A. Avantaggiati (Ed.)

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Pseudodifferential Operators with Applications

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FONDAZIONE CIME ROBERTO CONTI

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Lectures given at a Summer School of the Centro Internazionale Matematico Estivo (C.I.M.E.), held in Bressanone (Bolzano), Italy, June 16-24, 1977

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CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

II Ciclo - Bressanone dal 16 al 24 giugno 1977

PSEUDODIFFERENTIAL OPERATORS WITH APPLICATIONS

Coordinatore: Prof. A. Avantaggiati

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

PSEUDO-DIFFERENTIAL OPERATORS ON HEISENBERG GROUPS

A. DYNIN

Corso tenuto a Bressanone dal 16 al 24 giugno 1977

Introduction

The convolution operators on the euclidean spaces are only a particular case of convolution operators on arbitrary Lie groups. Pseudo-differential operators are roughly speaking convolution operators with variable coefficients. In classical theory of such operators it is important to have standard dilations on a euclidean space. So we pay attention only to Lie groups with dilation i. e. with 1-dimensional group of automorphisms converging to infinity when real parameter increases to infinity.

It is well known that all Lie groups with dilations are nilpotent.

In this seminar I consider only Heisenberg Groups. These groups are the simplest non-abelian nilpotent groups. They appear in Complex Analysis and such operators on strictly pseudoconvex boundaries as J. Kohn sub-Laplacian, induced Cauchy-Riemann operators, singular integral operators of Cauchy-Henkintype can be locally considered as convolution operators wtih variable coefficients on Heisenberg groups. The characteristic feature of all these operators is an anisotropy of their singularities tight with complex tangent directions. Actually this is a contact structure. Generally a contact structure is given on 2n+1 -manifold by 1-form ω such that the (2n+1) -form $\omega \wedge d\omega \wedge \dots \wedge d\omega \neq 0$

If a strictly pseudoconvex boundary is defined by real function ρ with

$d\rho \neq 0$ then we may take $\omega = \frac{1}{i}(\partial\rho - \bar{\partial}\rho)$. By D-arboux theorem Heisenberg groups are local models for any contact manifolds.

I construct a theory of pseudo-differential operators which belong to the contact structure as classical pseudo-differential operators belong to the smooth structure.

It is impossible to derive a theory of pseudo-differential operators without an elliptic accompaniment. I introduce a kind of ellipticity which as I hope can elucidate some striking analogies between non-elliptic in usual sense $\bar{\partial}_b$-complex and elliptic complexes.

Note that the problem of pseudo-differential operators on homogeneous Lie groups was put forward by E. Stein at the Nice congress (cf /4/).

Our results were mainly announced in /2/ and /3/. Here we exposed them in more precise form.

1 Heisenberg Lie groups and algebras

Heisenberg algebra H_n, $n > 0$, can be obtain from the standard euclidean space $\mathbb{R}^{2n+1}$ if we supply it with such commutators:

if $x = (x_0, x^1, x^{11}) \in \mathbb{R}^{2n+1}$ where $x_0 \in \mathbb{R}$, $x^1, x^{11} \in \mathbb{R}^n$

$e_0 = (1; 0, 0,) \in \mathbb{R}^{2n+1}$ then

$$[x, y] = 4\,(\langle x^1, y^{11}\rangle - \langle x^{11}, y^1\rangle)\, e_0$$

This Heisenberg algebra is a Lie algebra of the step 2 all $[x,[y,z]]$ are zero. Let $\mathbb{H}_n$ be a corresponding simply-connected Lie group. As manifold it is identified with $\mathbb{R}^{2n+1}$; the multiplication is

$$x * y = (x_0 + y_0 + 2\,(\langle x', y''\rangle - \langle x'', y'\rangle),\ x' + y',\ x'' + y'')$$

Therefore O serves as unit and $x^{-1} = -x$ The group $\mathbb{H}_n$ is called the Heisenberg group.

The identity mapping $x \mapsto x$ coincides with the exponential

$$\exp: H_n \to \mathbb{H}_n$$

There are dilations

$$\delta_t x = (t^2 x_0, tx', tx'') \quad , \quad t > 0,$$

in the Heisenberg group $\mathbb{H}_n$. Of course $\delta_{t_1 + t_2} = \delta_{t_1} \delta_{t_2}$ and the δ_t are automorphisms of the Lie structures in $\mathbb{H}_n$ and H_n. Let $s(\mathbb{H}_n)$, $s'(H_n)$ be corresponding spaces on $\mathbb{R}^{2n+1}$.

