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Stochastic Differential Equations

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STOCHASTIC PROCESSES AND STOCHASTIC DIFFERENTIAL EQUATIONS

C. DOLEANS-DADE

STOCHASTIC PROCESSES AND STOCHASTIC DIFFERENTIAL EQUATIONS

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Introduction. Since Ito has defined the stochastic integral with respect to the Brownian motion, mathematicians have tried to generalize it. The first step consisted of replacing the Brownian motion by a square integrable martingale. Later H. Kunita and S. Watanabe in [10] introduced the concept of local continuous martingale and stochastic integral with respect to local continuous martingales which P. A. Meyer generalized to the non continuous case.

But in many cases one observes a certain process X and there are at least two laws P and Q on $(\Omega,\underline{\underline{F}})$. For the law Q, X is not a local martingale but the sum of a local martingale and a process with finite variation. We would like to talk about the stochastic integrals $\int_P \Phi_s dX_s$ and $\int_Q \Phi_s dX_s$ in the two probability spaces $(\Omega,\underline{\underline{F}},P)$ and $(\Omega,\underline{\underline{F}},Q)$. And of course we would like those two stochastic integrals to be the same.

This is why one should try to integrate with respect to semimartingales (sums of a local martingale and a process with finite variation), and this is what people have been doing for awhile (see chapters 5 and 6). Now the latest result in the theory is "one cannot integrate with respect to anything more general than semimartingales" (see chapter 3). So as it stands now the theory looks complete.

To end this introduction I wish to thank Professor J. P. Cecconi and

the C.I.M.E. for their kind invitation to this session on differential stochastic equations in Cortona; the two weeks of which I, and my family, found most enjoyable.

STOPPING TIMES AND STOCHASTIC PROCESSES

We shall list in this chapter some definitions and properties on stopping times and stochastic processes. The proofs can be found in [1] or [2].

In all that follows $(\Omega,\underline{\underline{F}},P)$ is a given complete probability space and $(\underline{\underline{F}}_t)_{t\geq 0}$ a family of sub-σ-fields of $\underline{\underline{F}}$ verifying the "usual" following properties

a) the family $(\underline{\underline{F}}_t)_{t\geq 0}$ is non decreasing and continuous on the right

b) for each t, $\underline{\underline{F}}_t$ contains all the P-null sets of $\underline{\underline{F}}$ (a P-null set is a set of P-measure zero).

The σ-fields $\underline{\underline{F}}_t$ should be thought of as the σ-field of the events which occurred up to time t.

We will sometimes consider other probabilities Q on the measurable space $(\Omega,\underline{\underline{F}})$. But we shall always assume that the probabilities P and Q are equivalent (i.e. they have the same null sets); and the family $(\underline{\underline{F}}_t)$ will still satisfy the "usual" conditions relatively to the probability Q.

STOPPING TIMES

Suppose a gambler decides to stop playing when a certain phenomenon has occured in the game. Let T be the time at which he will stop playing.

The event $\{T \leq t\}$ will depend only on the observations of the gambler up to time t. This remark leads to the natural following definition.

1.1. Definition. A non negative random variable T is a stopping time if for every $t \geq 0$ the event $\{T \leq t\}$ is in $\underline{\underline{F}}_t$. (We allow the random variable T to take the value $+\infty$)

1.2. Properties of stopping times:

1) if S and T are two stopping times so are $S \vee T$, $S \wedge T$ and $S+T$

2) if S_n is a monotone sequence of stopping times, the limit $T = \lim_{n\to+\infty} S_n$ is also a stopping time.

1.3. The σ-field $\underline{\underline{F}}_T$. If T is a stopping time, $\underline{\underline{F}}_T$ is the family of all the events $A \in \underline{\underline{F}}_\infty = \bigvee_t \underline{\underline{F}}_t$, such that for every $t \geq 0$ the event $A \cap \{T \leq t\} \in \underline{\underline{F}}_t$.

It is easy to check that $\underline{\underline{F}}_T$ is a σ-field; it is intuitevely the σ-field of all the events that occurred up to time T. In particular, if T is the constant stopping time t, $\underline{\underline{F}}_T = \underline{\underline{F}}_t$; if S and T are two stopping times, and if $S \leq T$ a.e., then $\underline{\underline{F}}_S \subset \underline{\underline{F}}_T$.

If T is a stopping time, and if $A \in \underline{\underline{F}}_T$, the r.v. T_A defined by $T_A = T$ on A, $T_A = +\infty$ on A^c, is also a stopping time (A^c denotes the complement of the set A).

