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FONDAZIONE CIME ROBERTO CONTI

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Potential Theory

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« POTENTIAL THEORY »

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CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C. I. M. E.)

M. BRELOT

"HISTORICAL INTRODUCTION"

Corso tenuto a Stresa dal 2 al 10 Luglio 1969

HISTORICAL INTRODUCTION

by

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As an introduction to the next courses, some historical notions seem to be necessary.

1. Old Period (till the first World War) : Ideas of Gauss-Dirichlet.

Until about 1800, potential theory was only a study of some questions about electrostatics and newtonian attraction. The Laplace equation was already much used, and was extended by POISSON who gave also his famous integral in a ball; the Green function was soon introduced, but the first important mathematical work was a paper of GAUSS, in 1840 ([20]) ; three problems were solved in R^3: the problem of equilibrium giving the distribution of a given mass on a conductor ("closed"surface), to make the potential constant on it; this corresponds to the minimum of the energy. A second problem starts from masses inside the conductor, and studies a distribution on a conductor which gives the same potential outside. Similar problem for given masses outside. The solution is realized in physics by the phenomenon of "influence", and the equation was called later "sweeping out process" or commonly now "balayage" process. A third problem is the (so called later by RIEMANN) Dirichlet problem where a harmonic function (i.e. solution of the Laplace equation) is studied inside the conductor for given continuous boundary values. These studies were based on the integral $\int (U^{\mu} - 2f)\, d\mu$, where $U^{\mu}(x)$ is the newtonian potential $\int d\mu(y)/|x-y|$ of the measure $\mu \geqslant 0$. Actually GAUSS considered only μ with a density, and assumed the existence of a μ giving a minimum of the integral. The developments of GAUSS were amazingly deep, powerful , rich , still useful today, but they could not be rigorous, lacking notions like the general Radon measure, and they needed actually some restrictions. Therefore they were first left aside, except for the Dirichlet problem which was studied in various ways, first not rigorous either. Let us mention a method used by RIEMANN ([30]), following GAUSS - W. THOMPSON (Lord KELVIN) - DIRICHLET ; it considers regular functions on the domain, taking given values at the boundary; when the Dirichlet integral $\int \text{grad}^2 u \, dx$ (dx, Lebesgue measure) is minimum, u is the solution of the Dirichlet problem . But we meet the similar difficulty of the attained minimum, which was solved, under suitable restrictions, only by HILBERT ([22]) about 1900. Other methods were given which were rigorous, but with various restrictions on the boundary (use of the alternating process of Schwarz, of potentials of double layer by NEUMANN, later with the Fredholm theory, famous balayage process of Poincaré ([29]) , Lebesgue solution, ...) . If so many great mathematicians gave different solutions of this problem , the reason is that the restrictions on the boundary were not satisfactory, and even seemed unnecessary, till ZAREMBA and LEBESGUE noticed they were necessary.

2. Second period (essentially that between the wars) : Use of Radon measure.

The use of Radon measure (defined in 1913) in potential theory first by EVANS, F. RIESZ, de LA VALLÉE POUSSIN renewed the theory. As for the Dirichlet problem we were speaking of, the non-general existence of a solution led LEBESGUE and chiefly WIENER ([34]) (1924) to define a generalized solution, then to study its behavior at the euclidian boundary . A previous solution of PERRON gave the best form by using the subharmonic or superharmonic functions of F. RIESZ ([31]), which are locally equal to a newtonian or logarithmic potential of a measure (resp. $\leqslant 0$ or $\geqslant 0$) up to a harmonic function. These notions allow the treatment of many problems, and I did so systematically, without using a kernel like $1/|x - y|$ in R^3, and this became valuable later in axiomatic theories without given kernels. Now for any real function f on the boundary $\partial\omega$ of a bounded domain ω, let us consider the envelopes of Perron-Wiener $\overline{H}_f = \inf v$, v superharmonic or $+\infty$ (we say hyperharmonic) satisfying : $\liminf v$ at the boundary $\geqslant f$ and $> -\infty$ and $\underline{H}_f = -\overline{H}_{-f}$. Always $\underline{H}_f \leqslant \overline{H}_f$, and in case of equality with a finite necessarily harmonic function (case of resolutivity) , the common envelope H_f is called the solution. WIENER proved it is realized when f is finite continuous; then $H_f(x)$ is a positive linear form which we may write $\int f \, d\rho_x$, where $d\rho_x$ is a positive Radon measure called harmonic measure (with an interpretation in balayage theory) .

