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Model Theory and Applications

Bressanone, Italy 1975

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FONDAZIONE CIME ROBERTO CONTI

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Model Theory and Applications

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To the Memory of Abraham Robinson

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

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MODEL THEORY AND APPLICATIONS

Coordinatore: Prof. P. MANGANI

Centro Internazionale Matematico Estivo

"Model Theory and Applications"

(Second 1975 C. I. M. E. Session)

Lecture Notes for Course (a)

Theories of Algebraic Type

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1. Fundamentals

Buon giorno. This is the first of eight lectures on the model theoretic notion of theory of algebraic type. Some examples of the notion are the theories of algebraically closed fields of characteristic p ($p \geq 0$), real closed fields and differentially closed fields of characteristic 0. The last example is the most important for two reasons. First, it is the only one known whose complexity matches that of the general case. Second, several results about differential fields, results which hold for all theories of algebraic type, were first proved by model theoretic means.

The key definition is quite compact, but five lectures will be needed to unpack it. A theory T is said to be of <u>algebraic</u> <u>type</u> if T is complete, T is the model completion of a universal theory, and T is quasi-totally transcendental. In the brief time left before the onset of formalities, let me indicate why the theory of algebraically closed fields of characteristic 0 (ACF_0) is of algebraic type. The completeness of ACF_0 means that the same first order sentences are true in all algebraically closed fields of characteristic 0. Thus a first order sentence in the language of fields is true of the complex numbers if and only if it is true of the algebraic numbers.

ACF_0 is the model completion of TF_0, the theory of fields of characteristic 0. To say TF_0 is a universal theory is equivalent to saying a subset of a field of characteristic 0 closed under $+, \cdot$, etc. is a field of characteristic 0. To see the meaning of model completion, let $\mathfrak{A}$ be any field of characteristic 0 and let F be any first order sentence in the language of fields with parameters in $\mathfrak{A}$. (For example, F might say that some finite set of polynomials in several variables with coefficients in $\mathfrak{A}$ has a common zero.) To claim that ACF_0 is the model completion of TF_0 amounts to claiming F is true in all or in none of the algebraically closed extensions of $\mathfrak{A}$.

The property of quasi-total transcendality is too complex to elucidate in a lecture on fundamentals. For the moment think of it as a density condition on simply generated extensions of structures weakly exemplified by the density of the rationals in the reals. If T is quasi-totally transcendental, then each substructure $\mathfrak{A}$ of a model of T has a prime model extension, and all prime model extensions of $\mathfrak{A}$ are isomorphic over $\mathfrak{A}$. In the case of ACF_0, this means each field $\mathfrak{A}$ of characteristic 0 has a unique prime algebraically closed extension, namely the algebraic closure of $\mathfrak{A}$.

And now the fundamentals of model theory. A <u>similarity type</u> τ is a 5-tuple $\langle I, J, K, \theta, \psi \rangle$ such that $\theta : I \to N$ and $\psi : J \to N$, where N is the set of positive integers. A <u>structure</u> $\mathfrak{A}$ of type τ consists of:

(i) A nonempty set A called the universe of $\mathfrak{A}$.

(ii) A family $\{R_i^{\mathfrak{A}} \mid i \in I\}$ of relations. Each $R_i^{\mathfrak{A}}$ is a subset of $A^{\theta(i)}$.

(iii) A family $\{f_j^{\mathfrak{A}} \mid j \in J\}$ of functions. Each $f_j^{\mathfrak{A}}$ maps $A^{\psi(j)}$ into A.

(iv) A subset $\{c_k^{\mathcal{A}} \mid k \in K\}$ of A called the set of distinguished elements of A.

One often writes

$$\mathcal{A} = \langle A, R_i^{\mathcal{A}}, f_j^{\mathcal{A}}, c_k^{\mathcal{A}} \rangle_{i \in I,\, j \in J,\, k \in K}\,.$$

The cardinality of $\mathcal{A}$ is by definition the cardinality of A. Structures will be denoted by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$, and their universes by $A, B, C, \ldots$.

Consider the structure

$$\mathcal{A} = \langle A, +, \cdot, -, {}^{-1}, 0, 1 \rangle\,,$$

where $+$ and $\cdot$ are 2-place functions on A, $-$ and ${}^{-1}$ are 1-place functions on A, and 0 and 1 are distinguished elements of A. The concept of field can be formulated so that every field has the same similarity type as $\mathcal{A}$, but $\mathcal{A}$ need not be a field since the relations, functions and distinguished elements of $\mathcal{A}$ need not satisfy the axioms for fields.

