A. Pignedoli (Ed.)

Some Aspects of Diffusion Theory

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FONDAZIONE CIME ROBERTO CONTI

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Some Aspects of Diffusion Theory

Lectures given at a Summer School of the
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FONDAZIONE CIME ROBERTO CONTI

C.I.M.E. Foundation
c/o Dipartimento di Matematica "U. Dini"
Viale Morgagni n. 67/a
50134 Firenze
Italy
cime@math.unifi.it

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"SOME ASPECTS OF DIFFUSION THEORY"

Coordinatore : Prof. A. PIGNEDOLI

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CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

V.C.A. FERRARO

DIFFUSION OF IONS IN A PLASMA WITH APPLICATIONS TO THE IONOSPHERE

Corso tenuto a Varenna dal 19 al 27 settembre 1966

DIFFUSION OF IONS IN A PLASMA WITH APPLICATIONS TO THE IONOSPHERE

by

V. C. A. Ferraro
(Queen Mary College, University of London)

I. Derivation of the diffusion equations in plasmas

1. The term 'plasma' was first used by Langmuir for the state of a gas which is fully ionised (for example, the high solar atmosphere) or only partially ionised, (for example, the ionosphere). Our main interest in this course will be the diffusion of ions in such a plasma, arising from non-uniformity of composition, of pressure gradients or electric fields.

We begin by considering the simple case of a fully ionised gas and for simplicity restrict ourselves to the case when only one type of ion and electrons are present.

2. The velocity distribution function

We make the familiar assumption of molecular chaos, in which it is supposed that particles having velocity resolutes lying in a certain range are, at any instant, distributed at random. It is therefore most convenient to use six dimensional space in which the coordinates are the resolutes of the position vector $\underline{r}$ and velocity $\underline{v}$. The state of the plasma can then be specified by the distribution functions $f_\alpha(t, \underline{r}, \underline{v})$, where t is the time, that characterise each particle component α, for example, the ions or the electrons The quantity

$$f_\alpha(t, \underline{r}, \underline{v})\ d\underline{r}\ d\underline{v} \tag{1}$$

then represents the number of particles in the six dimensional volume element $d\underline{r}\ d\underline{v}$. In the simplest case, the plasma consists of single ions (α = i) and electrons (α = e) . In more complicated cases, the plasma may consist of several ion species in addition to neutral particles (α= n) such as atoms, molecules, exited atoms, etc. The total number of particles of constituent α in the element $d\underline{r}$ is obtained by integrating (1) throughout the velocity space. This number is, by hypothesis, $n_\alpha \, d\underline{r}$ and thus $n_\alpha = \int f_\alpha(t, \underline{r}, \underline{v}_\alpha)\, d\underline{v}_\alpha$ (2) . The behaviour of the ionised gas is described by a system of equations (Boltzmann equations) which can be derived as follows. Suppose that each particle of mass m_α is acted on by force $m_\alpha \underline{F}_\alpha$, then in a time dt in which the particles of constituent α suffer no collisions, the same particles that occupy the volume of phase space $d\underline{r}\, d\underline{v}_\alpha$ at time t would occupy the volume of phase $(\underline{r} + \underline{v}_\alpha dt)(\underline{v} + F_\alpha dt)$ at time t + dt . The number in this set is

$$f_\alpha(t + dt,\ r + \underline{v}_\alpha dt,\ v + \underline{F}_\alpha dt)$$

and the difference

$$f_\alpha(t + dt,\ \underline{r} + \underline{v}_\alpha dt,\ \underline{v}_\alpha + \underline{F}_\alpha dt) - f_\alpha(t,\ \underline{r},\ \underline{v}_\alpha)\ d\underline{r}\ d\underline{v}_\alpha$$

therefore represent the difference in the gain of particles by collisions to this final set and the loss of the particle to the original set in time dt. This must be proportional to $d\underline{r}\ d\underline{v}_\alpha dt$; and we denote it by $C_\alpha d\underline{r}\, .\ d\underline{v}_\alpha dt$. Taking the limit as $dt \to 0$, we arrive at Boltzmann's equation for f_α , viz

$$(3) \qquad \frac{f_\alpha}{t} + (\underline{v}_\alpha . \nabla) f_\alpha + (\underline{F}_\alpha . \nabla_{\underline{v}_\alpha}) f_\alpha = C_\alpha$$

where $\nabla_{\underline{v}_\alpha}$ stands for the gradient operator $\frac{\partial}{\partial u_\alpha}$, $\frac{\partial}{\partial v_\alpha}$, $\frac{\partial}{\partial w_\alpha}$ in velocity space.

3. Charge neutrality and the Debye distance

In general a plasma will rapidly attain a state of electrical neutrality; this is because the potential energy of the particle resulting from any space charge would otherwise greatly exceed its thermal energy. Small departures from strict neutrality will occur over small distances whose order of magnitude can be obtained as follows. The electrostatic potential V satisfies Poisson's equation.

$$\nabla^2 V = -4\pi(Zn_i - n_e)e \tag{4}$$

Here Ze is the charge on an ion and $-e$ that of the electrons. In thermodynamic equilibrium, the number densities of the ions and elections respectively are given by

$$n_i = n_i^{(o)} \exp(-ZeV/kT_i), \quad n_e = n_e^{(o)} \exp(eV/kT_e), \tag{5}$$

where k is the Boltzmann constant, T_i, T_e are the ion and electron temperatures and $n_i^{(o)}$ and $n_e^{(o)}$ are the values of n_i and n_e for strict neutrality so that $n_e^{(o)} = Zn_i^{(o)}$. In general, departures from neutrality are small so that we may expand the exponential to the first power of the arguments only. We have approximately

$$Zn_i - n_e \simeq Zn_i^{(o)}\left(1 - \frac{ZeV}{uT_i}\right) - n_e^{(o)}\left(1 + \frac{eV}{uT_e}\right)$$

and hence

$$\nabla^2 V = \frac{V}{D^2}, \tag{6}$$

where

$$D = \left\{\frac{kT_eT_i}{4\pi Ze^2(n_i^{(o)}T_i + n_e^{(o)}T_e)}\right\}^{1/2} \tag{7}$$