Any stopping time can be approached strictly on the right by the sequence of stopping times $T_n = T + \frac{1}{n}$ (knowing everything up to the near future you know the present); the similar property on the left is false (knowing the strict past is not enough to know the present); the stopping times which can be thus announced are called predictable times.

1.4. Predictable times. A predictable time T is a stopping time T for which there exists a non decreasing sequence $(T_n)_{n\geq 0}$ of stopping times such that

$$\lim_{n\to+\infty} T_n = T \text{ a.e., and } \forall n \quad T_n < T \text{ a.e. on } \{T > 0\}.$$

We shall say that such a sequence (T_n) announces the stopping time T.

Let T be a predictable time and (T_n) a sequence announcing T; the σ-field $\underline{\underline{F}}_{T-} = \bigvee_n \underline{\underline{F}}_{T_n}$ is independent of the choice of the announcing sequence. It is the σ-field of the events occurring strictly before the time T. If $A \in \underline{\underline{F}}_{T-}$ the stopping time T_A is also a predictable time. The σ-field $\underline{\underline{F}}_{T-}$ is contained in $\underline{\underline{F}}_T$, and if S is a stopping time and $S < T$ a.e., then $\underline{\underline{F}}_S \subset \underline{\underline{F}}_{T-}$.

1.5. Graph of a stopping time. If T is a stopping time, its graph $[\![T]\!]$ is the subset of $\mathbb{R}_+ \times \Omega$:

$$[\![T]\!] = \{(t,\omega) ; \ t = T(\omega) < +\infty\} .$$

1.6. Accessible time. An accessible time is a stopping time T, such that its graph $[\![T]\!]$ is contained in a countable union of graphs of predictable times. So there exists a partition (A_n) of Ω such that on each A_n, the time T can be announced by a sequence $(S_{n,m})_{m\geq 0}$. But the sequence $(S_{n,m})$ depends on the set A_n. The time T is predictable if one can make the $(S_{n,m})$ independent of n.

1.7. Totally inaccessible time. A totally inaccessbile time is a stopping time T such that for every predictable time S, we have $P(T = S < +\infty) = 0$. In other words, one just cannot announce a totally inaccessbile time except on sets of measure zero.

1.8. Decomposition of stopping time. Let T be a stopping time; there exists a set $A \in \underline{\underline{F}}_T$ (unique in the sense that the difference of two such sets is of measure zero) such that T_A is an accessible time, T_{A^c} is a totally inaccessible time and $A \subset \{T < +\infty\}$.

STOCHASTIC PROCESSES

A stochastic process X is a real valued function $(t,\omega) \to X_t(\omega)$ defined on $\mathbb{R}_+ \times \Omega$.

1.9. A stochastic process Y is a version of a process X if $\forall t \geq 0$ $P(Y_t \neq X_t) = 0$. If one looks at the values of two such processes X and Y at a countable number of times (which is the best one can do in reality) one can't tell them apart.

1.10. Two processes X and Y are indistinguishable if $P(\omega; \exists\, t$ such that $X_t(\omega) \neq Y_t(\omega)) = 0$. This is a much stronger property than the preceding one. In the following chapters we shall state theorems of the kind: "there exists a unique process such that...". It will mean, two processes having this property are indistinguishable.

1.11. A process X is measurable if the application $(t,\omega) \to X_t(\omega)$ is $\underline{\underline{B}}(\mathbb{R}_+) \times \underline{\underline{F}}$ measurable ($\underline{\underline{B}}(\mathbb{R}_+)$ is the borelian σ-field on $\mathbb{R}_+$).

1.12. A process X is adapted if for every $t \geq 0$ the application $\omega \to X_t(\omega)$ is $\underline{\underline{F}}_t$-measurable.

1.13. A process X is progressively measurable if for each $t \geq 0$ the restriction of the application $(s,\omega) \to X_s(\omega)$ to the set $[0,t] \times \Omega$ is $\underline{\underline{B}}([0,t]) \times \underline{\underline{F}}_t$- measurable. Such a process is an adapted process.