A monomorphism $m : \mathcal{A} \to \mathcal{B}$ is a one-one map $m : A \to B$ such that:

(i) $R_i^{\mathcal{A}}(a_1, \ldots, a_n)$ iff $R_i^{\mathcal{B}}(ma_1, \ldots, ma_n)$ ($i \in I$ and $n = \theta(i)$).

(ii) $mf_j^{\mathcal{A}}(a_1, \ldots, a_n) = f_j^{\mathcal{B}}(ma_1, \ldots, ma_n)$ ($j \in J$ and $n = \psi(i)$).

(iii) $mc_k^{\mathcal{A}} = c_k^{\mathcal{B}}$ ($k \in K$).

(It is assumed that $\mathcal{A}$ and $\mathcal{B}$ are both of type τ.)

$\mathcal{A}$ is a <u>substructure</u> of $\mathcal{B}$ ($\mathcal{A} \subset \mathcal{B}$) if $A \subset B$ and the inclusion map $i_A : A \subset B$ is a monomorphism. An <u>isomorphism</u> is an onto monomorphism. An isomorphism is indicated by $m : \mathcal{A} \overset{\approx}{\to} \mathcal{B}$ or by $\mathcal{A} \approx \mathcal{B}$.

Each similarity type τ gives rise to a <u>first order language</u> $\mathcal{L}_\tau$ whose sentences are interpretable in structures of type τ. The primitive symbols of $\mathcal{L}_\tau$ are:

(i) first order variables $x, y, z, \ldots$;

(ii) logical connectives $\sim$ (not), & (and), E (there exists), and = (equals);

(iii) a $\theta(i)$-place relation symbol R_i $(i \in I)$;

(iv) a $\psi(j)$-place function symbol f_j $(j \in J)$;

(v) an individual constant $\underline{c}_k$ $(k \in K)$.

The <u>terms</u> of $\mathcal{L}_\tau$ are generated by two rules: all variables and individual constants are terms; if f_j is an n-place function symbol and $t_1, \ldots, t_n$ are terms, then $f_j(t_1, \ldots, t_n)$ is a term.

The <u>atomic formulas</u> are: equations such as $t_1 = t_2$, where t_1 and t_2 are terms; and $R_i(t_1, \ldots, t_n)$, where R_i is an n-place relation symbol and $t_1, \ldots, t_n$ are terms.

The <u>formulas</u> are generated from the atomic formulas as follows: if F and G are formulas, then $\sim$F, F & G and (Ex)F are formulas, where x is any variable.

$\vee$ (or), $\rightarrow$ (implies), $\leftrightarrow$ (if and only if), and (x) (for all x) are abbreviations: F $\vee$ G for $\sim(\sim$F & $\sim$G), F $\rightarrow$ G for ($\sim$F) $\vee$ G, F $\leftrightarrow$ G for (F $\rightarrow$ G) & (G $\rightarrow$ F), and (x)F for $\sim$(Ex)$\sim$F.

The predicate, x is a <u>free variable</u> of the formula F, is defined by recursion on the number of steps needed to generate F: if F is atomic and x occurs in F, then x is a free variable of F; if x is a free variable of F, then x is a free variable of $\sim F$, of F & G and of G & F; if x is a free variable of F and y is a variable distinct from x, then x is a free variable of (Ey)F.

The only way to kill a free variable x of F is to prefix F with (Ex). A useful convention is: all the free variables of G(x, y, z) lie among x, y, z.

A sentence is a formula with no free variables.

Each sentence of $\mathcal{L}_\tau$ has a definite truth value in each structure $\mathfrak{A}$ of type τ. As an aid in defining truth, consider the language $\mathcal{L}_{\tau A}$ obtained by adding a new individual constant $\underline{a}$ for each $a \in A$ to the language $\mathcal{L}_\tau$. The formulas of $\mathcal{L}_{\tau A}$ are merely the formulas of $\mathcal{L}_\tau$ with some of the free variables replaced by individual constants naming elements of A. Each constant term (no variables) t of $\mathcal{L}_{\tau A}$ names some element σt of A as follows:

(i) $\sigma\underline{a} = a$ and $\sigma\underline{c}_k = c_k^{\mathfrak{A}}$.

(ii) $\sigma f_j(t_1, \ldots, t_n) = f_j^{\mathfrak{A}}(\sigma t_1, \ldots, \sigma t_n)$.

Let H be a sentence of $\mathcal{L}_{\tau A}$. The relation $\mathfrak{A} \models H$ (H is true in $\mathfrak{A}$) is defined by recursion on the number of steps needed to generate H from the atomic formulas of $\mathcal{L}_{\tau A}$:

$\mathfrak{A} \models t_1 = t_2$ iff $\sigma t_1 = \sigma t_2$.