The quantity D has the dimensions of a length and is called the Debye distance. The solution of (6) for spherical symmetry is

$$V = \frac{e_\alpha}{r} \exp\left(-\frac{r}{D}\right) \tag{8}$$

where e_α is the charge on the particle. For small distances r from the origin ($r << D$), (8) reduces to the pure Coulomb potential of the charged particle. For large distances ($r >> D$), $V \rightarrow 0$ exponentially. Thus in a neutral plasma in thermodynamical equilibrium the Coulomb field of the individual charge is cut off (shielded) at a distance of order D. Hence, we may assume that the particles do not interact in collisions for which the impact parameter is greater than D. The Debye shielding is not established instantaneously ; oscillations of the space charge will have a frequency $\omega_o = \left(4\pi n_e^2/m_e\right)^{\frac{1}{2}}$ (since the displacemente of the electrons (or ions) bodily by a distance x gives rise to an electric field of intensity $4\pi n_e ex$ lending to restore neutrality). Thus the time required to establish shielding is of the order

$$\tau \sim \frac{1}{\omega_o}$$

4. Diffusion of test particles in a plasma

A particular particle, which we call 'test particle', in a plasma will suffer collisions with the other particles in the plasma, which we call 'field particles'. Electrostatic forces between the particles have a greater range than the forces between neutral molecules in an ordinary gas. Consequently, the cumulative effect of distant encounters will be far more important than the effect of close collisions, which change completely the particle velocities. We shall therefore suppose that the deflections which the test particles undergo are mostly small. The motion of the test particle is most conveniently descibed in the <u>velocity space,</u> i.e., a space in which the velocity vector $\underline{v}$ is taken as the position vector and the apex of this vector is called the velocity point of the particle. Referred to Cartesian coordinates the coordinates of these points will be denoted by v_x, v_y, v_z .

As the test particle changes its position in ordinary space, its position in velocity space changes either continuously or discontinuously due to encounter with fixed particles. In general the displacement is complicated. (Fig. 1)

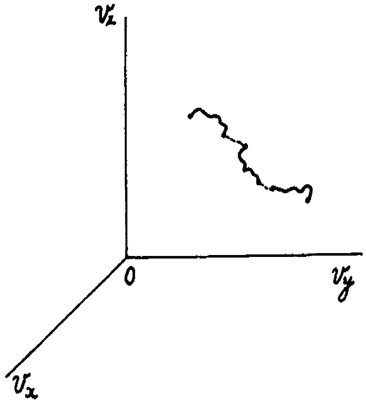

FIG. 1.

It is clearly impossible, and indeed futile, to trace the motion of a single particle and we are forced to consider a statistical description of the motion. In this, instead of a single particle, we consider an assembly containing a large number of test particles which have the <u>same</u> velocity $\underline{v}_o$ initially.

Suppose these are concentrated around the point $\underline{v}_o$ in the velocity space . At subsequent times the cloud will spread, changing both its size and shape, as a result of successive encounters.

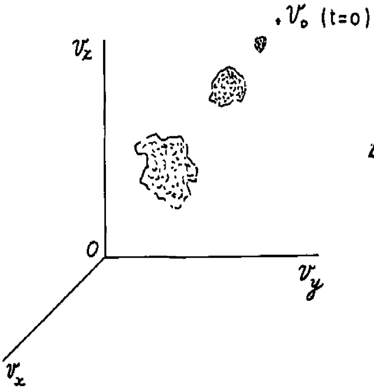

FIG. 2.

We now require to find quantities which will adequately describe the process. One such quantity is the change in velocity $\Delta \underline{v}$ of a test particle produced by the encounters. Suppose that $\underline{v}_o$ is parallel to the z-axis and consider the resolutes $\Delta v_x, \Delta v_y, \Delta v_z$ of $\Delta \underline{v}$. Suppose that $(\Delta v_x)_i$ is the change in Δv_x produced by the ith encounter. Then after N encounters,

$$\Delta v_x = \sum_{i=1}^{N} (\Delta v_x)_i$$

We assume that all the encounters are random, but as we have already seen, we cannot predict the change Δv_x for a single test particles. However, we can define an average value of Δv_x, say $\overline{\Delta v_x}$ for the large assembly of particles under consideration. If the distribution of velocities is isotropic, then $\overline{\Delta v_x} \equiv 0$, by symmetry , and likewise $\overline{\Delta v_y} \equiv 0$. But $\overline{\Delta v_z}$ need not vanish since the assembly (or cloud) has an initial velocity in the z-direction. However the mean square of $\overline{\Delta v_x^2}$ will not vanish . This mean value will contain terms of the form $\overline{(\Delta v_x)_i^2}$ and $\overline{(\Delta v_x)_i(\Delta v_x)_j}$: If the collisions are small we may expect that successive collisions will produce, on the average, the same average change as the first collisions. Thus the N terms $\overline{(\Delta v_x)_i^2}$ are all equal . But the mixed products $\overline{(\Delta v_x)_i(\Delta v_x)_j}$ will vanish when averaged over all particles considered since successive collisions are uncorrelated . Hence

$$(9) \qquad \overline{\Delta v_x^2} = N\overline{(\Delta v_x)_i^2}$$

The dispersion of the points in Fig. 2 will therefore increase like $\sqrt{N}$, but not, in general, equally in all directions. But the centre of gravity may be displaced by an amount proportional to N. (Fig. 2)