The operators of the left and of the right shifts by elements $y \in \mathbb{H}_n$

$$L(y) : x \mapsto y^{-1} \times x, \quad R(y) : x \mapsto x \times y, \quad x \in \mathbb{H}_n,$$

are continuous in this spaces. Moreover the Lebesgue measure is bilaterally invariant on the $\mathbb{H}_n$

We may identify the Heisenberg algebra H_n with the Lie algebra of left-invariant 1-st order differential operators. Pick out generators of the complexification of this algebra

$$X_0 = \frac{1}{i} \frac{\partial}{\partial x_0} \quad , \quad X_j' = \frac{1}{i} \left(\frac{\partial}{\partial x_j'} - 2x_j'' \frac{\partial}{\partial x_0} \right)$$

$$X_j'' = \frac{1}{i} \left(\frac{\partial}{\partial x_j''} + 2 x_j' \frac{\partial}{\partial x_0} \right) \quad , \quad j = 1 \, \ldots , n$$

Adopt the notation

$$X' = (X_1', \ldots, X_n') \; , \; X'' = (X_1'', \ldots X_n''), \; X = (X_0, X', X'')$$

The contact structure on the $\mathbb{H}_n$ is defined by a left-invariant 1-form

$$\omega(x) = dx_0 + 2 \langle x', dx'' \rangle - 2 \langle x'', dx' \rangle$$

3. H. Weyl quantization.

The modern theory of pseudo-differential operators took its shape in the sixties however we can find its origin as early as in the beginning of the thirties: there was a problem of quantization in the Quantum Mechanics and H. Weyl gave a general solution. The problem is to construct of non-cummuting operators of multiplication:

$$\hat{x} : u \mapsto x u \; ; \qquad x = (x_1, x_2, \dots, x_n)$$

and of differentiation

$$\hat{p} : u \mapsto \frac{1}{i} \frac{\partial}{\partial x} u , \quad \frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right).$$

This operators generate a 2-step Lie subalgebra W_n in the Lie algebra $\mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ of all continuous linear operators in the $\mathcal{S}(\mathbb{R}^n)$.

Let $\mathbb{W}_n$ be the symple-connected Lie group of this W_n. Then the $\mathbb{W}_n$ can be realized as a Lie group of continuous linear operators in the $\mathcal{S}(\mathbb{R}^n)$. We have the exponential map $\exp : W_n \mapsto \mathbb{W}_n$

Therefore for a $f \in \mathcal{S}(\mathbb{R}^{2n}_{x,p})$ we can define an operator

$$(*) \qquad f(\hat{x}, \hat{p}) = (2\pi)^{-2n} \int \tilde{f}(\xi, \eta) \exp i \left[\langle \xi, \hat{x} \rangle + \langle \eta, \hat{p} \rangle \right] d\xi \, d\eta$$

where

$$\tilde{f}(\xi, \eta) = \int e^{i [\langle x, \xi \rangle + \langle p, \eta \rangle]} f(x, p) \, dx \, dy$$

is the Fourier transform. The expression $(*)$ is similar to the inverse Fourier transform of the $\tilde{f}(\xi, \eta)$.

Let us justify this definition.

It is easy to see that the integral $(*)$ converges in the space $\mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ under the strong topology and actually is an integrål operator with the Schwartz kernel

$$\mathcal{K}_f(x,y) = \frac{1}{(2\pi)^{2n}} \int e^{i\langle \xi, \frac{x+y}{2}\rangle} \tilde{f}(\xi, y-x)\, d\xi .$$

We can rewrite this formula as follows

$$\mathcal{K}_f(x,y) = \mathcal{F}^{-1}_{p\to y-x} f\left(\frac{x+y}{2}, p\right)$$

where $\mathcal{F}^{-1}_{p\to z}$ is the inverse Fourier transformation (from the p to the z)
In this form the definition of $\mathcal{K}_f$ for every $f\in\mathcal{S}'(\mathbb{R}^{2n}_{x,p})$ is valid. If $f\in\mathcal{S}(\mathbb{R}^{2n}_{x,p})$ then $f(\hat{x},\hat{p})$ is a continuous operator from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$
If $f\in\mathcal{S}'(\mathbb{R}^{2n}_{x,p})$ then $f(\hat{x},\hat{p})$ is a continuous operators from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$

Chose moderate operators (cf /5/): Let $m\in\mathbb{R}$. Take (cf. [5])

$W^m(\mathbb{R}^n) =$

$$\left\{ f\in C^\infty(\mathbb{R}^{2n}_{x,p}) : \forall \alpha,\beta\in\mathbb{Z}^m_+ ,\ \exists \text{ constant } C_{\alpha,\beta} \right.$$

such that $\left.\left|\left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial p}\right)^\beta f(x,p)\right| \le C_{\alpha,\beta}(1+|x|+|p|)^{m-|\alpha|-|\beta|}\right\}.$