A boundary point x_0 is said to be regular, if $H_f(x)$ tends to $f(x_0)$ $(x \to x_0)$, $\forall f$ finite continuous . When all points are regular , H_f is the "classical solution". Only in 1933 , EVANS [17] (after KELLOGG in R^2) proved that the set of irregular points is a locally polar set, i.e. such that there exists locally (or in a bounded domain containing $\bar{\omega}$) a superharmonic > 0 function (or a potential of measure > 0) which is $+\infty$ on the set (notion introduced later by BRELOT , 1941) . At this time, it was called a set of capacity zero in the sense of inner capacity (notion without difficulty in R^n, $n \geqslant 3$, inspired by electrostatics, made precise by WIENER - EVANS-de LA VALLÉE POUSSIN ; actually, as H. CARTAN showed later, polar set = set of outer capacity zero). A little later by studying the case of any f , I proved ([4]) that the resolutivity is equivalent to the $d\rho_x$-summability (independent of x). These features and key results are preserved at least partly in modern axiomatic theories, as they will be considered in the courses of BAUER and BONY.

About at the same time in 1935, FROSTMAN [18] managed to make rigorous and precise the famous work of GAUSS. He used Radon measure, and weakened the results of GAUSS (actually valid with restrictions on the boundary), thanks to "exceptional sets" of inner capacity zero (actually even locally polar sets). His proofs were based on two still important principles: the principle of energy saying that the energy $\int U^{\mu}\, d\mu$ of any μ (with compact support and newtionian potential U^{μ}) is $\geqslant 0$, and zero only when $\mu = 0$; and a maximum principle saying that U^{μ} for $\mu \geqslant 0$ is majorized by the sup on the compact support of μ. Moreover, the notion of capacity was deepened, and the potentials with kernel $|x - y|^{-\alpha}$ studied too, as M. RIESZ did before him.

We arrive about at 1940. One could think potential theory was over. Actually the last thirty years have been extraordinarily fruitful.

3. Third period (about 1940-1955) : Role of topologies and extreme elements, energy and Schwartz distributions.

First further improvements were made. Note a key-convergence theorem on decreasing superharmonic $\geqslant 0$ functions ; the inf of such a sequence becomes superharmonic by changing the value on an "exceptional" set. It was known that this set has Lebesgue measure zero, and I proved in 1938 (C.R. Acad. Sc. Paris, t. 207, 1938, p. 1157) it is locally of inner capacity zero. CARTAN [8] improved that by changing inner to outer (i.e. the set is locally polar) and the sequence to any family. We have now results and proofs which are valid (and more or less strong with more or fewer hypotheses) under general axiomatic conditions, but the proof of CARTAN was based on the case of potentials of finite energy, on the use of a norm-energy and of a corresponding scalar product. This idea gave an interpretation of the balayage process in the basic case as a projection in a prehilbertian space. This opened the way to a deep study of the role of energy, even under larger conditions. Finally, DENY [11] developed a potential theory in R^n with finite energy, where the kernel is a Schwartz distribution (notion introduced in the context of a problem of potential theory) ; the given masses become a variable similar distribution, and the potential is given by convolution of both distributions. Under some restrictions, the Cartan theory may be adapted. This is connected with the so called BL and BLD (Beppo-Levi-Deny) functions, generalizing regular functions with finite Dirichlet integrals. Finally, BEURLING, then BEURLING-DENY (first in [2]) were led ten years ago to a theory of Dirichlet spaces which is an axiomatic of energy, that will be developed in the course of DENY .

Another axiomatic effort began in about 1940 with general kernel-functions N(x, y) (and later kernel-measures) in general topological space. It was obvious many classical arguments were valid under conditions much larger than the newtonian kernel $1/|x - y|$ in R^3 , by supposing as axioms some properties or "principles" of the classical theory. Much research was made in France and independently, chiefly, in Japan (KUNUGUI, KAMETANI, NINOMYA, ...), and continues till now. They have not stopped, because of the complexity of all principles, and because the use of nonsymmetric kernels introduces difficulties. That will not be developed here, because another type of kernel appeared which is more important, as we shall see later.