The dispersion of the points in the velocity space produced by collisions of the test particles with the field particles is analogous to the diffusion of particles in an ordinary gas. To measure the rate of diffusion in the v_x direction, we consider the average value of (9) per unit time. The resultant value of $\overline{\Delta v_x^2}$, measuring the increase of velocity of dispersion of a group of particles per second, will be denoted by $< \Delta v_x^2 >$ and called a 'diffusion coefficient', a term due to Spitzer . If the velocity distribution of the field particles is isotropic, the diffusion coefficients $< \Delta v_x >$ and $< \Delta v_x \Delta v_y >$ vanish identically,

The encounters between test and field particles which we are considering are assumed to be binary encounters only .(*)
Let $\underline{v}$ be the velocity of a field particle relative to a test particle. Then there will be only three independent diffusion coefficients, namely, $< \Delta v_{\parallel} >$, $< \Delta v_{\parallel}^2 >$ and $< \Delta v_{\perp}^2 >$, where $v_{\parallel}$ and $v_{\perp}$ are measured respectively parallel and perpendicular to $\underline{v}$. Their values will depend on the velocity distribution function of the field particles.

(*) The justification for this will be given in Section 7.

5. Binary encounter of two charged particles

(Hyperbolic orbit)

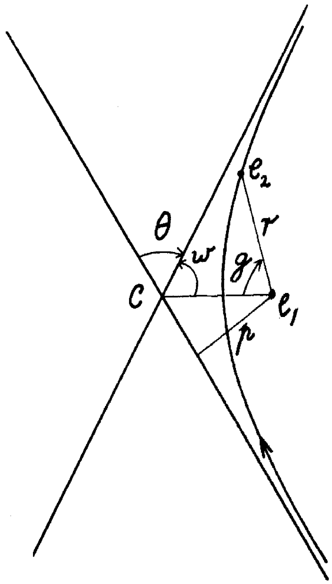

FIG. 3.

Consider the motion of charge e_2 relative to charge e_1; let $\underline{r}_1$ and $\underline{r}_2$ be the position vectors of e_1 and e_2 relative to a Newtonian origin. Then the equation of motion of the charges are respectively

$$m_1 \ddot{\underline{r}}_1 = + \frac{e_1 e_2 \underline{r}}{r^3}, \quad m_2 \ddot{\underline{r}}_2 = - \frac{e_1 e_2 \underline{r}}{r^3}$$

where $\underline{r} = \underline{r}_2 - \underline{r}_1$ and m_1 and m_2 are the masses of the charges. Hence

$$\ddot{\underline{r}} = \ddot{\underline{r}}_2 - \ddot{\underline{r}}_1 = -e_1 e_2 \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \frac{\underline{r}}{r^3}$$

that is, the relative motion is the same as that of a particle under a central force at A varying inversely as the square of the distance whose strength is $\frac{e_1 e_2}{m_{12}}$, where

$m_{12} = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass. (Fig. 3).

Let v_∞ be the relative velocity of the charges at infinity and p the impact parameter. The energy integral is, with the usual notation,

$$v^2 = \frac{e_1 e_2}{m_{12}} \left(\frac{2}{r} + \frac{1}{a}\right)$$

whence

$$v_\infty^2 = \frac{e_1 e_2}{m_{12} a} \tag{10}$$

The polar equation of the orbit is

(11) $$r = \frac{\ell}{1 + e \cos\varphi}$$

where ℓ is the semi-latus rectum and e the eccentricity. As $r \to \infty$, $\varphi \to \pi - w$ so that (11) gives

$$\cos w = \frac{1}{e}$$

Also AC = ae ; hence

$$\sin w = \frac{p}{ae}$$

Thus $1 = \sin^2 w + \cos^2 w = \frac{1}{e^2} + \frac{p^2}{a^2 e^2}$ or $e^2 = 1 + \frac{p^2}{a^2}$ giving

$$\cos w = \frac{1}{\sqrt{1 + \frac{p^2}{a^2}}}, \quad \sin w = \frac{p}{a\sqrt{1 + \frac{p^2}{a^2}}}, \quad \tan w = \frac{p}{a}$$

or using (1)

(12) $$\tan w = \frac{p v_\infty^2 m_{12}}{e_1 e_2}$$

6. Calculation of diffusion coefficients

Consider the scattering of test particles (α) by a flux of field particles (β) . The spatial density of the latter is

$$dn_\beta = f_\beta(\underline{v}')\, d\underline{v}'$$

where $\underline{v}'$ is the velocity of the particles and f_β the distribution function of the field particles. Consider the collision of a test particle α with a field particle β of this flux. Then the velocity $\underline{v}_\alpha$ of the test particle is related to the velocity $\underline{v}_g$ of the centre of mass of the two particles and their relative velocity $\underline{u}$ by

$$\underline{v}_\alpha = \underline{v}_g + \frac{m_\beta}{m_\alpha + m_\beta}\,\underline{u}\ ;$$

hence , since $\underline{v}_g$ is unaltered by the encounter,

(13)
$$\Delta \underline{v}_\alpha = \frac{m_{\alpha\beta}}{m_\alpha}\Delta\underline{u}$$

where $m_{\alpha\beta}$ is the reduced mass of m_α and m_β, and $\Delta\underline{u}$ the change in $\underline{u}$ produced by the encounter.

