I. N. Herstein (Ed.)

CIME Summer Schools

Some Aspects of Ring Theory

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FONDAZIONE CIME ROBERTO CONTI

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Some Aspects of Ring Theory

Lectures given at a Summer School of the
Centro Internazionale Matematico Estivo (C.I.M.E.),
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"SOME ASPECTS OF RING THEORY"

Coordinatore : Prof. I.E. HERSTEIN

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C.I.M.E.)

S.A. AMITSUR

ASSOCIATIVE RINGS WITH IDENTITIES

Corso tenuto a Varenna (Como) dal 23 al 31 agosto

1965

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

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S. A. AMITSUR

ASSOCIATIVE RINGS WITH IDENTITIES

Contents:

ASSOCIATIVE RINGS WITH IDENTITIES

by

S.A Amitsur

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The lectures given in the 1965 Summer meeting of the C.I.M.E. have been an attempt to summarize and survey the development of the theory of polynomial identities since they first appeared in a paper by Dehn (19229 on Desarguian Geometriés till their recent application to Geometry (1965) - giving an almost complete solution to the problem which arose from the paper of Dehn on Desarguian Non-Pappian Geometries.

But the survey is for from be being complete; applications to group representations, Jacobsons' rings the Kurosh problem and other aspects of the theory are missing in particular, it lacks completely-references. Some of the results appear in the book "Structure of Ring" by N. Jacobson and in the Lecture Notes on rings given by I.N. Herstein at the University of Chicago . Other results appear in various papers by Amitsur , Herstein, Kaplansky, Levitzki , Posner, Shyrshov and others Many recent extensions, in particular the results on polynomial identities with coefficients in arbitrary domains with appear in forthcoming papers by the author.

The proofs, as for as they were given in these notes are just outlines of proofs and were made only indicate their basic ideas. They are far from completeness and the rigor required .

1. Notations and elementary remarks :

Let R be an arbitrary (always associative) ring and Ω a ring of operators of R into a two-sided R-module T satisfying :

$$\omega(r_1 + r_2) = \omega r_1 + \omega r_2 \qquad \omega \in \Omega, \ r_i \in R .$$

$$(\omega r_1) r_2 = r_1 (\omega r_2) = \omega(r_1 r_2)$$

$$(\omega_1 + \omega_2)r = \omega_1 r + \omega_2 r$$

and $\Omega[x] = \Omega[x_1, x_2 \ldots]$ be the free ring generated by an infinithe set of non-commutative indeterminates $\{x_i\}$ over Ω. Every polynomial $p[x] \in \Omega[x]$ can be uniquely expressed in the form $p[x] = \sum \omega_{(i)} x_{i_1} \ldots x_{i_n}$ where the monomials $x_{i_1} \ldots x_{i_n}$ are all different

Definition : A ring R is said to satisfy an identity $p[x_1, \ldots x_n] = 0$ if for every substitution $x_i = r_i \in R$, $p[r_1, \ldots, r_n] = 0$ (in T). If $T=R$; this is equivalent that for every homomorphism $\varphi : \Omega[x] \to R$, $\varphi(p) = 0$

Examples :

1) $nx = 0$ (n an integer) is satisfied by all rings of characteristic n.

2) The commutative rings satisfy the identity $x_1 x_2 - x_2 x_1 = 0$.

3) A ring R which is a K-algebra, and a finitely generated K-module, i.e. $R = \sum_{i=1}^{n} K r_i$ - is a ring which satisfies a polynomial identity , which is a straightforward extension of the commutative law. Namely the following .

Let $S_n[x_1, \ldots, x_n] = \sum \pm x_{i_1} \ldots x_{i_n}$, where the sum ranges over all permutation $(i_1, \ldots, i_n)$ of the n letters $1, 2, \ldots, n$

and the sign is + for even permutations and - for odd permutations.

In fact a ring R will be said to be of <u>commutativity rank n</u> if R satisfies $S_n[x_1, \dots, x_n] = 0$ and does not satisfy $S_{n-1}[x_1, \dots, x_{n-1}] = 0$ (Put $S_1[x_1] = x_1$) we shall refer to the polynomials $S_n[x]$ as the <u>standard - polynomials.</u> They have the same properties as determinants, i.e.

i) Each is homogeneous and multilinear

ii) $S_n[x_{j_1}, \dots, x_{j_n}] = \pm S_n[x_1, \dots, x_n]$ if $(j_1, \dots, j_n)$ is a permutation of $(1, 2, \dots, n)$.

iii) $S_n[r_1, \dots, r_n] = 0$ if two of the r_i are equal.