Why is the notion of progressive measurability of any interest?

a) If X is a stochastic process and T is a stopping time, denotedby X_T the r.v. $X_T(\omega) = X_{T(\omega)}(\omega)$; this r.v. is defined only on $\{T < +\infty\}$ (unless X_∞ is defined in which case we take $X_T = X_\infty$ on $\{T = +\infty\}$). Assume that X is an adapted process; is then $X_T I_{\{T<+\infty\}}$ a $\underline{\underline{F}}_T$-measurable function? No, in general; but if X is progressively measurable, the r.v. $X_T I_{\{T<+\infty\}}$ is $\underline{\underline{F}}_T$-measurable.

b) Let A be a progressively measurable set (i.e. I_A is a progressively measurable process); then the r.v.

$$D_A(\omega) = \inf(t;\ (t,\omega) \in A)$$

is a stopping time (here we adopt the convention $\inf \phi = +\infty$). This last result is far from being trivial.

1.14. Càdlàg processes. A process X is càdlàg if each of its trajectory $t \to X_t(\omega)$ is a right continuous function with finite left limits. For such a process we will denote by X_{t-} the left limit at time t, and by $\Delta X_t = X_t - X_{t-}$ the jump at time t. The jumpsize will be $|\Delta X_t|$.

Any càdlàg adapted process is progressively measurable, and two càdlàg versions of the same process are indistinguishable.

Take a càdlàg process X, and define the r.v.

$$T_{1,0} = 0$$
$$T_{1,1} = \inf(t;\ |\Delta X_t| \geq 1)$$
$$T_{1,2} = \inf(t;\ t > T_{1,1},\ |\Delta X_t| \geq 1)$$
$$\cdots$$
$$T_{k,0} = 0$$
$$\cdots$$
$$T_{k,n} = \inf(t;\ t > T_{k,n-1}\ \frac{1}{k} \leq |\Delta X_t| < \frac{1}{k-1})$$
$$\cdots$$

In other words $T_{k,n}$ is the time of the n^{th} jump of size $|\Delta X_t| \in [\frac{1}{k}, \frac{1}{k-1}]$. The processes X_t and X_{t-} are progressively measurable therefore the $T_{k,n}$ are stopping times. Each of the trajectories $t \to X_t(\omega)$ is a right continuous function with left limits; in a compact interval $[0,s]$ it has only a finite number of jumps of size bigger than a given $\varepsilon > 0$, and the set $U = \{(t,\omega);\ \Delta X_t(\omega) \neq 0\}$ is exactly the countable union of the graphs $\bigcup_{\substack{k \geq 1 \\ n \geq 1}} [T_{k,n}]$.

Each stopping time can be split into its totally inaccessible part and its accessible part. Each graph of an accessible time can be covered by a countable union of graphs of predictable times. And in the end we can find a countable number of totally inaccessible times T_n, and a countable number of predictable times S_n such that

$$U = \{(t,\omega);\ \Delta X_t(\omega) \neq 0\} \subset \bigcup_n([\![S_n]\!] \cup [\![T_n]\!]).$$

Moreover we can always assume that $P(T_n = T_m < +\infty) = 0$ and $P(S_n = S_m < +\infty) = 0$ $n \neq m$. So we can cover the jump times of a càdlàg adapted process by a countable number of totally inaccessible, or predictable times. Note that at the totally inaccessible times T_n we have $\Delta X_{T_n} \neq 0$ on $\{T_n < +\infty\}$, but at the predictable times S_n, ΔX_{S_n} can be zero on part of $\{S_n < +\infty\}$. This is what comes from using predictable times instead of accessible times.

1.15. Predictable σ-field. The predictable σ-field is the σ-field on $\mathbb{R}_+ \times \Omega$ generated by the left continuous adapted processes. This σ-field will be essential in stochastic integration (see chapters 3 and 5). A subset of $\mathbb{R}_+ \times \Omega$ is predictable if it belongs to the predictable σ-field. A process X is predictable if the function $(t,\omega) \to X_t(\omega)$ is measurable with respect to the predictable σ-field. Any predictable process is progressively measurable.

It is handy to have some other systems of generators for the predictable σ-field. Here are two:

a) it is generated by the process of the form $\varphi_0^*(\omega)I_{\{0\}}(t) + \sum_{i=0}^{n-1}\varphi_i(\omega)I_{]t_i,t_{i+1}]}(t)$, where $0 \le t_0 < t_1 <\ldots< t_n < +\infty$, φ_0^* is a bounded $\underline{\underline{F}}_0$-measurable r.v., and the r.v. φ_i are bounded and $\underline{\underline{F}}_{t_i}$-measurable

b) it is also generated by the process of the form $\varphi_0^*(\omega)I_{\{0\}}(t) + \sum_{i=1}^{n-1}\varphi_i(\omega)I_{]\!]T_i,T_{i+1}]\!]}(t,\omega)$, where the (T_i) form a nondecreasing finite sequence of stopping times, φ_0^* is a bounded, $\underline{\underline{F}}_0$-measurable r.v., the φ_i are $\underline{\underline{F}}_{T_i}$-measurable, bounded r.v. and $]\!]T_i,T_{i+1}]\!]$ is the stochastic interval $\{(t,\omega);\ T_i(\omega) < t \leq T_{i+1}(\omega)\}$.