$\mathfrak{A} \models R_i(t_1, \ldots, t_n)$ iff $R_i^{\mathfrak{A}}(\sigma t_1, \ldots, \sigma t_n)$.

$\mathfrak{A} \models F \,\&\, G$ iff $\mathfrak{A} \models F$ and $\mathfrak{A} \models G$.

$\mathfrak{A} \models \sim F$ iff it is not the case that $\mathfrak{A} \models F$.

$\mathfrak{A} \models (Ex)F(x)$ iff $\mathfrak{A} \models F(\underline{a})$ for some $a \in A$.

If the sentence H is not true in $\mathfrak{A}$, then it is said to be false. $\langle a_1, \ldots, a_n \rangle$ <u>satisfies</u> (or realizes) $F(x_1, \ldots, x_n)$ in $\mathfrak{A}$ if $\mathfrak{A} \models F(\underline{a}_1, \ldots, \underline{a}_n)$.

It is now quite simple to say what a field is. The similarity type of a field is exemplified by the structure $\mathfrak{A}$:

$$\langle A, +^{\mathfrak{A}}, \cdot^{\mathfrak{A}}, -^{\mathfrak{A}}, (-1)^{\mathfrak{A}}, 0^{\mathfrak{A}}, 1^{\mathfrak{A}} \rangle .$$

The nonlogical primitive symbols of the language associated with the similarity type of fields are: $+, \cdot, -, {}^{-1}, 0$ and 1. The theory of fields (TF) is the following set of sentences:

$$(x)(y)(z)[(x+y)+z = x+(y+z)].$$

$$(x)[x+0 = x].$$

$$(x)[x+(-x) = 0].$$

$$(x)(y)[x+y = y+x].$$

$$(x)(y)(z)[(x\cdot y)\cdot z = x\cdot (y\cdot z)].$$

$$(x)[x\cdot 1 = x].$$

$$(x)[x \neq 0 \rightarrow x\cdot x^{-1} = 1].$$

$$(x)(y)[x\cdot y = y\cdot x].$$

$$(x)(y)(z)[x\cdot (y+z) = (x\cdot y)+(x\cdot z)].$$

$$0 \neq 1.$$

$\mathfrak{A}$ is a field iff it has the similarity type specified above and every sentence of TF is true in $\mathfrak{A}$.

$\mathfrak{A}$ is <u>first order</u> (or <u>elementarily</u>) equivalent to $\mathfrak{B}$ ($\mathfrak{A} \equiv \mathfrak{B}$) means: $\mathfrak{A} \models F$ iff $\mathfrak{B} \models F$ for every sentence F. (It is assumed that

$\mathfrak{A}$ and $\mathfrak{B}$ belong to the same similarity type τ, and that F is a sentence of $\mathcal{L}_\tau$.) In the next lecture it will be seen that any two algebraically closed fields of the same characteristic are first order equivalent. More generally it will be observed that any two models of a theory of algebraic type are first order equivalent.

An elementary monomorphism $m : \mathfrak{A} \overset{\equiv}{\to} \mathfrak{B}$ is a map of A into B such that

$$\mathfrak{A} \models F(\underline{a}_1, \ldots, \underline{a}_n) \text{ iff } \mathfrak{B} \models F(\underline{ma}_1, \ldots, \underline{ma}_n)$$

for every formula $F(x_1, \ldots, x_n)$ and every sequence $a_1, \ldots, a_n \in A$. An elementary monomorphism m is necessarily a monomorphism, since

$$\mathfrak{A} \models \underline{a}_1 = \underline{a}_2 \text{ iff } \mathfrak{B} \models \underline{ma}_1 = \underline{ma}_2 .$$

Note that a map m of A into B is an elementary monomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ iff

$$\langle \mathfrak{A}, a \rangle_{a \in A} \equiv \langle \mathfrak{B}, ma \rangle_{a \in A} .$$

(The similarity type of $\langle \mathfrak{A}, a \rangle_{a \in A}$ is $\mathcal{L}_{\tau A}$.)

Proposition 1. Suppose $f : \mathfrak{A} \to \mathfrak{B}$ and $g : \mathfrak{B} \to \mathfrak{C}$.

(i) If f and g are elementary, then gf is elementary.

(ii) If g and gf are elementary, then f is elementary.

$\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ (or $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$) if $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and the inclusion map $i_A : A \subset B$ is an elementary monomorphism ($\mathfrak{A} \prec \mathfrak{B}$). In Lecture 3 it will be shown that every monomorphism between models of a theory of algebraic type is elementary. Grazie, e buon giorno.

2. Existence of Models

B. g. Today I will describe two approaches to the construction of models, the first via the extended completeness theorem of first order logic, and the second via direct limits. (A structure $\mathfrak{A}$ is said to be a model of a set S of sentences if $\mathfrak{A} \models G$ for every $G \in S$.)