We say that such f are symbols of **order** m

let

$$O_p W^m = \left\{ f(\hat{x},\hat{p}) : f\in W^m(\mathbb{R}^n)\right\}$$

If $f\in W^m(\mathbb{R}^n)$ then $f(x,\hat{p}) : \mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$

and these operators are called Weyl operators. In particular it is possible to take a product of Weyl operators If $f_j\in O_p W^{m_j}$, $j=1,2$, then there exists a $f\in W^{m_1+m_2}(\mathbb{R}^n)$ such that

$$f(\hat{x}, \hat{p}) = f_1(\hat{x}, \hat{p})\, f_2(\hat{x}, \hat{p})$$

and

$$f(x, p) = (f_1 f_2)(x, p) + \frac{1}{2}\{f_1, f_2\}(x, p)$$ + terms of the lower order. This a simple consequence of the Hausdorff-Campbell formula for the product of exponents.

Moreover

$$f(\hat{x}, \hat{p})^* = \bar{f}(\hat{x}, \hat{p})$$

Let $W_o^m(\mathbb{R}^n)$ be the subspace of $W^m(\mathbb{R}^n)$ whose elements have the following property: there exists $f_o \in W^m(\mathbb{R}^n)$ such that

(i) $f_o(tx, tp) = t^m f_o(x, p)$, $\forall\, t > 1$, $|x| + |p| > 1$

(ii) $f - f_o \in W^{m-1}(\mathbb{R}^n)$

We say that such f are symbols with principal part f_o.

An operator $f(\hat{x}, \hat{p}) \in O_p W_o^m$ is called <u>elliptic</u> if $f_o(x, p) \neq 0$ for $|x| + |p| > 1$; it is known that the elliptic operators are hypoelliptic (cf /5/). Let $Ell\, W_o^m$ be the class of all elliptic operators of order m. If $f(\hat{x}, \hat{p}) \in Ell\, W_o^m$ then there exists its parametrix $r(\hat{x}, \hat{p}) \in Ell\, W_o^{-m}$ with principal part $r_o(x, p) = (f_o(x, p))^{-1}$ for $|x| + |p| > 1$

<u>Example</u>: the Hermite operator $E = -\frac{\partial^2}{\partial x^2} + x^2 \in Ell\, W_o^2$.

Let

$$\mathcal{E}^k(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_k = \|E^{k/2} u\|_{\mathcal{L}^2} < \infty \right\}$$

Then $f(\hat{x}, \hat{p}) \in Ell\, W_o^m$ if and only if $f(\hat{x}, \hat{p})$ is a Fredholm operator from $\mathcal{E}^k(\mathbb{R}^n)$ to $\mathcal{E}^{k-m}(\mathbb{R}^n)$ for some (and therefore for any) $k \in \mathbb{R}$.

4. Pseudo - differential operators on $\mathbb{H}_n$

Apply the Weyl quantization for construction of pseudo-differential operators on $\mathbb{H}_n$.

First of all the operators of multiplication by coordinate functions.

$$\hat{x} : u(x) \to x\, u(x) \;, \qquad x = (x_0, x', x'')$$

and left- invariant operators

$$\hat{X} : u \to X u \qquad\qquad X = (X_0, X', X'')$$

generate a 3-step Lie subalgebra in Lie algebra $\mathcal{L}(\mathcal{S}(\mathbb{H}_n))$. Treating this Lie algebra as above enables to define

$$f(\hat{x}, \hat{X}) =$$

$$(**) \qquad \frac{1}{(2\pi)^{4n+2}} \int \tilde{f}(\xi, \eta) \exp i[\langle \xi, \hat{x} \rangle + \langle \eta, \hat{X} \rangle]\, d\xi\, d\eta$$

If $f \in \mathcal{S}(\mathbb{R}^{4n+2}_{x,X})$ then $(**)$ is an integral operator with the Schwartz Kernel

$$\mathcal{K}_f(x, y) = \mathcal{F}^{-1}_{X \to x * y^{-1}} f\left(\frac{x+y}{2}, X\right).$$

This formula is valid for every $f \in \mathcal{S}'(\mathbb{R}^{4n+2}_{x,X})$ and we can use it for definition of $f(\hat{x}, \hat{X})$

If $f \in \mathcal{S}(\mathbb{R}^{4n+2}_{x,X})$ then $f(\hat{x}, \hat{X}) : \mathcal{S}'(\mathbb{H}_n) \to \mathcal{S}(\mathbb{H}_n)$