If X is a predictable process and T a predictable time, the r.v. $X_T I_{\{T<+\infty\}}$ is $\underline{\underline{F}}_{T-}$-measurable (it is obvious for left continuous processes and extend easily to predictable processes).

1.16. Predictable times and predictable σ-fields. A r.v. T is a predictable time if and only if its graph $[\![T]\!]$ is a predictable set (this is another non trivial result).

If A is a predictable set, the r.v. $D_A(\omega) = \inf(t;\ (t,\omega) \in A)$ is a stopping time (1.13 and 1.15). If the graph $[\![D_A]\!]$ is included in the set A, $[\![D_A]\!] = A \setminus]\!]D_A + \infty[\![$ is a predictable set and D_A is a predictable time.

1.17. Càdlàg predictable processes. In particular, if (X_t) is a càdlàg predictable process, the time $T_{k,n}$ of the n^{th} jump of size $|\Delta X_t| \in [\frac{1}{k}, \frac{1}{k-1}]$ is a predictable time and $U = \{(t,\omega);\ \Delta X_t \neq 0\} = \bigcup_{\substack{n\geq 1 \\ k\geq 1}} [\![T_{n,k}]\!]$. Furthermore the r.v. $X_{T_{n,k}}$ are $\underline{\underline{F}}_{T_{n,k}-}$-measurable (1.15).

1.13. Increasing processes and processes with finite variation. A process A is an increasing process if

a) A is adapted and càdlàg

b) $A_0 = 0$

c) $A_s \leq A_t$ for $s \leq t$.

A process B is a process with finite variation if

a) B is adapted and càdlàg

b) $B_0 = 0$

c) for each ω, the trajectory $\omega \to B_t(\omega)$ has finite variation on compact intervals.

One can show that a process is a process with finite variation if and only if it is the difference of two increasing processes.

If B is a process with finite variation, the Stieltjes integrals

$\int_0^t f(s)dB_s(\omega)$ exist for any bounded (or non negative) borelian function $f(s)$. The symbol $\int f(s)|dB_s|$ will denote the integral of $f(s)$ with respect to the variation of B_s. In particular $\int_0^t |dB_s(\omega)|$ is the variation of $B_s(\omega)$ on $[0,t]$.

An increasing process A is <u>integrable</u> if $E[\int_0^\infty dA_t] < +\infty$. A process B has <u>integrable variation</u> if $E[\int_0^\infty |dB_s|] < +\infty$.

If B is a process with finite variation, the sums $\sum_{s\leq t} |\Delta B_s| \leq \int_0^t |dB_s|$ are finite; and the process B is of the form

$$B_t = B_t^c + \sum_{s\leq t} \Delta B_s$$

where B^c is a continuous process with finite variation (if B is an increasing process, so is B^c). Using 1.14 we can write $\sum_{s\leq t} \Delta B_s$ in the form $\sum_n \Delta B_{T_n} I_{\{t\geq T_n\}}$, where T_n is a sequence of stopping times.

1.19. <u>Predictable processes with finite variation</u>. Suppose now that B is a predictable process with finite variation, the stopping times T_n can be taken predictable, and the ΔB_{T_n} are $\underline{\underline{F}}_{T_n-}$-measurable (see 1.17). Any <u>predictable process with finite variation</u> is therefore of the form

$$B_t = B_t^c + \sum \varphi_n I_{\{t\geq T_n\}}$$

where B^c is a continuous process with finite variation, the T_n are predictable times, the r.v. φ_n are $\underline{\underline{F}}_{T_n-}$-measurable, and $\sum_n |\varphi_n| I_{\{t\geq T_n\}}$ exists for any t. The reader can check that conversely any process of this form is a predictable process with finite variation.

CHAPTER II: MARTINGALES, LOCAL MARTINGALES AND SEMIMARTINGALES

We shall just give here the results necessary for Theorem 3.1 of chapter 3 which shows why semimartingales are important. The machinery on martingales and local martingales needed to construct the stochastic integrals will be seen in chapter 4.

