty, we set again $\sup \mathcal{A} = X$ and $\inf \mathcal{A} = \emptyset$.

Example 1.12. Let X be a topological space and let $\mathcal{F}(X)$ be the family of all closed subsets of X. $(\mathcal{F}(X), \supseteq)$ provides a complete lattice. The infimum and supremum of a nonempty subset $\mathcal{A} \subseteq \mathcal{F}(X)$ are given as

$$\inf \mathcal{A} = \operatorname{cl} \bigcup_{A \in \mathcal{A}} A \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A.$$

If $\mathcal{A}$ is empty, we set again $\sup \mathcal{A} = X$ and $\inf \mathcal{A} = \emptyset$.

Example 1.13. Let X be a set, L a complete lattice and $\mathcal{L}(X)$ be the set of all L-valued functions on X. A partial ordering on $\mathcal{L}(X)$ is defined by

$$l_1 \leq l_2 :\Longleftrightarrow \forall x \in X : l_1(x) \leq l_2(x).$$

Then $(\mathcal{L}(X), \leq)$ provides a complete lattice. The infimum and supremum are given by the pointwise infimum and supremum.

1.2 Conlinear spaces

Convexity is one of the most important concepts in optimization. A *convex set* C is usually defined to be a subset of a vector space X satisfying the condition

$$[\lambda \in [0,1] \wedge x, y \in C] \implies \lambda x + (1-\lambda) y \in C. \tag{1.2}$$

A very important special case of a convex set is a *convex cone*, where a *cone* is defined to be a set K satisfying

$$[\lambda \in \mathbb{R}_+, x \in K] \implies \lambda x \in K, \tag{1.3}$$

where $\mathbb{R}_+ := \{\lambda \in \mathbb{R} |\ \lambda \geq 0\}$.

We observe that neither of the definitions require a multiplication by a negative real number. It is therefore consistent to define convexity on more general spaces. This can be motivated by the examples below showing convex sets and convex cones which cannot be embedded into a linear space (vector space). The natural framework for convexity seems to be a *conlinear space* rather than a linear one.

Definition 1.14. A nonempty set Z equipped with an addition $+ : Z \times Z \to Z$ and a multiplication $\cdot : \mathbb{R}_+ \times Z \to Z$ is said to be a *conlinear space* with the *neutral element* $\theta \in Z$ if for all $z, z^1, z^2 \in Z$ and all $\alpha, \beta \geq 0$ the following axioms are satisfied:

(C1) $z^1 + \left(z^2 + z\right) = \left(z^1 + z^2\right) + z$,
(C2) $z + \theta = z$,
(C3) $z^1 + z^2 = z^2 + z^1$,
(C4) $\alpha \cdot (\beta \cdot z) = (\alpha\beta) \cdot z$,
(C5) $1 \cdot z = z$,
(C6) $0 \cdot z = \theta$,
(C7) $\alpha \cdot \left(z^1 + z^2\right) = \left(\alpha \cdot z^1\right) + \left(\alpha \cdot z^2\right)$.

An instance of a conlinear space is given in Example 1.31 below. The axioms of a conlinear space $(Z, +, \cdot)$ are appropriate to deal with convexity. A convex set and a cone in a conlinear space are defined, respectively, by (1.2) and (1.3). The *convex hull* $\operatorname{co} A$ of a subset A of a conlinear space $(Z, +, \cdot)$

is the intersection of all convex sets in Z containing A. The convex hull of a set A coincides with set of all (finite) convex combinations of elements of A (Hamel, 2005, Theorem 3).

Additionally to the axioms (C1) to (C7), it is sometimes useful to use a second distributive law, that is, for all $z \in Z$ and all $\alpha, \beta \geq 0$ we can suppose additionally that

(C8) $\alpha \cdot z + \beta \cdot z = (\alpha + \beta) \cdot z.$

In a conlinear space, singleton sets are not necessarily convex, see Example 1.17 below. Indeed this requirement is equivalent to (C8).

Proposition 1.15. *For every conlinear space, the following statements are equivalent:*

(i) Every singleton set is convex,
(ii) (C8) holds.

Proof. This is obvious. □

Definition 1.16. An element $\bar{z}$ of a conlinear space Z is said to be *convex*, if the set $\{\bar{z}\}$ is convex.