If $f \in \mathcal{S}'(\mathbb{R}^{4n+2}_{x,X})$ then $f(\hat{x}, \hat{X}) : \mathcal{S}(\mathbb{H}_n) \to \mathcal{S}'(\mathbb{H}_n)$

Let $[y]$ be a γ_t - homogeneous function on the $\mathbb{H}_n$

$$[y] = (y_0^2 + (y'^2 + y''^2)^2)^{1/4}$$

For any $\beta \in \mathbb{Z}_+^{2n+1}$ put $|\beta| = 2\beta_0 + \beta_1 + \cdots + \beta_n$.

Define $\psi^m(\mathbb{H}_n)$ as

$$\psi^m(\mathbb{H}_n) = \{ f \in C^\infty(\mathbb{R}^{4n+2}) : \forall \alpha, \beta \in \mathbb{Z}_+^{2n+1}, \exists \text{ const. } C_{\alpha,\beta}$$

such that $\left| \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial y}\right)^\beta f(x,y) \right| \leq C_{\alpha,\beta} (1 + [y])^{m - [\beta]} \}$.

We say that such f are symbols of order m

Let

$$O_p \psi^m = \{ f(\hat{x}, \hat{X}) : f \in \psi^m(\mathbb{H}_n) \}.$$

Indicate the main formulas of the Symbolic Calculus:

(I) If $f \in \psi^m(\mathbb{H}_n)$ then $f(\hat{x}, \hat{X})^* = \bar{f}(\hat{x}, \hat{X}) \in O_p \psi^m$;

(II) Let $f_j \in O_p \psi^{m_j}(\mathbb{H}_n), j = 1, 2$. Then there exists $f \in \psi^{m_1 + m_2}(\mathbb{H}_n)$

such that

$$f(\hat{x}, \hat{X}) - f_1(\hat{x}, \hat{X}) f_2(\hat{x}, \hat{X}) \in \psi^{m_1 + m_2 - 1}(\mathbb{H}_n)$$

and

$$\left(f(\hat{x}, \hat{X}) u \right)(x) = \left[f_1(x, \hat{X}) f_2(x, (\hat{X})) u \right](x);$$

(III) Assume that $f \in \psi^m(\mathbb{H}_n)$ has the compact support with respect to x. Consider a diffeomorfism χ of a neighborhood of the support. Assume that χ conserves the contact form ω up to a functional multiplier (i. e. χ is a contact mapping).

Let

$$f^\chi(x, y) = f(\chi(x), [d\chi(x)]^* y).$$

Then

$$\chi^{*} f(\hat{x}, \hat{X}) \chi^{*-1} \in O_p\Psi^m$$

and

$$f^{\chi}(\hat{x}, \hat{X}) - \chi^{*} f(\hat{x}, \hat{X}) \chi^{*-1} \in O_p\Psi^{m-1}$$

The last property permits the standard extension of $O_p\Psi^m$ to any contact manifold M. Thus we have $O_p\Psi^m(M)$ with a symbolic Calculus as above.

Now we introduce a subclass $\Psi_0^m(\mathbb{H}_n)$ of $f \in \Psi^m(\mathbb{H}_n)$ such that for every f there exists a $f_0 \in \Psi^m(\mathbb{H}_n)$ such that

(i) $f_0(x, \delta_t X) = t^m f(x, X), \quad \forall t > 1, \quad \forall |X| > 1.$

(ii) $f - f_0 \in \Psi^{m-1}(\mathbb{H}_n)$

We say that the symbols f have principal parts f_0.

The formula (II) of the Symbolic Calculus shows that in principal parts the product of pseudo-differential operators is the product of left-invariant operators depending on x as a parameter. For study of such products it is very convenient to use the Heisenberg-Fourier transform on the $\mathbb{H}_n$. It is defined by means of the non-degenerate series of unitary representations of the group. By the Stone-von Neumann theorem they are equivalent to the representations π_μ depending on non-zero real parameter μ in the space $\mathcal{L}^2(\mathbb{R}^n_\tau)$ such that

$$\pi_\mu(X_0) = \mu 1, \quad \pi_\mu(X') = 2\mu\hat{\tau}, \quad \pi_\mu(X'') = 2i\frac{\partial}{\partial\tau}.$$