MARTINGALE, SUBMARTINGALE AND SUPERMARTINGALE

This section is just a summary of the classical results in martingale theory. The reader who is not familiar with the subject should consult [6] or [12].

2.1. Martingales. A martingale is an adapted process M such that

a) $E[|M_t|] < +\infty \quad \forall t \geq 0$

b) $E[M_t|F_s] = M_s$ a.e. $\forall t \geq s$.

2.2. Sub and supermartingales. A super (resp. sub) martingale is an adapted process M such that

a) $E[|M_t|] < +\infty \quad \forall t \geq 0$

b) $E[M_t|\underline{\underline{F}}_s] \leq M_s$ (resp. $\geq M_s$) a.e. $\forall t \geq s$.

If M_t is the capital of a gambler a time t the notion of martingale (resp. sub, resp. super) corresponds to the notion of fair (resp. favorable, resp. unfavorable game).

2.3. Càdlàg versions of martingales. Any martingale M has a càdlàg version; therefore the term "martingale" will from now on mean "càdlàg martingale".

2.4. If X is a supermartingale (non necessarily càdlàg), for almost all ω, the two limits

$$X_{t_+} = \lim_{\substack{s\to t \\ s>t \\ s\in \mathbb{Q}}} \quad \text{and} \quad X_{t_-} = \lim_{\substack{s\to t \\ s<t \\ s\in \mathbb{Q}}} X_s$$

exist for each $t \in \mathbb{R}_+$ (the limits are taken over the set $\mathbb{Q}$ of the rational numbers). The process (X_{t_+}) is then indistinguishable from a càdlàg supermartingale. The supermartingale (X_t) has a right continuous version if and only if the function $t \to E[X_t]$ is right continuous. We shall always, except when otherwise specified, consider càdlàg supermartingales, and call them supermartingales for short.

2.5. A martingale M is said to be uniformly integrable if the family of r.v. $(M_t)_{t\geq 0}$ is uniformly integrable. For any uniformly integrable martingale M, the limit $M_\infty = \lim_{t\to+\infty} M_t$ exists a.e.; and for any stopping time T, we then have $M_T = E[M_\infty|\underline{\underline{F}}_T]$. Apply this result to a sequence S_n announcing a predictable time S. We get $M_{S_n} = E[M_\infty|\underline{\underline{F}}_{S_n}] = E[M_S|\underline{\underline{F}}_{S_n}]$ for any n; and by taking limits on both sides, $E[M_S|\underline{\underline{F}}_{S-}] = M_{S-}$. So if M is a uniformly integrable martingale and S a predictable time, the jump at time S verifies $E[\Delta M_S|\underline{\underline{F}}_{S-}] = 0$.

2.6. Let X be a non negative supermartingale, and take $X_\infty = 0$, then $(X_t)_{0\leq t\leq+\infty}$ is a supermartingale, and for any two stopping times T and S such that $S \leq T$ we have X_S and $X_T \in L^1$, and $E[X_T|\underline{\underline{F}}_S] \leq X_S$.

2.7. Let M be a non negative martingale, and let $T = \inf(t; M_t \text{ or } M_{t-} = 0)$, then $M = 0$ a.e. on $[\![T, +\infty]\!]$. In particular if $M_\infty = \lim_{t\to+\infty} M_t$ exists and if $M_\infty > 0$ a.e., we have $T = +\infty$ a.e. that is $P\{(\omega;\ \exists t \text{ such that } M_t(\omega) \text{ or } M_{t-}(\omega) = 0)\} = 0$.

2.8. Jensen's inequality. If M is a martingale and $f(x)$ is a convex function, the process $f(M)$ is a submartingale provided $E[|f(M_t)|] < +\infty,\ \forall t \geq 0$.

2.9. Doob's decomposition theorem.

If M is a uniformly integrable martingale, and A is an integrable increasing process, the process $X = M - A$ is a supermartingale, satisfying the strong following integrability condition: the family of r.v. $\{X_T I_{\{T<+\infty\}},\ T \text{ stopping time}\}$ is a uniformly integrable family. We shall call those supermartingales, supermartingales of class (D). Doob's decomposition theorem is just the converse statement: any supermartingale X of class (D) is of the form

$$X = M - A$$

where M is a uniformly integrable martingale, and A is a predictable, integrable, increasing process. And this decomposition is unique. See [12], [4] and [14] for three different proofs of this theorem.