Form these properties we can conclude that

iv) $S_n \quad a_1, \dots, a_{n+1} \quad = 0$ if each a_i can be expressed as a linear combination $a_i = \quad k_{i_j} r_j$ of n elements $r_1, \dots, r_n$ $(k_{i_j} \quad K$ - K - the commutative ring).

Indeed, $S_{n+1}[a_1, \dots, a_{n+1}] = \sum p_{(i)} \quad S_{n+1}[r_{i_1}, \dots, r_{i_{n+1}}]$

$p_{(i)} \in K$, since $S_{n+1}[x]$ is multilinear and homogeneous,

but each $S_{n+1}[r_{i_1}, \dots, r_{i_{n+1}}] = 0$ for some of the r_i must be equal.

Hence - if R is of the example (3) and it is generated by n elements that it will satisfy $S_{n+1}[x] = 0$. So, it has a rank of commutativity' n+1.

An interesting example of these rings are the matrix rings F_n over a field F. From the previous remarks we conclude that F_n satisfies $S_{n^2+1}[x] = 0$. Let $\rho(n)$ be the minimal integer so that F_n satisfies $S_{\rho(n)}[x] = 0$, then our result shows that $\rho(n) \leq n^2 + 1$. On the other hand, it is to verify that $\rho(n) \geq \rho(n-1) + 2$

and since $\rho(1) = 2$, we have $\rho(n) \geq 2n$. In fact, we shall see that $\rho(n) = 2n$.

The problems which arise in relation with identities are of two types: 1) Given a ring R - to determine the set I(R) of all identities of R, and 2) Given an identity- what are the structure properties of the rings R which satisfy this identity.

Some elementary observations are :

A) The set of identities I(R) is a T-ideal in the ring $\Omega[x]$. That is : if p, q $\in$ I(R) and $g \in \Omega[x]$ then p-q $\in$ I(R) and so are also pg and gp . Furthermore, if $p[x_1, \dots, x_n] \in \Omega[x]$ and $\tau : x_i \to t[x]$ is an endomorphism of $\Omega[x]$ then $p^{\tau} = p[t_1[x], \dots, t_n[x]]$ is also an identity for R.

B) If $p[x_i, \dots, x_n] = 0$ is an identity for R , then by replacing x_1 by x+y , we get $p[x+y, \dots, x_n] - p[x, x_2, \dots, x_n] - p[y, x_2, \dots, x_n] = q[x, y, x_2, \dots x_n]$ which also an identity for R . Generally, for arbitrary rings Ω, the new identity q, which is of lower height, may produce less information with respect to structure of R. But the following, which in obtained by repeating this procedure is interesting

<u>Lemma 1.1</u> If R satisfies $p[x] = 0$ and $\omega x_{i_1} \dots x_{i_2}$ is a monomial which appears in $p[x]$, then R satisfies also an identity of the form $p_r[x] + p_{r+1}[x] + \dots + p_d[x]$, where $p_i[x]$ are homogeneous of degree i and have the same coefficients as the i-th homogeneous part of $p[x]$, and $p_r[x] = \omega x_1 x_2 \dots x_r$ is also multilinear.

In particular, if r = d in the degree of $p[x]$ - it follows that R satisfies a multilinear homogeneous identity

C) If R satisfies an identity $p[x] = p_0[x] + \ldots + p_n[x]$, and each p_j homogeneous of degree j in an indeterminate x_i, then R also satisfies an identity $\left[\prod_{i \neq j} (\omega_i - \omega_j)\right] p_\lambda[x] = 0$ for any $n+1$ elements $\omega_1, \ldots, \omega_{n+1} \in \Omega$. If Ω is an infinite field then $\prod(\omega_i - \omega_j) \neq 0$ and, therefore each $p_\lambda[x]$ will also be an identity for R, i.e. the ideal $I(R)$ is homogeneous.

D) If R satisfies an identity in R (i.e. $T=R$) then every subring and every homomorphic image of R satisfy the same identity.

For every commutative ring K, $R \otimes_Z K$ will satisfy the multilinear identities of R, and in fact, if R satisfies $p[x] = 0$ then $R \otimes K$ will satisfy an identity of the form $\omega\, p[x] = 0$ where ω is some element of Ω, and in general can be taken of the form $\left[\prod_{i \neq j} (\omega_i - \omega_j)\right]^r$.