For $\varphi \in \mathcal{S}(\mathbb{H}_n)$ the Fourier-Heisenberg transform is by definition

$$\pi_\mu(\varphi) = \int_{H_n} \pi_\mu(x)\varphi(x)\,dx, \qquad (\mu \neq 0)$$

This is an integral operator with Schwartz kernel

$$(***)\quad \mathcal{K}_\varphi(\tau,\vartheta,\mu)=\int_{\mathbb{H}_n} e^{i\mu[x_0+(\tau+\vartheta)x']}\varphi(x_0,x',\frac{\tau-\vartheta}{2})\,dx_0\,dx'$$

so that

$$\pi_\mu(\varphi):\ \mathcal{S}'(\mathbb{H}_n)\to\mathcal{S}(\mathbb{H}_n)$$

We see that this formula ($***$) is valid for any distribution from $\mathcal{S}'(\mathbb{H}_n)$ so we can extend the definition of π_μ to all $\mathcal{S}'(\mathbb{H}_n)$.

This leads to a representation of left-invariant operators $f(\hat{X})\in O_p\Psi_0^m$ by operators.

$$\pi_\mu(f(\hat{X}))=f(\mu 1,2\mu\hat{\tau},2i\frac{\partial}{\partial\tau})\in O_p W_0^m .$$

Moreover

$$\pi_\mu(f_1(\hat{X})f_2(\hat{X}))=\pi_\mu(f_1(\hat{X}))\,\pi_\mu(f_2(\hat{X}))$$

and the principal part of $\pi_\mu(f(\hat{X}))$ is given by means of the principal part of $f(\hat{X})$:

$$(\pi_\mu(f(\hat{X})))_0=f_0(0,2\mu\hat{\tau},2i\frac{\partial}{\partial\tau})$$

Let $f_{oo}(X)$ be the δ_τ-homogeneous function on the $\mathbb{R}^{2n+1}$ which coincide with $f_o(X)$ far from origin . Consider the operators $\pi_\mu(f_{oo}(\hat{X}))$ By δ_t-homogenity we have

$$\pi_\mu(f_{oo}(\hat{X}))=|\mu|^{-m/2}V_{\sqrt{|\mu|}}\,\pi_{\operatorname{sgn}\mu}\,V^{-1}_{\sqrt{|\mu|}}\qquad \text{where } V_t u(\tau)=u(\frac{\tau}{t})$$

Therefore there are significant only $\mu=\pm 1$

Finally we define a σ_m-symbol of an operator $f(\hat{x},\hat{X})\in O_p\Psi_0^m$ as an operator valued function on the manifold of contact directions

$$\sigma_m(f(\hat{x},\hat{X}))_{\omega(x)}=f_{oo}(x,1,2\hat{\tau},2i\frac{\partial}{\partial\tau}),$$

$$\sigma_m (f (\hat{x}, \hat{X}))_{-\omega(x)} = f_{oo} (x, -1 - 2\hat{\tau}, 2i\frac{\partial}{\partial\tau})$$

$$\sigma_m (f (\hat{x}, \hat{X}))_{\pm\omega(x)} \in O_p W_o^m$$

The σ_m-symbol reveals usual properties:

(i) if $f_1 \in \Psi^{m_1}(\mathbb{H}_n)$, $f_2 \in \Psi^{m_2}(\mathbb{H}_n)$,

then $\sigma_{m_1+m_2}(f_1(\hat{x}, \hat{X}), f_2(\hat{x}, \hat{X})) = \sigma_{m_1}(f_1(\hat{x},\hat{X}))\sigma_{m_2}(f_2(\hat{x},\hat{X}))$

(ii) $\sigma_m (f (\hat{x}, \hat{X})^*) = \sigma_m (f (\hat{x}, \hat{X}))^*$

(iii) $\sigma_m (f (\hat{x}, \hat{X})) = 0$ if and only if $f (\hat{x}, \hat{X}) \in O_p\Psi^{m-1}$

(iiii) the σ_m-symbol belongs to the contact structure so it can be transfered to any $f (\hat{x}, \hat{X}) \in O_p\Psi^m(M)$.

Examples: Everywhere M is a strongly pseudoconvex boundary in $\mathbb{C}^{n+1}$.

(a) Let $\Box_M$ be the Kohn sub - Laplacian on the space of (o, q) -forms. Then

$$\Box_M \in O_p \Psi^2(M) \quad \text{and} \quad \sigma_2(\Box_M) = \left(-\frac{\partial^2}{\partial\tau^2} + \tau^2 + n - 2q\right) 1_{T^{0,q}}$$

The operator $-\frac{\partial^2}{\partial\tau^2} + \tau^2$ is the energy operator of the harmonic oscillator of Quantum Mechanics. Actually this example has appeared in /4/ and by the way it served an origin of our study.