A formula F is a logical consequence of S ($S \vdash F$) if F is among the formulas generated from S as follows: $F \in S$; F is an axiom of first order logic; F is the result of applying some rule of inference of first order logic to $F_1, \ldots, F_n$ when $S \vdash F_i$ $(1 \leq i \leq n)$.

The axioms and rules of first order logic with equality are consonant with common sense, so they need not be listed here. One point worth noting is the finite character of logical consequence: if $S \vdash F$, then $S_0 \vdash F$ for some finite $S_0 \subset S$.

S is logically consistent if no sentence of the form $F \,\&\, {\sim}F$ is a logical consequence of S.

Theorem 2.1. S is consistent iff S has a model.

Proof. If S has a model, then no contradiction is a logical consequence of S, because every consequence of S is true in every model of S.

Suppose S is consistent. The construction of the model is in two steps. First S is Henkinized by adding new individual constants to the language of S, and new sentences to S that force the new constants to be witnesses.. Let $S_0 = S$. S_{n+1} consists of S_n together with all sentences of the form

$$(1) \qquad (Ex)F(x) \rightarrow F(\underline{c}_{F(x)}),$$

where $F(x)$ is a formula in the language of S_n, and $\underline{c}_{F(x)}$ is a new individual constant. Let S_∞ be $\bigcup\{S_n \mid n < \omega\}$. The consistency of S_∞ is easily checked by induction on n. For example, suppose S_0 $(= S)$ together with (1) yield a contradiction. Then S_0 yields the negation of (1), and so

$$S_0 \vdash (Ex)F(x) \mathbin{\&} \sim F(\underline{c}_{F(x)}).$$

Since $\underline{c}_{F(x)}$ does not occur in S_0, the derivation of $\sim F(\underline{c}_{F(x)})$ from S_0 is equivalent to the derivation of $\sim F(y)$ from S_0, where y is some variable not occurring in S_0. Since S_0 in no way limits y, the derivation of $\sim F(y)$ from S_0 is equivalent to one of $(y)\sim F(y)$ from S_0. But then

$$S_0 \vdash (Ex)F(x) \mathbin{\&} (y)\sim F(y),$$

an impossibility since S_0 is consistent.

The second step is an application of Zorn's lemma made possible by the finite character of the logical consequence relation $\vdash$. S_∞ is extended to S_∞^*, a maximal consistent set of sentences in the language of S_∞. Note that every sentence or its negation belongs to S_∞.

S_∞^* defines a model $\mathcal{A}$ of S as follows. With each individual constant $\underline{c}$ of the language $\mathcal{L}_\infty$ of S_∞, associate the equivalence class

$$[\underline{c}] = \{\underline{d} \mid \underline{c} = \underline{d} \in S^*_\infty\}.$$

The universe A of $\mathcal{A}$ consists of $\{[\underline{c}] \mid \underline{c} \in \mathcal{L}_\infty\}$. The relations, functions and distinguished elements of $\mathcal{A}$ are defined by:

$$R_i^{\mathcal{A}}([\underline{c}_1], \ldots, [\underline{c}_n]) \text{ iff } R_i(\underline{c}_1, \ldots, \underline{c}_n) \in S^*_\infty,$$

$$f_j^{\mathcal{A}}([\underline{c}_1], \ldots, [\underline{c}_n]) = [\underline{c}] \text{ iff } f_j(\underline{c}_1, \ldots, \underline{c}_n) = \underline{c} \in S^*_\infty,$$

$$c_k^{\mathcal{A}} = [\underline{c}_k].$$

An induction on the complexity of (i.e. number of steps needed to generate) F shows $\mathcal{A} \models F$ iff $F \in S^*_\infty$, where F is any sentence of $\mathcal{L}_\infty$. It follows $\mathcal{A}$ is a model of S. $\square$

Theorem 2.1 is the work of K. Gödel, T. Skolem and A. Tarski; the proof given is due to L. Henkin.

A <u>theory</u> T is (according to the definition I prefer) a consistent set of sentences. Thus every theory has a model by 2.1. $T_1 \subset T_2$ means every logical consequence of T_1 is also one of T_2. $T_1 = T_2$ means $T_1 \subset T_2$ and $T_2 \subset T_1$. T is <u>complete</u> if either $T \vdash F$ or $T \vdash \sim F$ for every sentence F in the language of T. By 2.1 T is complete iff all models of T are elementarily equivalent.

Corollary 2.2 (compactness). Suppose S is a set of sentences such that each finite subset of S has a model. Then S has a model.

Countability Proviso: From now on assume every structure has at most