Also, in taking the average of the change of velocity over the test particles in the assembly, the summation reduces to a summation over all particles of the flux incident on a <u>fixed</u> scattering centre. The number of particles moving through an area $dA = p\,dp\,d\varphi$ of a plane π perpendicular to $\underline{u}$ in unit time is

(14)
$$dn_\beta \left|\underline{u}\right| d\,A = f_\beta\,(\underline{v}')\,d\underline{v}'\,u\,dA$$

Multiply this by the components of the vector $\Delta\,\underline{v}_\alpha$ given by (13) and integrate over all the plane π and then over the velocities of the field particles we find

(15)
$$<\Delta\,v_k> = \int f_\beta\,(\underline{v}')\,w_k\,d\underline{v}' \quad (k = x, y, z)$$

where
$$w_k = \frac{m_{\alpha\beta}}{m_\alpha}\int_{\text{plane}} \Delta u_k\ \ u dA\ ,$$

(16)
$$<\Delta v_k\,\Delta\,v_\ell> = \int f_\beta\,(\underline{v}')\,w_{k\ell}\,d\underline{v}'\ , \quad (k, l = x, y, z)$$

where
$$w_{k\ell} = \left(\frac{m_{\alpha\beta}}{m_\alpha}\right)^2 \int_{\text{plane}} \Delta u_k\,\Delta\,u_\ell\ \ u\,d\,A$$

(15) and (16) are the diffusion coefficients . It will be convenient to compute these integrals relative to a coordinate system in which the z-axis

is along $\underline{u}$.

For referring to Fig. a , and Fig. 4b .

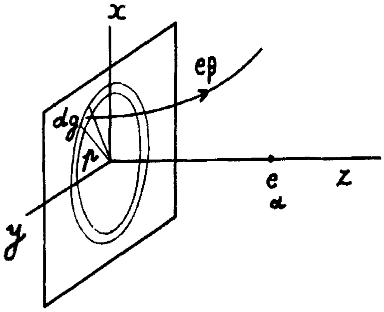

FIG. 4 a

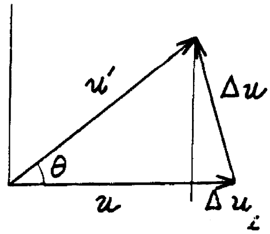

FIG. 4b

we have

$$\Delta u_x = u \sin\theta \cos\varphi$$
$$\Delta u_y = u \sin\theta \sin\varphi$$
$$\Delta u_z = -u(1 - \cos\theta)$$

Also from (12) and the fact that $\theta = \pi - 2w$ we have

(17) $$\tan\frac{\theta}{2} = \frac{p_\perp}{p}$$

where $$p_\perp = \frac{e_\alpha e_\beta}{m_{\alpha\beta} u^2}$$

Then $$\Delta u_x = 2u \frac{p\, p_\perp}{p^2 + p_\perp^2} \cos\varphi \; , \quad \Delta u_y = 2u \frac{p\, p_\perp}{p^2 + p_\perp^2} \sin\varphi$$

$$\Delta u_z = -2u \frac{p_\perp^2}{p^2 + p_\perp^2}$$

Integration with respect to p and φ over the plane gives

$$w_x = 0 = w_y$$

whilst

$$w_z = \frac{m_{\alpha\beta}}{m_\alpha} \int_{\text{plane}} \Delta u_z \, u \, d A .$$

If the limits of integration for p are 0 and ∞, the integral diverges; however, we have already seen that the Coulomb field of individual charges is cut off at distance of order D, the Debye distance. Hence we can take D as the upper limit for p in the integral. We then find

$$
\begin{aligned}
w_z &= -\frac{1+\frac{m_\alpha}{m_\beta}}{4\pi u^2}\left(\frac{4\pi e_\alpha e_\beta}{m_\alpha}\right)^2 \int_0^{D} \frac{p\,dp}{p^2+p_\perp^2} \\
&= -\lambda\,\frac{1+\frac{m_\alpha}{m_\beta}}{4\pi u^2}\left(\frac{4\pi e_\alpha e_\beta}{m_\alpha}\right)^2
\end{aligned}
\tag{18}
$$

where

$$
\lambda = \int_0^{D} \frac{p\,dp}{p^2+p_\perp^2} = \log_e \frac{(D^2+p_\perp^2)^{\frac{1}{2}}}{p_\perp}
\tag{19}
$$

It has been tacitly assumed that $D \gg p_\perp$; to illustrate that this is likely to be the case in general, let us take $T_i = T_e = 1$ keV ($10^{7\circ}$ K), $n_i = n_e = 10^5\ \mathrm{cm}^{-3}$, and $Z = 1$, then

$$
D = \left(\frac{kT}{8\pi n e^2}\right)^{\frac{1}{2}} \sim \frac{1}{2}\times 10^{-3}\ \mathrm{cm}
$$

and

$$
p_\perp = \frac{e^2}{3kT} \sim \frac{1}{2}\times 10^{-10}\ \mathrm{cm}
$$

so that $D/p_\perp \sim 10^7$. Hence in (19) we may neglect $p_\perp$ compared with D in the numerator and write

$$
\lambda \sim \log_e \frac{D}{p_\perp} = \log_e \left\{ \frac{3}{2e^3}\left(\frac{k^3T^3}{2\pi n}\right)^{\frac{1}{2}} \right\}
\tag{19a}
$$

Likewise

$$w_{xx} = \left(\frac{m_{\alpha\beta}}{m_\alpha}\right)^2 \int (\Delta u_x)^2 u \, dA = \left(\frac{m_{\alpha\beta}}{m_\alpha}\right)^2 \int_{\text{plane}} \left(2u \frac{p\,p_\perp}{p^2+p_\perp^2} \cos\varphi\right)^2 updpd\varphi$$

$$= \frac{1}{4\pi u}\left(\frac{4\pi e_\alpha e_\beta}{m_\alpha}\right)^2 \int_0^D \frac{p^3\,dp}{(p^2+p_\perp^2)^2}$$

$$\simeq (\lambda - \tfrac{1}{2}) \frac{1}{4\pi u}\left(\frac{4\pi e_\alpha e_\beta}{m_\alpha}\right)^2 \text{ neglecting terms of order} \left(\frac{p_\perp}{D}\right)^2$$