Problem : If R, S satisfies identities -does $R \otimes S$ satisfy an identity? It seems that this in true if one of the rings does not have nilpotent ideals

2. Nil semi-group of rings with polynomial identities.

Let $M \subseteq R$ be a multiplicative semi-group of elements of the ring R, and M_0 be a fixed two-sided ideal in M. We define the lower Radical of M modulo M_0 :

Let $N_0(M) = M_0$, and define the sequence $N_\lambda(M)$ of ideals by induction on λ :

For limit ordinals λ : $N_\lambda(M) = \bigcup_{\nu < \lambda} N_\nu(M)$

For $\lambda = \rho + 1$: $N_{\rho+1}(\ )$ is the union of all ideals P of M containing M_0 which are nilpotent modulo $N_\rho(M)$.

Then there exist an ordinal τ such that $N_\tau(\mathcal{M}) = N_{\tau+1}(\mathcal{M})$ and this ideal is the Lower Radical $L(\mathcal{M})$ of $\mathcal{M}$ modulo M_0. It is also characterised as the minimal ideal P containing $\mathcal{M}_0$ and such that $\mathcal{M}$ does not contain nilpotent ideals mod P, except P itself.

The ideal $L(\mathcal{M})$ satisfies the following lemma :

Lemma 2.1. 1) If $\mathcal{M}/L(\mathcal{M}) \neq 0$ and nil (i.e. contains only nilpotent elements) - then there exist $m_1, m_2, \ldots$ a sequence of elements in $\mathcal{M}$ such that $m_1 m_2 \ldots m_k \notin L(M)$ for every k, but every other product of the m_i's belongs to $L(M)$. 2) $L(M)$ is locally nilpotent mod. $\mathcal{M}_0$.

(Proof omitted)

We use this lemma to obtain the main properties of nil semi-groups of rings with polynomial identities :

Theorem 2.2. Let R satisfy a polynomial identity $p[x] = 0$ of degree d , and $\mathcal{M}$ be a semi-group in R and $\mathcal{M}_0$ the ideal in R which is annihilated by all coefficients of R. Then

i) If $\mathcal{M}$ is nilpotent modulo $\mathcal{M}_0$ of index n then $\mathcal{M}^{\left[\frac{d}{2}\right]}$ generates a nilpotent ideal in R modulo $\mathcal{M}_0$ of index $f(n,d) \leq \left((d+1)^{\,n-\left[\frac{d}{2}\right]} \right)$.

ii) If $\mathcal{M}$ is nil mod $\mathcal{M}_0$, then $M^{\left[\frac{d}{2}\right]} \subseteq N_1(R; M_0) =$ the sum of all nilpotent ideals in R mod $\mathcal{M}_0$.

iii) The result (i i) is best possible.

We shall review the proof only for the case R satisfies a polynomial identity $p[x] = x_1 x_2 \ldots x_d + \ldots\ldots$ which is homogeneous and multilinear (in this case $\mathcal{M}_0 = 0$) :

Let R^* be the ring with a unit adjoined to R, and say

$M^{m} = 0$ If $t > \left[\frac{d}{2}\right]$, and $R^{*}M^{t}R^{*}$ is nilpotent of index $\mu = \mu(t)$, then consider the sets :

$$T_1 = M^{t-1}R^{*} \qquad T_3 = M^{t-2}R^{*}M, \; \dots, \; T_{2j-1} = M^{t-j}R^{*}M^{j-1}$$
$$T_2 = M^{t-1}R^{*}M \qquad T_4 = M^{t-2}R^{*}M^2, \; \dots, \; T_{2j} = M^{t-j}R^{*}M^{j}$$

By taking $a_i \in T_i$, then $a_i a_j \in R^{*}M^{t}R^{*}$ if $j < i$; furthermore if the a_i range over the elements of T_i - then the products $a_1 a_2 \dots a_r$ will range on set of generators of the additive set $(M^{t-1}R^{*})^{r}M^{j}$, where $r = 2j$ or $= 2j-1$. Writing the identity in the form : $(*)$ $\quad x_1 x_2 \dots x_d = -\sum \omega_{(i)} x_{i_1} \dots x_{i_d}$ and setting $x_i = a_i$ we get that $a_1 a_2 \dots a_d \in R^{*}S^{t}R^{*}$. Hence it follows from the previous remarks that $(M^{t-1}R^{*})^{d}M^{j} \subseteq R^{*}M^{t}R^{*}$, and by multiplying both sides by R we get :

$$(R^{*}M^{t-1}R^{*})^{d+1} \subseteq R^{*}M^{t}R^{*}$$

and hence $(R^{*}M^{t-1}R^{*})^{(d+1)\mu} = 0$. Consequently , if $t > \left[\frac{d}{2}\right]$ $R^{*}M^{t-1}R^{*}$ - the ideal generated by M^{t-1} in also nilpotent of index $\mu(t-1) \leq (d+1)\mu(t)$.