This introduction of fundamental topological spaces in potential theory takes place in a general and varied use of topologies. CARTAN used various topologies on measures. A notion of thinness (1940), I introduced and continued to deepen till now, generalizing the regular boundary points and unstable ones in a kind of Dirichlet problem for compact sets, led CARTAN to the equivalent notion of fine topology, the coarsest one making continuous all superharmonic functions. This gave final improvements in potential theory and general results on the behaviour of superharmonic functions and of functions of a complex variable. For instance, if v is superharmonic $\geqslant 0$ on an open set ω, x_0 an irregular boundary point (that means actually $C\omega$ is thin at x_0), and h equal to $|x_0 - x|^{2-n}$ or $\log 1/|x_0 - x|^{2-n}$ or $\log 1/|x_0 - x|$ in R^2, then v and v/h have at x_0 fine limits (i.e. limits according to the fine topology), and that means ordinary limits outside a suitable set thin at x_0. On the other hand, new boundaries were introduced, for instance, by completion of a metric compatible with the topology, after some particular cases in the previous period. The most important one is the Martin boundary, introduced in 1941 ([24]). Consider the normalized Green function of a bounded domain Ω of R^n (or even of a "Green space" which is, for example, connected locally euclidean with a Green function) which will be $K(x, y) =$

$= G(x, y)/G(x, y_0)$ $(y_0 \in \Omega)$. There exists a unique compact space $\hat{\Omega}$ (up to an homomorphism) in which Ω is dense, such that all functions $x \to K(x, y)$ may be continued continuously and separate the new boundary $\Delta = \hat{\Omega} - \Omega$. We denote by Δ_1 the set of points X such that the corresponding $K(X, y)$ is a minimal harmonic function, i.e. such that any other smaller harmonic >0 function is proportional. Now any harmonic $h > 0$ has a unique representation

$$\int K(x, y) \, d\mu_h(x) \quad ,$$

where μ_h is a Radon measure $\geqslant 0$ on Δ, but supported by Δ_1. If we consider the cone of the positive harmonic functions, and the base B of the functions equal to 1 at y_0, Δ_1 corresponds to the set of the extreme points of B in the vector space of the differences of positive harmonic functions. Later, this led CHOQUET to a general and deep study of the extreme points and of a corresponding integral representation, probably the most important discovery in analysis in the last twenty years.

The Martin topology allows a Dirichlet problem with Δ, but is not sufficient for a study of behaviour at the boundary. We shall come back later on that point with recent results.

Let us complete the period 1940-1955. Another axiomatic effort is the study of CHOQUET ([9]) of the notion of capacity, which has become a basic and general tool in analysis.

Let us mention finally an attempt by TAUTZ of an axiomatization of harmonic function, which is valid for solutions of equations of elliptic type, by starting from an abstract Poisson integral. That was the beginning of important researches I shall now speak of.

4. Modern period (from 1955) : Probabilistic interpretation, Hunt's kernels, axiomatics of harmonic and superharmonic functions, Dirichlet spaces, boundary behaviour of functions.

I already mentioned the researches on Dirichlet spaces starting from [2], to be developed here by DENY , and also the work which continues on kernel-functions and deep discussion on principles (CHOQUET [10] , CHOQUET-DENY [11] , NINOMYA, KISHI, FUGLEDE [20] , DURIER,...) .

But the most striking new field in potential theory is the rich connection with probability. It is not surprising, when comparing the mean value property of harmonic functions, and the fact that in a brownian motion the probability of the motion from a point is the same in all directions. DOOB deepened this remark, and founded the modern field of probability-potential theory. Let us mention only that, starting from a few axioms, a little like TAUTZ, he defines ([14]) axiomatic harmonic functions in a locally compact metrizable space, and considers a sequence $x_1, x_2, \ldots, x_n, \ldots$ and open sets (regular, i.e. allowing a Dirichlet problem, with a unitary harmonic measure) $\omega_1 \ni x_1$ with $\partial\omega_1 \ni x_2$, $\omega_2 \ni x_2$ with $\partial\omega_2 \ni x_3, \ldots$. The harmonic measure on $\partial\omega_{n-1}$ at x_{n-1} will be taken as probability of choice of $x_n \in \partial\omega_{n-1}$. A suitable Markov process corresponds to this "transition probability". Under some conditions, DOOB studies the values of any corresponding superharmonic function along the corresponding trajectories, and finds the existence of a limit for "almost all" trajectories.