Again $w_{yy} = w_{xx}$ and $w_{\alpha\beta} = 0$, $\alpha \neq \beta$: Finally

$$w_{zz} = \left(\frac{m_{\alpha\beta}}{m_\alpha}\right)^2 \int (\Delta u_z)^2 u \, dA = \left(\frac{m_{\alpha\beta}}{m_\alpha}\right)^2 \int_{\text{plane}} \left(-2u \frac{p_\perp^2}{p^2+p_\perp^2}\right)^2 updp\,d\varphi$$

and the integration with respect to p can, in fact, be carried out from 0 to ∞ since the integral is finite ;

we find

$$w_{zz} = 4\pi\left(\frac{e_\alpha e_\beta}{m_\alpha u}\right)^2$$

Since this is λ times smaller than w_{xx} or w_{yy} it may be set to zero . We can now express w_k and $w_{k\ell}$ as a vector or tensor respectively. In fact,

$$w_k = -\lambda \frac{1+\frac{m_\alpha}{m_\beta}}{4\pi u^2}\left(\frac{4\pi e_\alpha e_\beta}{m_\alpha}\right)^2 \frac{u_k}{u} \tag{20}$$

$$w_{k\ell} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix} - \begin{pmatrix} 0, & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix}$$

$$= A\left(\delta_{k\ell} - \frac{u_k u_\ell}{u^2}\right) \tag{21}$$

where

(22) $$A = (\lambda - \frac{1}{2}) \frac{1}{4\pi u} \left(\frac{4\pi e_{\alpha} e_{\beta}}{m_{\alpha}} \right)^2$$

Finally

(23) $$< \Delta v_k > = - (1 + \frac{m_{\alpha}}{m_{\beta}}) Q_{\alpha\beta} \int \frac{u_k}{u^3} f_{\beta}(\underline{v}') \, d\underline{v}'$$

(24) $$< \Delta v_k \Delta v_{\ell} > = \frac{Q_{\alpha\beta}}{4\pi} \int \left(\frac{\delta_{k\ell}}{u} - \frac{u_k u_{\ell}}{u^3} \right) f_{\beta}(\underline{v}') \, d\underline{v}'$$

where $\underline{u} = \underline{v} - \underline{v}'$

(25) $$Q_{\alpha\beta} = \lambda \left(\frac{4\pi e_{\alpha} e_{\beta}}{m_{\alpha}} \right)^2 .$$

It can be shown that the third and higher diffusion coefficients $<\Delta v_k \Delta v_{\ell} \Delta v_m ..>$ are smaller than the first two diffusion coeffents by a factor of λ. This means that the motion of Coulomb particles can be visualized as a diffusion in velocity space. The approximation in which only the first two diffusion coefficients are considered is called the Fokker- Planck approximation.

7. Justification of the assumption of binary encounters in the theory.

The assumption is certainly justified for short-range force. If the interaction range d(effective diameter of the molecule) is much smaller thatn the mean distance between the particles, $n^{-1/3}$, where n is the density of the gas, the sphere of action, of volume $\sim d^3$, will contain only a small number of particles N_d, that is

$$N_d = nd^3 << 1.$$

Under these conditions the probability of multiple collisions, involving three or more particles simultaneously, is very small. A description in terms of binary collisions is adequate.

Coulomb forces acting between particles of a plasma are not short-range forces. The potential energy between two such charges e_1 and e_2 is

$$\frac{e_1 e_2}{r} \exp\left(-\frac{r}{D}\right) \tag{26}$$

where r is the distance apart of the charges. Thus the interaction between them extends at least as far as the Debye distance D, and for conditions in which we are interested $D \gg n^{-1/3}$ and the sphere of action contains many particles, i.e.,

$$N_D = nD^3 \gg 1 \tag{27}$$

In this case a given particle will interact simultaneously with many particles and the results derived earlier on the basis of binary collisions is suspect. A rigorous analysis shows that the formulae derived yield logarithmic accuracy. However, a non-rigorous, but plausible discussion can be given along the following lines.

Let us consider a test particle moving through the plasma and suppose that it is so massive that its velocity can be treated as constant. Draw a cylinder of radius p with the trajectory as axis (Fig. 5)

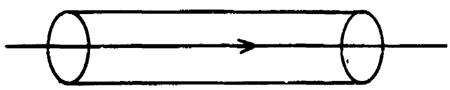

FIG. 5.

Collisions of the test particles with field particles for which $p >> n^{-1/3}$ will be many-body collisons. Those characterized by immact parameters $p << n^{-1/3}$ are binary collisions . We shall show that the method used to treat binary collisions need not be restricted to collisions with impact parameters $p << n^{-1/3}$, but can be extended to parameters $>> n^{-1/3}$.

Now, when $r << D$, the potential energy between the charges is simply $e_1 e_2/r$ so that the presence of other particles has no effect on the interaction between two particles separated by a distance smaller than D. Thus results derived on the hypothesis of binary collisions apply for all impact parameters smaller than the Debye radius, i.e., $p << D$. Because $D >> n^{-1/3}$, in the present case the collisions can be regarded as binary interactions even when $p >> n^{-1/3}$ as long as $p << D$. Accordingly, even if $p \sim D$, the difference between the exact interaction formulae which takes account of other particles, and a pure Coulomb interaction, is small (by a factor of order 1) . Thus, cutting off the Coulomb interaction for the impact parameter $p = D$ provides an approximate method of taking into account the effect of multiple collisions for which $p >> n^{-1/3}$.