Now since for $t = n$, $\mu(n) = 1$, and if $n > \left[\frac{d}{2}\right]$, one can use the above procedure to show that

$$\mu\left(\left[\tfrac{d}{2}\right]\right) \leq (d+1)\mu\left(\left[\tfrac{d}{2}\right]+1\right) \leq \dots \leq (d+1)^{\left(n-\left[\frac{d}{2}\right]\right)} \mu(n) = (d+1)^{\left(n-\left[\frac{d}{2}\right]\right)}$$

which shows that $R^{*}M^{\left[\frac{d}{2}\right]}R^{*}$ is nilpotent.

To prove (ii) : let M be nil and $L(M)$ be the Lower Radical of M . By lemma 2.1, if $M \neq L(M)$ then if follows that there exists $m_1, m_2, \dots, m_d, \dots$ such that $m_1 \dots m_d \notin L(M)$ and every other product $m_{j_1} \dots m_{j_d} \in L(M)$. Setting $x_i = m_i$ in the identity (*) we get $m_i m_2 \dots m_d \in \Omega L(M)$ from which one concludes that the product $m_1 \dots m_d \in L(M)$ which is a contradiction. Consequently, $M = L(M)$ and thus M is locally nilpotent. This implies that if $a_1, \dots, a_r \in M$ then they generate a nilpotent ideal and the - by the first part of the theorem it will follow that $a_1, \dots, a_{\left[\frac{d}{2}\right]}$ generates a nilpotent ideal and thus $\subseteq N_1(R)$. This being true for every. $r = \left[\frac{d}{2}\right]$ elements yields that $a_1 a_2 \dots a_{\frac{d}{2}}$ $N_1(R)$ which proves (ii) .

The proof of (iii) is omitted.

The previous result has some interesting corollaries :

Corollary 2.3. In F_n every nil multiplicative set M is nilpotent (of index n)

Proof. $N_1(F_n) = 0$ since F_n has no nilpotent ideals and so (i) of theorem 2.3 implies the corollary . Noting that F_n satisfies an identity of degree 2n, it follow that the index of nilpotent is exactly $\left[\frac{2n}{2}\right] = n$.

Corollary 2.4 . The identities of matrix rings F_n are of degree $\geq 2n$; and hence the ring of alla finite matrices does not satisfy an identity.

Indeed, F_n contains nilpotent rings of index n exactly, hence the preceding result yields that $\left[\frac{d}{2}\right] \geq n$, or $d \geq 2n$.

3. Primitive rings with identities :

A ring R is (left) primitive if it has an irreducible faithful left module $V = {}_R V$.

In this case we know that $\mathrm{Hom}_R(V,V) = D$ is a division ring D, and R is a dense ring (in the finite topology) of the ring of linear transformations of V as a vector space over D.

The existence of an identity for primitive ring R introduces a strong restriction on the ring and we can determine the ring completely. The properties of primitive rings with identities is summarized in the following theorem :

<u>Theorem 3.1:</u>

i) A primitive ring satisfies a proper[1] identity of degree d if and only if R is a central simple algebra of dim $n \leq \left[\frac{d}{2}\right]^2$ over its center C.

ii) A minimal identity of R is $S_{2n}[x_1, \ldots, x_{2n}] = 0$.

iii) With the exceptions $R = GF(2)$, $[GF(2)]_2$, the minimal identities of R are linear combinations of the standard polynomials in $2n$ indeterminates.

iv) The identities of R are the same as that of C_n.

Part (i) is known as Kaplansky's theorem (1948). the bound for n was given by Levitzki. Parts (ii) - (iii) are known as the Amitsur - Levitzki identity.

The proof of parts (ii) and (iii) is long and tedious (although the original proof can be shortened) The second part has been shown by Kostant to be connected with other subjects in Mathematics : Representations of the alternative group and Cohomology of the orthogonal group.