(b) The induced Cauchy-Riemann operator $\bar{\partial}_M$ on the (o, q) -forms belongs to $O_p\Psi^1(M)$ with

$$\sigma_1(\bar{\partial}_M)_{\pm\omega(x)} = \left(-\frac{\partial}{\partial\tau} \pm \hat{\tau}\right) 1_{T^{0q}}$$

(c) The Cauchy-Henkin integral can be considered as operators S on M if we take their boundary values; they belong to $O_p\Psi^0(M)$

and their symbol is

$$\sigma_1(S)_{\omega(x)} = \text{ orthogonal projector on the linear span of } e^{-\tau^2/2}$$

$$\sigma_1(S)_{-\omega(x)} \doteq \text{ zero}$$

We say that an operator $f(\hat{x}, \hat{X}) \in O_p\Psi^m$ is σ-elliptic if

(Ell-1) $\sigma_m(f(\hat{x}, \hat{X})_{\pm\omega(x)} \in \mathcal{E}\ell\ell\, W_0^m$

(Ell-2) The operators $\sigma_m(f(\hat{x}, \hat{X}))_{\pm\omega(x)}$ are invertible in $\mathcal{S}(\mathbb{R}^n_\tau)$

Let M be a compact contact manifold. We can introduce a scale of anisotropic spaces of functions and distributions on M

$$\mathcal{S}^k_p(M) \quad , \quad 1 < p < \infty \quad , \quad -\infty < k < +\infty$$

of E. Stein (cf. /4/).

If $f(\hat{x}, \hat{X}) \in O_p\Psi^m$ then $f(\hat{x}, \hat{X})$ is bounded from $\mathcal{S}^k_p(M)$ to $\mathcal{S}^{k-m}_p(M)$ for any k

The following properties are equivalent for $f(\hat{x}, \hat{X}) \in O_p\Psi^m$:

a) $f(\hat{x}, \hat{X})$ is a σ-elliptic operator.

b) The apriori estimate

$$\|u\|_{k+m} \leq Const\, (\|f(\hat{x}, \hat{X})u\|_k + \|u\|_{k'})$$

is valid in Stein norms for a (and therefore for any) $k \in \mathbb{R}$, $k' < k$

c) $f(\hat{x}, \hat{X})$ is a Fredholm operator from $\mathcal{S}^k_p(M)$ to $\mathcal{S}^{k-m}_p(M)$ for a (and therefore for any) $k \in \mathbb{R}$.

Remark

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

AN INDEX FORMULA FOR ELLIPTIC BOUNDARY PROBLEMS

A. DYNIN

Corso tenuto a Bressanone dal 16 al 24 giugno 1977

I give an analytical formula for index of elliptic boundary problems for scalar differential operator and for some systems of differential operators of even order in bounded domains with smooth boundaries in euclidean space.

1. Notation.

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \dots, \xi_n) \in (\mathbb{R}^n)^*$,

$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, $D = (D_1, \dots, D_n)$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$,

$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

The term "smooth" always means C^∞.

Let U be an open bounded domain in $\mathbb{R}^n$ with smooth boundary Y . Points of Y are denoted y . Cotangent vectors at y with length 1 are denoted τ_y . The (2n-3) - manifold S (Y) of all such τ_y is supplied with canonical orientation: the manifold $T^*(Y)$ of all cotangent vectors of Y is $\mathbb{R}^{n-1} \times (\mathbb{R}^{n-1})^*$ locally. Let $(y_1, \dots, y_{n-1})$ be any system of coordinates on $(\mathbb{R}^{n-1}$ and $(\eta_1, \dots, \eta_{n-1})$ the dual system of coordinates on $(\mathbb{R}^{n-1})^*$. Then the orienting (2n-2) - form $\Omega = (dy_1 \wedge d\eta_1) \wedge \dots \wedge (dy_n \wedge d\eta_n)$ does not depend on the choice of le local coordinates and therefore gives an orientation of $(\mathbb{R}^{n-1})^*$. Now let $\tau : T^*(Y) \to \mathbb{R}$ be the euclidean metric, so that $S(Y) = \tau^{-1}(1)$. Then the orienting $(2n-3)$- form ω on the $S(Y)$ is defined by its property $d\tau \wedge \omega = \Omega$.