8. Diffusion in velocity space.

From a microscopic point of view , the change of spatial coordinates of a particle during a collision can always be neglected . Hence, as far as the spatial part of the phase space is concerned, the motion of a particle corresponds to a continuous point to point variation.

On the other hand, collisions have a marked effect on the continuity of motion in the velocity space. The velocity can be changed abruptly by a single near collision, essential in a vanishingly small time interval. Hence, a particular velocity point $\underline{v}$ in a cloud of particles in velocity space can be 'annihilated' by a collision and 'recreated' at some remote point without passing through intermediate points in the velocity space. Thus in general the effect of collisions cannot be expressed in the kinetic theory by introducing a term describing the divergence of flux in velocity space. But this will certainly only be the case for near collisions in which the velocity of the particle is changed abruptly. In the case of coulomb forces, the change in velocity, characterised by the quantities $<\Delta v_k>$ and $<\Delta v_k \Delta v_\ell>$ is due to the

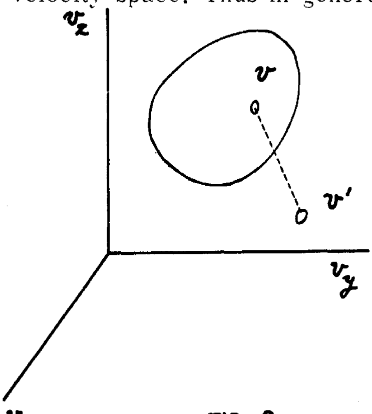

FIG. 6

effect of remote interactions and the changes in velocity are small. For example, if $\lambda = 15$, then the relative change in particle velocity

$$\frac{|\Delta \underline{v}|}{v} = \frac{p_\perp}{p} = e^{-\lambda} \frac{D}{p} \sim 10^{-6} \left(\frac{D}{p}\right),$$

and so very small.

If these interactions are referred in velocity space, the whole process may be regarded as a form of diffusion. The motion can be regarded as nearly continuous.

9. Calculation of the diffusion coefficient for a Maxwellian distribution of velocities.

The expressions (23) and (24) may be expressed more conveniently by introducing the super-potentials. In fact, since

$$u = \sqrt{(\underline{v}_k - \underline{v}'_k)(\underline{v}_k - \underline{v}'_k)}$$

we have

$$\frac{\partial}{\partial v_k} \frac{1}{u} = - \frac{u_k}{u^3}$$

$$\frac{\partial^2}{\partial v_k \partial v_\ell} u = \frac{\partial}{\partial v_k} \frac{1}{2} \frac{2}{u} (v_\ell - v'_\ell) = \frac{\partial}{\partial v_k} \frac{v_\ell - v'_\ell}{u} = \frac{\partial}{\partial v_k} \frac{u_\ell}{u}$$

$$= - \frac{u_k u_\ell}{u^3} + \delta_{k\ell} \frac{1}{u}$$

Hence (23) and (24) can be written

(28) $$< \Delta v_k > = - (1 + \frac{m_\alpha}{m_\beta}) Q_{\alpha\beta} \frac{\partial \varphi_\beta}{\partial v_k}$$

(29) $$< \Delta v_k \Delta v_\ell > = - 2 Q_{\alpha\beta} \frac{\partial^2 \psi_\beta}{\partial v_k \partial v_\ell}$$

where

(30) $$\varphi_\beta = - \frac{1}{4\pi} \int \frac{f_\beta(\underline{v}') \, d\underline{v}'}{|\underline{v} - \underline{v}'|}$$

(31) $$\psi_\beta = - \frac{1}{8\pi} \int |\underline{v} - v'| f_\beta(\underline{v}') \, d\underline{v}'$$

which have been termed 'super-potentials' by Rosenbluth et al. For a Maxwellian distribution function

$$f(\underline{v}') = n(\frac{m}{2\pi n T})^{3/2} e^{-\frac{m v'^2}{2kT}}$$

Chandrasekhar found that

(32) $$< \Delta v_{\parallel} > = - \frac{1}{2} n_\beta Q_{\alpha\beta} (1 + \frac{m_\alpha}{m_\beta}) G(\frac{m_\beta v}{2kT_\beta})$$

$$<\Delta v_{\perp}^2> = <\Delta v_{xx}^2> + <\Delta v_{yy}^2>$$

(33) $$= \frac{1}{2v} n_\beta \, Q_{\alpha\beta} \left\{ \Phi\left(\frac{m_\beta v}{2kT_\beta}\right) - G\left(\frac{m_\beta v}{2kT_\beta}\right) \right.$$

(34) where $$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$$

is the usual error function and

(35) $$G(x) = \frac{\Phi(x) - x\Phi'(x)}{2x^2}$$

Values of G and $\Phi - G$ are given by Spitzer and others.

10. Relaxion times. (Collision interval)

The term "relaxation time" is used to denote the time in which collisions will alter the original velocity distribution; or again , the time that the ions and electrons in a gas will attain, through collisions, a Maxwellian distribution.