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[1] By a proper identity is meant by an identity in R (i.e. $R=T$), and that the elements fo R are not annihilated by all coefficients.

We shall skip the proof of (ii) and (iii) but we shall give a modern version of the proof of (i) :

Let F be a maximal field of D. Consider V as an RF space - then V is also RF irreducible, faithful and $\mathrm{Hom}_{RF}(V, V) = F$ since it must be contained in D and commute with F, but F is maximal commutative.

Now R will satisfy also a proper multilinear identity of degree d, and $R \otimes F$ and, therefore, also RF will satisfy this identity $p[x_1, \ldots, x_k] = 0$. Consider $p[x_1, \ldots, x_k]$ as a function on $\mathrm{Hom}_F(V, V) \supseteq RF$; then $p[x]$ is continuous and vanishes on the dense set RF - hence vanishes on all $\mathrm{Hom}_F(V, V)$. It follows, therefore, by theorem 2.2 that $\mathrm{Hom}_F(V, V)$ cannot contain a nilpotent subring of index $> \left[\frac{d}{2}\right]$. Clearly this requires that $n = (V: F) \leq \left[\frac{d}{2}\right]$. But then $(V: F) < \infty$ and so $RF = \mathrm{Hom}_F(V. V) \cong F_n$.

Furthermore, $(V:F) = (V:D)(D:F)$ - hence $(V;D) = m < \infty$ and consequently $R = \mathrm{Hom}_D(V, V) \cong D_m$. We also get that $(D:F) = r < \infty$, whence $(D:C) = r^2$. Thus $(R:C) = (mr)^2 = (V:F)^2 \leq \left[\frac{d}{2}\right]^2$.

Conversely, if R is central simple over C, and $(R:C) = n^2$ and F is a maximal field then $R \otimes_C F = F_n$ for some splitting field F, which shows that R will satisfy the standard identity $S_{2n}[x_1, \ldots, x_n] = 0$ (by part (ii) .

Finally, if the center C is a finite field then $R = C_n$ by the Wedderburn theorem, and if C is infinite then R and $R \otimes_C F$ will satisfy the same identities, and since the latter is F_n - it follows that the identities of central simple algebras are completely determined by the identities of matrix ring F_n. Now $F_n = C_n \otimes F$ so that R, F_n and C_n have the same identities.

From the above, one deduces immediately that

Corollary 3.1. The Jacobson radical of a ring with "an identity" coincides with the Brown-Mc Coy radical of the ring.

Since the primitive images in this case are simple rings with a unit.

4. Ultra-products and Ultra-powers.

Before proceeding with the structure of rings with identities, we pause to introduce the new tool of ultra-products, and we begin with filters :

Let $\mathfrak{X} = \{\alpha\}$ be a set. A set of subsets $\mathfrak{F} = \{S\}$ is a filter in $\mathfrak{X}$ if it satisfies :

i) $\phi \in \mathfrak{F}$

ii) If $S_1, S_2 \in \mathfrak{F}$ then $S_1 \cap S_2 \in \mathfrak{F}$; and hence every finite intersection of sets in $\mathfrak{F}$ lies in $\mathfrak{F}$

iii) If $S \in F$ and $S \subseteq T \subseteq \mathfrak{X}$ then $T \in \mathfrak{F}$.

If in addition the filter $\mathfrak{F}$ satisfies the property :

iv) For every $T \subseteq \mathfrak{X}$, either $T \in \mathfrak{F}$ or its complement $\mathfrak{X} - T \in \mathfrak{F}$

then we call $\mathfrak{F}$ an ultra-filter. (iv) is also equivalent to the condition that $\mathfrak{F}$ is a maximal filter ; i.e., it cannot be embedded in greater filter.

Given a set $\{S_\rho\}$ of non-empty subsets of $\mathfrak{X}$ which have the finite intersection property, that is :

(v) For every finite number of sets $S_{\rho_1}, S_{\rho_2}, \dots, S_{\rho_n}$ there exists $S_\rho \subset S_{\rho_1} \cap S_{\rho_2} \cap \dots \cap S_{\rho_n}$.

then one can construct a filter containing all S_ρ, and by Zorn's lemma , we can find an ultra-filter containing all S_ρ .