2. Elliptic Boundary Problems.

Let A be a scalar differential operator of order $2m$ with smooth coefficients

$$A : u(x) \to \sum_{|\alpha| < 2m} a_\alpha(x) D^\alpha u(x), \quad u \in C^\infty(\overline{U})$$

Its principal symbol is the function on $T^*(\overline{U}) = \overline{U} \times (\mathbb{R}^n)^*$

$$\sigma(A)(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x)$$

The operator A is assumed elliptic i.e.

$$\sigma(A)(x, \xi) \neq 0, \quad \forall x \in \overline{U}, \ \forall \xi \in (\mathbb{R}^n)^*$$

Consider λ-polynomial with coefficients from $C^\infty(S(Y))$ of order $2m$

$$\hat{\sigma}(A)(\tau_y, \lambda) = \sigma_A(y, \tau_y + \lambda \nu_y)$$

where ν_y is the inward unit conormal at y.

For each τ_y this polynomial has no root with zero imaginary part. If $n > 2$ there ~~is~~ are exactly m roots with positive imaginary part (this is an easy consequence of the connectivity of $S(Y)$). If $n = 2$ then we assume this property especially.

We can factorize the λ-polynomial into the product of two polynomials with smooth coefficients

$$\hat{\sigma}(A)(\tau_y, \lambda) = \sigma^+(\tau_y, \lambda)\, \sigma^-(\tau_y, \lambda),$$

where all roots of $\sigma^+(\tau_y, \lambda)$ are in the upper complex λ - halfplane and all roots of $\sigma^-(\tau_y, \lambda)$ are in the lower λ - halfplane.

Consider a boundary problem

$$(a)\begin{cases} Au = f \quad ,f \in C^{\infty}\ (\bar{U}) \\ B_j u = g_j\ g_j \in C^{\infty}\ (Y), \quad j = 1, \dots, m \end{cases}$$

Here B_j are boundary differential operators of order m_j with smooth coefficients

$$B_j : u \to \sum_{|\alpha| < m_j} b_\alpha(y)\ D^\alpha u \Big/ Y,$$

$$B_j : C^\infty\ (\bar{U}) \to C^\infty\ (Y) .$$

We suppose that the boundary problem satisfies the Shapiro-Lopatinsky condition of ellipticity which we take in the Agmon-Douglis-Nirenberg version /1/ (cf. lectures by F. Trèves):

Consider λ-polynomials of degree m_j with smooth coeffients on S (Y)

$$\hat{\sigma}\ (B_j)\ (\tau_y, \lambda) = \sigma(B_j)\ (y, \tau_y + \lambda \nu_y) \equiv$$

$$\equiv \sum_{|\alpha| = m_j} b_{j\alpha}(y)\ (\tau_y + \lambda \nu_y)^\alpha, \quad j = 1, \dots, m .$$

The Agmon-Douglis-Nirenberg condition is the linear independence of these polynomials modulo λ-polynomial $\sigma^+\ (\tau_y, \lambda)$ for every τ_y.

We can represent this condition in an equivalent form. Let $r_j\ (\tau_y, \lambda)$ be the remainder from division of λ-polynomial $\hat{\sigma}(B_j)\ (\tau_y, \lambda)$ by λ-polynomial $\sigma^+\ (\tau_y, \lambda)$:

$$\hat{\sigma}\ (B_j)\ (\tau_y, \lambda) = q_j\ (\tau_y, \lambda)\ \sigma^+(\tau_y, \lambda)) + r_j\ (\tau_y, \lambda)$$

where q_j and r_j are λ-polynomials and the degree of $r_j\ (\tau_y, \lambda)$ is less than m . Let

$$\tau_j(\tau_y, \lambda) = \sum_{k=0}^{m-1} \tau_{kj}(\tau_y)\lambda^k \quad , \quad j = 1, \dots, m$$

Consider the square (m x m) - matrix valued function on S (Y)

$$\tau(\mathcal{a}) = (\tau_{kj})$$

The Agmon-Douglis-Nirenberg condition is obviously equivalent to non-degeneracy condition

(ADN) $\quad \det \tau(\mathcal{a})\ (\tau_y) \neq 0, \quad \forall \tau_y \in S(Y)$

3. The Index Formula

As usual the elliptic boundary problem ($\mathcal{a}$) leads to a linear continuous operator

$$\mathcal{a} = (A, B_1, \dots, B_m) : C^\infty(\bar{U}) \to C^\infty(\bar{U}) \times (C^\infty(Y))^m$$

which is a Fredholm operator and therefore has a finite index

$$\operatorname{ind} \mathcal{a} = \dim \operatorname{Ker} \mathcal{a} - \dim \operatorname{coker} \mathcal{a}$$

(see e. g. / 2 / and / 8 /).