Various relaxation times can be defined ; the time between collisions (collision interval or the reciprocal of the collision frequency) may be defined as the time in which small deflections will deflect test particles through 90° . More precisely, if τ_D is the 'deflection time', we have

(36) $$<\Delta v_{\perp}^2> \tau_D = v^2$$

Substituting from (33) we find

(37) $$\tau_D = \frac{2 v^3}{n_\beta Q_{\alpha\beta} (\Phi_\beta - G_\beta)}$$

An energy exchange time τ_E can likewise be defined by the relation

(38) $$<\Delta E^2> \tau_E = E^2 ;$$

the change of energy

(39) $$\Delta E = \frac{1}{2} m(2v\Delta v_{11} + \Delta v^2_{11} + \Delta v^2_{\perp})$$

If only dominant terms are required

$$<\Delta E^2> = m^2 v^2 <\Delta v^2_{11}>$$

and (38) gives

(40) $$\tau_E = \frac{v^3}{4 n_\beta Q_{\alpha\beta} G_\beta}$$

An important special case is that of a group of ions, or a group of electrons, interacting amongst themselves. If we consider such a group whose velocity has the root mean square value for the group , then $\left(\frac{mv^2}{2nT}\right)^{\frac{1}{2}} = 1.225.$

In this case we find that $\tau_D / \tau_E = 1{,}14$ so that $\tau_D \sim \tau_E$ and is a measure of both the time required to reduce substantially any anisotropy in the velocity distribution function and the time for the kinetic energies to approach a Maxwellian distribution. We shall call this particular value of τ_D the 'self-collision interval' for a group of particles and will be denoted by τ_c From (37) we have

(41) $$\tau_c = \frac{m^{\frac{1}{2}} (3\ k\ T)^{3/2}}{5.71\ \pi n e^4\ Z^4 \log_e \lambda}$$

where T is in degrees K, m is the mass of a typical particle of the group. It may be written Am_H where m_H is the mass of a proton . For electrons, $A = \frac{1}{1825}$ so that the self-collision time for electrons is $\frac{1}{43}$ that for protons, provided the ions and electrons have the same temperatures.

We consider next the approach to equilibrium of a two component plasma; to fix our ideas we consider the case when the constituents are ions and electrons. There are three stages involved in the process. First, collisions between ions and electrons lead to an isotropic velocity distribution of electrons, and the same time collisions between electrons themselves establishes a Maxwellian distribution. Secondly, collisions between the ions themselves establishes an isotropic velocity distribution amongst the ions. Thirdly, the ions and electrons which have already attained Maxwellian distribution, but possibly at different temperatures T_i and T_e, will be brought to the same temperature by collisions between the ions and electrons.

To consider the last process we require the equation of energy $(\mathcal{E}_\alpha = \frac{1}{2} m_\alpha v^2)$

(42) $$\frac{d\mathcal{E}_\alpha}{dt} = \frac{1}{2} m_\alpha \frac{d}{dt} \overline{v_i v_i} = m_\alpha \left(\frac{1}{2} \langle \Delta v_i \Delta v_i \rangle + v_i \langle \Delta v_i \rangle\right)$$

using

(43) $$\overline{\Delta v_i \Delta v_i} = \overline{v_i v_i} - \bar{v}_i \bar{v}_i$$

Using (28) and (29) we find

(44) $$\frac{d\mathcal{E}_\alpha}{dt} = - m_\alpha Q_{\alpha\beta} \left[\varphi_\beta + \left(1 + \frac{m_\alpha}{m_\beta}\right) \underline{v} \cdot \nabla \varphi_\beta \right]$$

since

(45) $$\langle \Delta v_i \Delta v_i \rangle = - 2Q_{\alpha\beta} \nabla^2 \psi_\beta = - 2Q_{\alpha\beta} \varphi_\beta$$

Since the distribution of velocities are Maxwellian, this may be rewritten

(46) $$\frac{d\mathcal{E}_\alpha}{dt} = -\frac{2\mathcal{E}_\alpha}{\tau_{\alpha\beta}(\mathcal{E}_\alpha)}\left\{\frac{m_\alpha}{m_\beta}\mu(x_\beta) - \mu'(x_\beta)\right\}$$

where $x_\beta = \frac{1}{2}\frac{m_\beta v^2}{kT_\beta} = \frac{m_\beta}{m_\alpha}\frac{\mathcal{E}_\alpha}{kT_\beta}$, and $\mu(x) = \Phi(x) - x\,\Phi'(x)$

Also

(47) $$\tau_{\alpha\beta}(\mathcal{E}_\alpha) = \frac{4\pi v^3}{n_\beta Q_{\alpha\beta}} = \frac{(\frac{1}{2}m_\alpha)^{\frac{1}{2}}\mathcal{E}_\alpha^{3/2}}{\pi e_\alpha^2 e_\beta^2 n_\beta \log\lambda}$$

where $\mathcal{E}_\alpha = k(T_\alpha + \frac{m_\alpha}{m_\beta}T_\beta)$. After some algebra, (46) can be reduced to

(48) $$\frac{dT_\alpha}{dt} = \frac{T_\beta - T_\alpha}{\tau^*_{\alpha\beta}}$$

where

(49) $$\tau^*_{\alpha\beta} = \frac{3\, m_\alpha m_\beta k^{3/2}}{8(2\pi)^{\frac{1}{2}} n_\beta e_\alpha^2 e_\beta^2 \log\lambda}\left(\frac{T_\alpha}{m_\alpha} + \frac{T_\beta}{m_\beta}\right)^{3/2}$$

It is easily verified that

$$\tau^*_{ee} : \tau^*_{ii} : \tau^*_{ei} : \tau^*_{ie} = 1 : \sqrt{\frac{M}{m}} : \frac{M}{m} : \frac{M}{m}$$

where $T_\alpha \sim T_\beta$ and where M is the mass of the ion and m the electronic mass.

Equation (48) was first given by Spitzer ; it shows that if the mean square relative velocity, which is $\propto \left(\frac{T_\alpha}{m_\alpha} + \frac{T_\beta}{m_\beta}\right)$, does not change appreciably, , $\tau^*_{\alpha\beta}$ is nearly constant and departure from equipartitions decrease exponentially.