I was so much interested in the starting axioms of these general axiomatic developments that I deepened the question, and by changing more or less the axioms and sometimes adding one, I tried to develop a theory close to the classical one, as follows ([5]) : in a connected, locally compact, but not compact space Ω , we consider on any open set a vector space of finite real continuous functions (called harmonic). They must define a sheaf (axiom 1). As axiom 2, we suppose the existence of a base of "regular" domains, i.e. such that there exists a unique solution of a Dirichlet problem (increasing with the finite continuous boundary function) . As axiom 3, any increasing directed set (or equivalently sequence) of harmonic functions on a domain tends to $+\infty$ or to a harmonic function. Note that the quotients by a finite continuous $h > 0$ give another sheaf satisfying the axioms ; if h is harmonic Ω , we get a case where the constants are harmonic (as DOOB supposed) .

Easy definition of superharmonic function of potential (i.e. superharmonic with every harmonic minorant $\leqslant 0$) : By supposing the existence of a potential > 0 and often a countable base in Ω , a large development is possible as in the classical case (Dirichlet problem with resolutivity theorem ; lattice properties and extension of the Riesz-Martin representation ; thanks to extreme elements, Martin boundary in case of proportionality of the potentials with point-support and corresponding Dirichlet problem ...) ; with a supplementary "axiom D" (domination axiom) , it is possible to adapt the greater part of the classical theory (first the great convergence theorem with its consequences). See [5] . Many important parts or complements were given by Mme HERVÉ [21] with a theory of an adjoint sheaf, BOBOC-CONSTANTINESCU-CORNEA, GOWRISANKARAN, LOEB, B. WALSH (with the role of nuclear spaces and cohomology) , MOKOBODZKI, D. SIBONY, A. de LA PRADELLE (quasi-analyticity), TAYLOR, etc. See [6] . It is easy to see that the solutions of elliptic partial differential equations of 2nd order, with smooth coefficients, satisfy the previous axioms.

The same is true, but difficult, for discontinuous coefficients with suitable definitions, as was proved by Mme HERVÉ.

This allows simplification of the difficult direct study of these equations (see STAMPACCHIA [34]) .

Now the solutions of parabolic equations do not satisfy the previous axioms (actually 3 and D), whereas they did a least for the heat equation in the Doob's axiomatic. Therefore H. BAUER, in order to gather all these possible applications, weakened the previous axioms by replacing the third one by weaker versions of a Doob's conditions, by adjoining another one implying a maximum principle which is a key to our classical and axiomatic theories (see a final form of the Bauer's axiomatic, in [1]).

He succeeded in extending nearly all the previous results independent of D, except those depending on the Choquet theory of extreme elements. The corresponding integral representation (generalizing the Martin-Riesz one) was made later by MOKOBODZKI, but cannot be given in the same useful form. Further important complements were given by various pupils of BAUER (HANSEN, HINRICHSEN, GUBER, SIEVEKING, BLIEDTNER, ...), and weaker axiomatics were also considered (BOBOC-CONSTANTINISCU-CORNEA).

The course of BAUER will develop partly his axiomatic, and give shortly relations with Markov processes and probabilistic interpretations of some key tools of potential theory.

The research of sheaves satisfying these axiomatics, or even wheaker ones, has been undertaken by BONY ([3]) . For smooth functions in

R^n, it is very interesting to see the identity with solutions of a suita= ble partial differential equation of 2nd order. A deep discussion of relations between the previous axiomatics and partial differential equa= tions (and more precisely a characterisation of various axiomatics by different differential operators) will be given in the course of BONY.

The previous theories have a local character. It remains essentially to speak of the fundamental global Hunt's theory of kernels ([23]), published in 1957/58. Avoiding any details and restrictions and speaking roughly, let us consider for a space Ω (abstract or locally compact) a measure μ_x depending on a point $x \in \Omega$, that is written also $N(x, e)$, called a kernel.

Given a function $f \geq 0$, we associate the function $Nf = \int f \, d\mu_x$, or with another common notation $\int f(y) \, N(x, dy)$. Given a measure θ, we associate the measure

$$\theta N(e) = \int N(x, e) \, d\theta(x) ,$$

that contains nearly all basic notions of potential theory. For example, in the classical case (R^3, newtonian kernel-function $1/|x - y|$), let us choose $N(x, e) = \int_e (1/|x - y|) \, d\lambda(y)$ ($d\lambda$, Lebesgue measure). Now, $Nf = \int (f/|x - y|) \, d\lambda(y)$, which is the newtonian potential of the measure with density f (relative to $d\lambda$). Then

$$\theta N(e) = \int \left(\int_e (1/|x - y|) \, d\lambda(y) \right) d\theta(x) = \int_e \left(\int (d\theta(x)/|x - y|) \right) d\lambda(y).$$