It is known that the index depends only on the symbol

$$\sigma(\mathcal{a}) = (\sigma(A), \sigma(B_1), \dots, \sigma(B_m)).$$

We express it by means of the $\tau(\mathcal{a})$ which is defined by $\sigma(\mathcal{a})$

(1) $$\operatorname{ind} \mathcal{a} = \frac{(-1)^n}{(2\pi i)^{n-1}} \frac{(n-2)!}{(2n-3)!} \int_{S(Y)} \operatorname{Sp} \left[\tau(\mathcal{a})^{-1}\, d\tau(\mathcal{a})\right]^{2n-3}$$

The intengrand is the trace of (2n-3)-power of (m x m) -matrix valued differential 1-form $\tau(\mathcal{a})^{-1}\, d\tau(\mathcal{a})$ in the exterior algebra of matrix valued differential forms. So we integrate (2n-3)-form over the (oriented) manifold S (Y).

In particular the Index Formila shows that if $m < n-1$ then ind $\mathfrak{a} = 0$

The prof. of (1) involves a special homotopy of $\sigma(\mathfrak{a})$ in the space of symbols of elliptic boundary problems for the A with pseudo-differential boundary operators. (By the way, this is the first place where pseudo-differential operators of positive order were introduced as early as in 1961: see / 4 / and / 5 /.)

The homotopy is

$$\hat{\sigma}_{(t)}(B_j)(\tau_y,\lambda) = (1-t)\, q_j(\tau_y,\lambda)\, \sigma^{+}(\tau_y,\lambda) + \tau_j(\tau_y,\lambda)$$

Here $0 \leq t \leq 1$ and

$$\hat{\sigma}_{(0)}(B_j)(\tau_y,\lambda) = \hat{\sigma}(B_j)(\tau_y,\lambda),$$

$$\hat{\sigma}_{(1)}(B_j)(\tau_y,\lambda) = \tau_j(\tau_y,\lambda).$$

This homotopy may be covered by homotopy of boundary value problems ($\mathfrak{a}_t$) for the same operator A which all are elliptic by $\tau(\mathfrak{a}_t) = \tau(\mathfrak{a})$ and the condition (ADN) is satisfied (cf. /5/ and lectures by F. Treves).

Stability os Index under homotopies implies

(2) ind $\mathfrak{a}$ = ind $\mathfrak{a}_1$,

The (1x(m+1) -matrix $\sigma(\mathfrak{a}_1)$ can be factorized

(3) $\sigma(\mathfrak{a}_1) = \sigma(\mathcal{D})\,(1 \oplus \tau(\mathfrak{a}))$.

where

$$\sigma(\mathcal{D}) = (\sigma(A),\ 1, \nu, \ldots ; \nu^{m-1})$$

is the symbol of the (elliptic) Dirichlet problem for the operator A.

We consider $\tau(\mathfrak{a})$ as the symbol $\sigma(R_{\mathfrak{a}})$ of a system of pseudo-differential operators $R_{\mathcal{a}}$ in $(C^{\infty}(Y))^m$ which is elliptic by (ADN). (Strictly speaking the $R_{\mathcal{a}}$ is elliptic in the Douglis-Nirenberg sense only, otherwise we have to modify the Dirichlet problem, cf. /5/ and lectures by F. Treves.)

Now by algebraic properties of Index the equality (3) implies

$$\operatorname{ind} \mathcal{a}_1 = \operatorname{ind} \mathcal{D} + \operatorname{ind} (1 + R_{\mathcal{a}}) = \operatorname{ind} \mathcal{D} + \operatorname{ind} R_{\mathcal{a}}$$

But $\operatorname{ind} \mathcal{D} = 0$ (cf. /2/). Therefore (by (2))

$$\operatorname{ind} \mathcal{a} = \operatorname{ind} R_{\mathcal{a}}$$

Finally the Index Formula (1) coincides with the Index Formula discovered by A. Dynin and B. Fedosov (cf. /6/) for the elliptic pseudodifferential system $R_{\mathcal{a}}$ on the manifold without boundary Y . Of course such formula can be derived from the famous Atiyah - Singer formula and actually this was accomplished by the author (Proceedings of the Conference on the Mathematical Methods in Physics, Dubna, 1964) and by B. Fedosov /6/ . Nowadays B. Fedosov /7/ has found a completely analytical proof of the formula. Therefore we have an elementary proof of our formula (1).

4. The Index Formula for Systems.

Consider now an elliptic (NxN)-system A of order 2m. Suppose that we can again factorize its principal symbol

$$\hat{\sigma}(A)(\tau_y, \lambda) = \sigma^+(\tau_y, \lambda)\, \sigma^-(\tau_y, \lambda) \;, \quad \deg \sigma^{\pm} = m .$$

By a theorem of Lopatinsky this factorization exists if and only if the rank of the (NxmN) -matrix