11. Relaxation towards the steady state

The solution of Boltzmann's equation for non-uniform gases is found by successive approximation. We write

$$f = f_o(1 + \mathcal{E}),$$

where f_o is the Maxwellian distribution function and $\mathcal{E}$ is small compared with unity. We have seen that in a plasma of two constituents each constituent will approach its Maxwellian distribution in a time equal to the relaxation time $\tau^*_{\alpha\alpha}$ and the two constituents will attain equal temperatures in a relaxation time $\tau^*_{\alpha\beta}$. As a first approximation, therefore, we can take the collision term C_α to be of the form

$$-\frac{f - f_o}{\tau^*} \tag{50}$$

so that if f is the distribution function at time $t = 0$ and f_o the Maxwellian distribution function, then departure from a Maxwellian state $f - f_o \rightarrow 0$ with time as $e^{-t/\tau}$.

12. Equations of continuity and motion for a fully ionized gas

We consider the plasma to be a mixture of positive ions (i) and electrons (e) and denote their number densities by n_i and n_e, and their velocities by $\underline{v}_i$ and $\underline{v}_e$ respectively. Then

$$n_i = \int f_i \, d\underline{v}_i, \qquad n_e = \int f_e \, d\underline{v}_e \tag{51}$$

where f_i and f_e denote the velocity distribution functions for the ions and electrons respectively and $d\underline{v}_i$ and $d\underline{v}_e$ denote an element of volume in the velocity space for ions and electrons, respectively. Denoting their masses by m_i and m_e and the densities of the ion and electron gas by ρ_i and ρ_e respectively, we have

$$\rho_i = n_i m_i, \qquad \rho_e = n_e m_e \tag{52}$$

Denote by $\bar{\underline{v}}_i$ and $\bar{\underline{v}}_e$ the mean velocities of the ion and electron gas in a volume element of the plasma, then

(53) $$n_i \bar{\underline{v}}_i = \int \underline{v}_i f_i \, d\underline{v}_i, \qquad n_e \bar{\underline{v}}_e = \int \underline{v}_e f_e \, d\underline{v}_e$$

It is convenient to introduce the total number density n_o and total mass density ρ_o, defined as $n_o = n_i + n_e$

(54) $$\rho_o = \rho_i + \rho_e$$

and the mean velocity $\underline{v}_o$ of the plasma element defined by

(55) $$\rho_o \underline{v}_o = \rho_i \bar{\underline{v}}_i + \rho_e \bar{\underline{v}}_e$$

Let $\underline{V}_i$ and $\underline{V}_e$ be the peculiar or thermal velocities of the ions and electrons, respectively, defined by

(56) $$\underline{V}_i = \underline{v}_i - \underline{v}_o, \qquad \underline{V}_e = \underline{v}_e - \underline{v}_o$$

Then it follows from (55) that

(57) $$\rho_i \bar{\underline{V}}_i + \rho_e \bar{\underline{V}}_e = 0$$

The partial pressure for the ion and electron gases, and total pressures defined in a frame of reference moving with the mean velocity $\underline{v}_o$ are respectively given by

(58) $$p_i = \rho_i \overline{\underline{V}_i \underline{V}_i}, \qquad p_e = \rho_e \overline{\underline{V}_e \underline{V}_e}, \qquad p_o = p_i + p_e$$

The hydrostatic partial pressures for ions and electrons are defined by

(59) $$p_i = \frac{1}{3} \rho_i \overline{V_i^2}, \qquad p_e = \frac{1}{3} \rho_e \overline{V_e^2}$$

and the corresponding mean kinetic temperature by

(60) $$p_i = k n_i T_i, \qquad p_e = k n_e T_e$$

Boltzmann's equation for the two distribution functions f_i and f_e are

(61) $$\frac{\partial f_\alpha}{\partial t} + (\underline{v}_\alpha \cdot \nabla) f_\alpha + (\underline{F}_\alpha \cdot \nabla_{\underline{v}_\alpha}) f_\alpha = C_\alpha, \quad \alpha = i, e$$

where $m_i \underline{F}_i$ and $m_e \underline{F}_e$ are the forces acting on an ion and electron respectively. If these are produced by an electric field $\underline{E}$ and magnetic field $\underline{B}$, then

(62) $$\underline{F}_i = \frac{e_i}{m_i} (\underline{E} + \underline{v}_i \times \underline{B}), \quad \underline{F}_e = \frac{e_e}{m_e} (\underline{E} + \underline{v}_e \times \underline{B})$$

where e_i and e_e are the charges carried by an ion and electron respectively.

We next form the moment equations; if $\varphi_\alpha(\underline{v}_\alpha)$ be any function of molecular properties for the constituent α of the plasma, then by multiplying equation (61) by φ_α, integrating partially and remembering that

(63) $$n_\alpha \overline{\varphi}_\alpha = \int \varphi_\alpha f_\alpha \, d\underline{v}_\alpha$$

we find

(64) $$\frac{\partial (n_\alpha \overline{\varphi}_\alpha)}{\partial t} + \nabla \cdot (n_\alpha \overline{\varphi_\alpha \underline{v}_\alpha}) - n_\alpha \underline{F}_\alpha \cdot \overline{\nabla_{v_\alpha} \varphi_\alpha} = \int \varphi_\alpha C_\alpha \, d\underline{v}_\alpha$$

The right-hand side represents the change of the mean value of φ_α due to collisions. This vanishes if $\varphi_\alpha = 1$ and (64) gives

(65) $$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \underline{v}_o) + \nabla \cdot (n_\alpha \overline{\underline{V}}_\alpha) = \frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \overline{\underline{v}}_\alpha) = 0$$

which is the equation of continuity for the component α. Multiplying the equations of continuity for the ions and electrons (65) by m_i and