2.10. Corollaries.

1) Let X be a supermartingale of class (D), and B be the predictable increasing process in Doob's decomposition. The process B jumps only at predictable times; at such a predictable time T, ΔB_T is $\underline{\underline{F}}_{T-}$-measurable (see 1.19) and we have, (2.5), if M is the uniformly integrable martingale $M = X + B$

$$0 = E[\Delta M_T | \underline{\underline{F}}_{T-}] = E[\Delta X_T + \Delta B_T | \underline{\underline{F}}_{T-}] = E[\Delta X_T | \underline{\underline{F}}_{T-}] + \Delta B_T$$

And the jumps of B are easy to compute.

2) If A is an integrable increasing process the process $-A$ is a supermartingale of class (D); therefore there exists a unique integrable predictable increasing process B such that $B - A$ is a uniformly integrable martingale. The process B is called the compensator of A.

This generalizes to processes with integrable variation. If A is such a process, there exists a unique predictable process B with integrable variation such that $B - A$ is a uniformly integrable martingale.

The process B is again called the compensators of A. From part 1 of this corollary we get:

a) if T is a totally inaccessible time, and φ an $\underline{\underline{F}}_T$-measurable function in L^1, the compensator of $A_t = \varphi I_{\{t \geq T\}}$ is a continuous process with integrable variation.

b) if T is a predictable time, and φ is an $\underline{\underline{F}}_T$-measurable function in L^1, the compensator of $A_t = \varphi I_{\{t \geq T\}}$ is the process $B_t = E[\varphi | \underline{\underline{F}}_{T-}] I_{\{t \geq T\}}$

LOCAL MARTINGALES AND PROCESSES WITH LOCALLY INTEGRABLE VARIATION

Let X be a stochastic process, and T a stopping time. The symbol X^T will denote the process X stopped at time $T : X^T(\omega) = X_{t \wedge T}(\omega)$. A process M is a martingale if and only if, for any constant time n, the process M^n is a uniformly integrable martingale. And it is natural to let the constant times n be stopping times T_n:

2.11. Definition. A localizing sequence is a nondecreasing sequence (T_n) of stopping times such that $\lim_n T_n = +\infty$ a.e.

2.12. Definition. A process M is a local martingale if

a) $M_0 = 0$

b) there exists a localizing sequence (T_n) such that each process M^{T_n} is a uniformly integrable martingale.

Such a sequence (T_n) will be called a fundamental sequence for the local martingale M.

Remark. 1) Local martingales are necessarily càdlàg processes as we decided that here "martingale" means "càdlàg martingale".

2) The processes defined above should really be called "local martingales vanishing at time zero". We shall not use here the general

concept of local martingales. The interested reader can consult [3].

2.13. Definition. A stopping time T reduces a local martingale M if M^T is a uniformly integrable martingale.

2.14. Theorem. Let M be a local martingale then

1) a stopping time S reduces M if and only if the process M^S is of class (D) (i.e. the family of r.v. $\{M^S_T I_{\{T<+\infty\}}\ ;\ T$ stopping time$\}$ is uniformly integrable;

2) if T is a stopping time reducing M, if S is a stopping time and if $S \leq T$, then S reduces T.

3) if S and T are two stopping times reducing M, then $S \vee T$ reduces M.

Proof. Parts 1 and 2 are trivial. Part 3 comes from the fact that $M^{S\vee T} = M^S + M^T - M^{S\wedge T}$.

2.15. Theorem. If a process M is locally a local martingale, then it is a local martingale.

So there is no way one can get more general processes by localizing once more.

Proof. There exists a localizing sequence (T_n) such that each process M^{T_n} is a local martingale. Let $\underline{\underline{H}} = \{T;\ T$ stopping time, M^T is a uniformly integrable martingale$\}$. And take $R = \text{ess.}\sup_{T\in \underline{\underline{H}}} T$. There exists a sequence S_n of elements of $\underline{\underline{H}}$ which converges a.e. to R. Using part 3 of 2.14 we can make this sequence non decreasing. The r.v. R has to be bigger than or equal to any of the T_n (a.e.), so $R = +\infty$, and S_n is a fundamental sequence for the process M.

2.16. Definition. A process B has locally integrable variation if there exists a localizing sequence (T_n) such that each process B^{T_n} has integrable variation.

2.17. Theorem. Let B be a process with locally integrable variation, there exists a unique predictable process A with locally integrable variation such that $B - A$ is a local martingale. A is called the compensator of B.