Example 1.17. An element of a conlinear space can be nonconvex. Indeed, let $Z = \mathcal{P}(\mathbb{R})$ and consider the element $A := \{0, 1\} \in Z$. We have $\frac{1}{2}A + \frac{1}{2}A = \left\{0, \frac{1}{2}, 1\right\} \neq A$.

If $(Z, +, \cdot)$ is a conlinear space, we denote by Z_{co} the subset of all $z \in Z$ satisfying (C8).

Proposition 1.18. $(Z_{\text{co}}, +, \cdot)$ *is a conlinear space.*

Proof. Let $z^1, z^2 \in Z_{\text{co}}$. For $\alpha, \beta \geq 0$, we have

$$\begin{aligned}
\alpha(z^1 + z^2) + \beta(z^1 + z^2) &\overset{\text{(C7), (C3)}}{=} \alpha z^1 + \beta z^1 + \alpha z^2 + \beta z^2 \\
&\overset{\text{(C8)}}{=} (\alpha + \beta) z^1 + (\alpha + \beta) z^2 \\
&\overset{\text{(C7)}}{=} (\alpha + \beta)(z^1 + z^2),
\end{aligned}$$

i.e., $z^1 + z^2 \in Z_{\text{co}}$. Similarly we obtain $\gamma \cdot z^1 \in Z_{\text{co}}$ for $\gamma \geq 0$. □

If a conlinear space is equipped with an ordering relation, it is useful to require that this ordering relation is compatible with the conlinear structure. The same procedure is well-known for partially ordered vector spaces.

Definition 1.19. Let $(Z, +, \cdot)$ be a conlinear space and let $\leq$ be a partial ordering on the set Z. $(Z, +, \cdot, \leq)$ is called a *partially ordered conlinear space* if for every $z^1, z^2, z \in Z$ and every $\alpha \in \mathbb{R}_+$ the following conditions hold:

(O1) $z^1 \le z^2 \implies z^1 + z \le z^2 + z$,
(O2) $z^1 \le z^2 \implies \alpha z^1 \le \alpha z^2$.

Of course, a partially ordered vector space is a special case of a partially ordered conlinear space. Let us define convex functions in the general setting of conlinear spaces.

Definition 1.20. Let W be a conlinear space and let Z be a partially ordered conlinear space. A function $f : W \to Z$ is said to be *convex* if

$$\begin{aligned}&\forall \lambda \in [0,1], \forall w^1, w^2 \in W :\\ &\quad f\big(\lambda \cdot w^1 + (1-\lambda)\cdot w^2\big) \le \lambda \cdot f(w^1) + (1-\lambda)\cdot f(w^2).\end{aligned}$$

Concave functions are defined likewise.

1.3 Topological vector spaces

A well-known concept is that of a topological vector space, also called topological linear space or linear topological space (see e.g. Kelley *et al.*, 1963; Köthe, 1966; Schaefer, 1980). The idea is to equip a linear space with a topology and to require that the topology is compatible with the linear structure. Many useful results depend on this compatibility.

Definition 1.21. Let Y be a real linear space (vector space) and let τ be a topology on Y. The pair (Y, τ) is called a *topological vector space* (or *linear topological space*) if the following two axioms are satisfied:

(L1) $(y^1, y^2) \longmapsto y^1 + y^2$ is continuous on $Y \times Y$ into Y,
(L2) $(\lambda, y) \longmapsto \lambda y$ is continuous on $\mathbb{R} \times Y$ into Y.

If there is no risk of confusion, a topological vector space (Y, τ) is simply denoted by Y. We write $\operatorname{int} A$ and $\operatorname{cl} A$, respectively, for the interior and closure of a subset A of a topological vector space Y. The boundary of $A \subseteq Y$ is the set $\operatorname{bd} A := \operatorname{cl} A \setminus \operatorname{int} A$.

Proposition 1.22. *Let Y be a topological vector space.*

(i) *For any subset A of Y and any base $\mathcal{U}$ of the neighborhood filter of $0 \in Y$, the closure of A is given by*

$$\operatorname{cl} A = \bigcap_{U \in \mathcal{U}} A + U.$$

(ii) *If A is an open subset of Y and B is any subset of Y, then $A + B$ is open.*

Proof. See e.g. Schaefer (1980). □