It is a measure with a density which is the ordinary newtonian potential of θ. In a difficult theory, HUNT [23] shows that under cer= tain conditions (satisfied in our applications), there exists for an N a semi-group P_t $(t > 0)$ of kernels (i.e. satisfying $P_{s+t} = P_s . P_t$ with a suitable convention) such that

$$Nf = \int^{\infty} P_t \, f \, dt .$$

Corresponding excessive functions are defined for $f \geqslant 0$ by the conditions

$$P_t\, f \leqslant f \qquad \text{and} \qquad \lim_{t \to 0} P_t\, f = f \quad .$$

In case of equality, f is said to be invariant. When an excessive function $f \geqslant 0$ has no invariant minorant $\geqslant 0$, except 0, it is called a (probability) potential.

Now under suitable conditions, P_t may be interpreted as the "transition semi-group" of a Markov process. Details will be found also in the books of P.-A. MEYER ([25] , [26]), and given in BAUER's course.

Then MEYER proved that in the axiomatic I had developed, and BAUER will show it is the same in his one, hyperharmonic non-negative functions are the excessive functions corresponding to a suitable family $\{P_t\}$. Hence the probabilistic interpretation of the axiomatics.

The previous local or global theories study the cones of hyperharmonic or excessive functions. The inverse problem of starting from a cone of functions, and studying when they are hyperharmonic functions in a local axiomatic, or excessive functions in a suitable even extended Hunt's theory, was studied by MOKOBODZKI and D. SIBONY.

The first problem ([27]) is closely connected to the minimum principle, the second one will be deepened in the course of MOKOBODZKI.

We are in the heart of the latest general researches in potential theory.

There are important questions that were mentioned very slightly or not at all in this survey, for instance further connections with probabilities (see MEYER [25] , [26] , DYNKIN, GETOOR, BLUMENTHAL, K. ITO, ...) and applications to function theory.

(See old classical results in the book of TSUJI [33] , and modern developments in the lecture of DOOB, Colloquium on potential theory, Paris-Orsay (1964) , and in a survey of BRELOT, Colloquium of Erevan (1965).)

Let us emphasize only, among the roles of topology, the question of the behaviour of some types of functions connected with potential theory at a suitable boundary. A course on that subject would have been desirable too, because of the possible improvements, complements and applications.

But that would require a large knowledge in potential theory, and basic courses had first to be developed.

However I would like to give an idea of this question by means of examples. Aside of the use of the so called Choquet boundary and Kuramochi boundary, let us consider first the classical case, the Martin space $\hat{\Omega}$, the Martin boundary Δ , and its minimal part Δ_1 .

Thanks to a notion of thinness of a set at any $x \in A$ (NAIM [28]), the fine topology introduced on Ω may be continued on $\Omega \cup \Delta_1$, in such a way that v/h ($\forall$ v superharmonic ≥ 0, h harmonic > 0) has a fine limit at any $x \in \Delta_1$, except on a set of μ_h - measure 0 (DOOB [15] , [16]). That is true also for v = BLD function , h = 1 , or v = h - BLD function in a suitable sense.

As a smooth euclidean boundary is homeomorphic to the Martin boundary, the general results imply and extend old Fatou theorems for the disk ; in the case of the disk, the general results yield angular limits for harmonic functions, radial limits for superharmonic functions. There are of course applications to and contacts with func= tions of a complex variable, and maps between Riemann surfaces (for these maps, see CONSTANTINESCU-CORNEA in pure potential theory, and DOOB in probability), and let us suggest that the detailed theory of cluster sets had to be adapted with the previous fine topology (Sy= stematic adaptations were made in the frame of the axiomatic of Bre= lot, with applications to the correspondence between two such "harmo= nic spaces" (CONSTANTINESCU-CORNEA, D. SIBONY) and to partial differential equations, but probabilistic interpretations are incomplete.); see [6] and also a detailed survey, with an abstract axiomatic intro= duction and bibliography in [7] .

May these preliminares help lecturers and audience, and suggest also new research.

M. Brelot

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Let us mention the Seminars on potential theory or probability, in Paris, Strasbourg, Erlangen, and the Annales de l'Institut Fourier , which have been publishing, for a long time, many important papers on potential theory, for example those of a Colloquium on this field in Paris-Orsay (1964), vol. 15, 1967, n° 1.

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

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HARMONIC SPACES AND ASSOCIATED MARKOV PROCESSES

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