Proof. Easy consequence of the existence and uniqueness of the compensator of a process with integrable variation.

2.18. Remark. It is important to remark the following fact. If B is a predictable process with finite variation, then the variation of B is locally bounded: define the stopping times $T_n = \inf(t; \int_0^t |dB_s| \geq n) \wedge n$. The variation of B on $[\![0,T_n[\![$ is bounded by n, but we know nothing on the jump of B at time T_n. Now each time T_n is predictable and can be announced by a sequence $(S_{n,m})_{m\geq 0}$; on $[\![0,S_{n,m}]\!]$ the variation of B is bounded by n. Take the stopping times $R_k = \sup_{\substack{n\leq k\\ m\leq k}} S_{n,m}$. The sequence (R_k) is a localizing sequence and on $[\![0,R_k]\!]$ the variation of B is bounded by k.

Let A be a process with finite variation. If there exists a predictable process B with finite variation such that $A - B$ is a local martingale, then B has locally bounded variation, and A itself has locally integrable variation. We can rewrite theorem 2.17 in a stronger form

2.19. Theorem. A process A with finite variation has a compensator if and only if its variation is locally integrable.

Here is now an easy way to verify that the variation is locally integrable.

2.20. Lemma. Let A be a process with finite variation; we assume that there exists a localizing sequence (T_n) such that for each n, $\sup_{s\leq T_n} |\Delta A_s| = Y_n \in L^1$. Then the variation of A is locally integrable.

Proof. Take $S_n = T_n \wedge \inf(t; \int_0^t |dA_s| \geq n)$. The sequence (S_n) is localizing, the variation of A on $[\![0,S_n[\![$ is bounded by n, and

$$\int_{[\![0,S_n]\!]} |dA_s| \leq n + |\Delta A_{S_n}| \leq n + Y_n \in L^1.$$

We shall now state the fundamental lemma for local martingales.

2.21. Fundamental lemma. Let M be a local martingale then

1) the increasing process $M_t^* = \sup_{s\leq t} |M_s|$ is locally integrable

2) the local martingale M can be written in the form $M = U + V$ where U is a local martingale, the jumps of U are bounded by 1 in size, and V is both a local martingale and a process with finite variation. (The bound 1 for the jump size could have been replaced by another strictly positive constant). In particular there exists a localizing sequence (T_n) such that each U^{T_n} is a bounded process.

Proof. 1) Let R_n be a fundamental sequence for M. We can always assume that the R_n are finite (otherwise use the fundamental sequence $R_n \wedge n$); we consider the following stopping times

$$S_n = R_n \wedge \inf\{t; |M_t| \geq n\}.$$

The martingales M^{R_n} are uniformly integrable, and $S_n \leq R_n$, therefore the r.v. M_{S_n} is integrable (2.5). Furthermore on $[\![0,S_n[\![$, we have $|M_t| \leq n$; and

$$M^*_{S_n} \leq n + |M_{S_n}| \in L^1.$$

As the sequence (S_n) is a localizing sequence, the increasing process M_t^* is locally integrable.

2) Let $A_t = \sum_{s\leq t} \Delta M_s I_{\{|\Delta M_s| > \frac{1}{2}\}}$. This sum is, for each ω, a finite sum. Take the sequence (S_n) constructed above and consider the stopping

times

$$T_n = S_n \wedge \inf\{t; \int_0^t |dA_s| \geq n\}.$$

The sequence (T_n) is a localizing sequence, and $\int_{[\![0,T_n]\!]} |dA_t| \leq n + |\Delta A_{T_n}| \leq n + 2M^*_{T_n} \in L^1$. There exists therefore a compensator B for A. Take $V = A - B$, V is both a local martingale and a process with finite variation.

The jumps of B occur only at predictable times T. Stop all the processes at the time S_n, and remember that M^{S_n} is a uniformly integrable martingale. If T is a predictable time, we get

$$|\Delta B_T^{S_n}| = |E[\Delta A_T^{S_n}|\underline{\underline{F}}_{T-}] = |E[\Delta M_T^{S_n} - \Delta M_T^{S_n} I_{\{|\Delta M_T|<\frac{1}{2}\}}|\underline{\underline{F}}_{T-}]| \leq 0 + \frac{1}{2}.$$

And the jumps of U verify

$$|\Delta U_t| \leq |\Delta(M - A)_t| + |\Delta B_t| \leq 1$$

It is now easy to see that the sequence (R_n), $R_n = \inf(t; |U_t| \geq n)$, is a localizing sequence and that U^{R_n} is bounded by $n + 1$.

SEMIMARTINGALES

2.22. Definition. A process X is a semimartingale if it is of the form $X = X_0 + M + A$, where X_0 is an $\underline{\underline{F}}_0$-measurable r.v., M is a local martingale and A is a process with finite variation (remember that by definition both processes M and A vanish at $t = 0$).

This definition contains no local integrability condition. It should not if we want the semimartingales to remain semimartingales when the probability P is replaced by an equivalent probability Q.

Here again one cannot get more general processes by localizing the notion of semimartingale

2.23. Theorem. Any process which is locally a semimartingale is a semimartingale.

Proof. We shall need the following useful lemma

2.24. Lemma. Let X be a semimartingale, assume that the size of the jumps of X is bounded by a $(a > 0)$. Then one can write X in a unique way as

$$X = X_0 + M + A$$

where X_0 is an $\underline{\underline{F}}_0$-measurable r.v., M is a local martingale, and A is a predictable process with finite variation (in fact, locally bounded variation by 2.18).

Proof. The semimartingale is of the form $X = X_0 + N + B$ where N is a local martingale and B a process with finite variation. The r.v. X_0 is uniquely determined. The jumps of the process B verify

$$|\Delta B_s| \le |\Delta M_s| + |\Delta X_s| \le 2M_s^* + a.$$

The increasing process $Y_t = \sup_{s\le t}|\Delta B_s|$ is locally integrable and by 2.20, the variation of B is locally integrable. Let A be the compensator of B, and let $M = N + B - A$; M is a local martingale, A is a predictable process with finite variation, and $X = X_0 + M + A$.

If $X = X_0 + M + A = X_0 + M' + A'$ are two such decompositions of the semimartingale X, $A - A'$ is a local martingale, so A' is the compensator of A. But A being predictable is its own compensator and $A = A'$.

Proof of 2.23. The process X is locally a semimartingale. That is, there exists a localizing sequence of stopping times (T_n), such that each X^{T_n} is a semimartingale.

Take $Y_t = \sum_{s\le t} \Delta X_s I_{\{|\Delta X_s|\ge 1\}}$. As X is càdlàg the process Y has finite variation; for each n the process $X^{T_n} - Y^{T_n}$ is a semimartingale with jump size smaller than 1. It can therefore be written in a unique way as $X^{T_n} - Y^{T_n} = X_0 + M_n + A_n$, where M_n is a local martingale and A_n is a predictable process with finite variation. The uniqueness of the A_n gives

$$A_n = A_{n+1}, \quad \text{and} \quad M_n = M_{n+1} \quad \text{on} \quad [\![0,T_n]\!].$$

One can patch the A_n together, and the M_n together to get the processes $A = \sum_n A_n I_{]\!]T_{n-1},T_n]\!]}$ $M = \sum_n M_n I_{]\!]T_{n-1},T_n]\!]}$. The process A is predictable and has finite variation, the process M is locally a local martingale, therefore it is a martingale (2.15) and $X = X_0 + M + A + Y$ is a semimartingale.

The following lemma is important.

2.25. Lemma. *Let X be a càdlàg adapted process; we assume that there exists a sequence of stopping times T_n and a sequence of semimartingales Y_n such that*

1) $\lim_{n\to+\infty} T_n = +\infty$ a.e.
2) $X_n = Y_n$ on $[\![0,T_n[\![$.

Then X is a semimartingale.

Proof. $X^{T_n} = Y_n^{T_n} - Y_{n,T_n} I_{\{t \geq T_n\}} + X_{T_n} I_{\{t \geq T_n\}}$. So for each n the process X^{T_n} is a semimartingale. If the sequence T_n is non decreasing, 2.23 says that X is a semimartingale. Otherwise, make the sequence non decreasing by remarking that if S and T are two stopping times, and if X^S and X^T are both semimartingales, then $X^{S\wedge T}$ and $X^{S\vee T} = X^S + X^T - X^{S\wedge T}$ are both semimartingales.

2.26. Examples of semimartingales.

1) any supermartingale (therefore any submartingale) is a semimartingale: let X be a supermartingale, by 2.24 we just have to show that each X^n is a semimartingale. But $X_t^n = E[X_n|\underline{\underline{F}}_t] + X_t^n - E[X_n|\underline{\underline{F}}_t]$,