Examples :

1) $\mathfrak{X} = \{n\}$ the set of all integers, then the set of all complements of finite sets is a filter

2) $\mathfrak{X}$ any set, then the set of all subsets of $\mathfrak{X}$ containing a fixed element $\alpha \in \mathfrak{X}$, is an ultra-filter known as the principal ultra-filter containing α, We shall be mainly interested in non-principal ultra filters.

The following example in widely used in the Zaviski-Topology of an affine space.

3) Let $\mathfrak{X} = C^n$ be the n-dim. affine space over an infinite field C, i.e. $\mathfrak{X}$ is all n-tuples $(\xi_1, \dots, \xi_n)$, $\xi_i \in C$.

Let $C[x_1, x_2, \dots, x_n]$ be the ring of polynomials in n <u>commutative indeterminates</u> over C. For every $f[x_1, \dots, x_n] \neq 0$ we denote by $S_f = \{(\xi) \in C^n \,;\, f[\xi] \neq 0\}$; that is the set of all points on which $f[x]$ does not vanish .

Since C is infinite $S_f \neq \emptyset$, and $S_f \cap S_g = S_{fg}$ and also $fg \neq 0$ if both $f \neq 0$ and $g \neq 0$. Thus , the sets $\{S_f\}$ satisfy the finite intersection property and, therefore, they generate a filter which is the one used for the open sets in the Zarishi's topology of $\mathfrak{X}$

We turn now to the ultra products :

Let $\mathcal{R} = \prod_{\alpha \in \mathcal{X}} R_\alpha$ be the complete product of the rings R. (or of algebraic structures: groups, etc.) That is; $\mathcal{R}$ consists of all functions f defined on $\mathcal{X}$ and such that $f(\alpha) \in R_\alpha$. Addition and multiplication is defined component wise.

To each filter $\mathcal{F} = \{S\}$, we make correspond a homomorphic image of $\prod R_\alpha$, and denote it by $\prod R_\alpha / \mathcal{F}$ as follows: Put $f \equiv g \pmod{\mathcal{F}}$ if $\{\alpha \mid f(\alpha) = g(\alpha)\} \in \mathcal{F}$, and in particular a function f is zero mod $\mathcal{F}$ if it vanish on a set of the filter

It is easily seen that this defines a congruence relation in $\prod R_\alpha$, and the set of all functions congruent to zero, is an ideal in $\prod R_\alpha$. Thus the set of all classes will form the ring $\prod R_\alpha / \mathcal{F}$ which is isomorphic with the quotient ring $\prod R_\alpha$ mod the ideal of zero functions mod $\mathcal{F}$.

<u>Definition</u>. If $\mathcal{F}$ is an ultra-filter then $\prod R_\alpha / \mathcal{F}$ is called an ultra-product; and if all $R_\alpha = R$ then $\prod R_\alpha = R^{\mathcal{X}}$ - all functions $f : \mathcal{X} \to R$ and $R^{\mathcal{X}}/\mathcal{F}$ is called an ultra power.

The importance of the ultra-product lies in the fact (proved in Logic) that the ultra-product $\prod R_\alpha / \mathcal{F}$ will satisfy all "elementary statements" which holds in all ring R_α where the index α range over a set in the filter ; and, in particular, the ultra-power $R^{\mathcal{X}}/\mathcal{F}$ has the same "elementary" properties as R.

The proof of this fact is straight forward; and in its application to algebra it is quite often worthwhile to try the proof of the special properties used in the applications.

Here are some examples of elementary properties, preserved under ultra products:

1) If all R_α are (ordered) division rings, then the ultra product $\prod R_\alpha/\mathcal{F}$ is also an (ordered) division ring.

2) If all R_α satisfy an identity $p[x] = 0$, then so does $\prod R_\alpha/\mathcal{F}$.

3) If all R_α are (real) algebraically closed fields, then so $\prod R_\alpha/\mathcal{F}$

For our sake the following examples is interesting :

Theorem 4.1: Let R_α be all primitive rings, with faithful irreducible modules V_α, centralizer $D_\alpha = \mathrm{Hom}_{R_\alpha}(V_\alpha, V_\alpha)$ and centers C_α of D_α. Then :

1) The ultra product $\mathcal{R} = \prod R_\alpha/\mathcal{F}$ is primitive with an irreducible R-module $\mathcal{V} = \prod V_\alpha/\mathcal{F}$; a centralizer (isomorphic-with) $\mathcal{D} = \prod D_\alpha/\mathcal{F}$ whose center is $\mathcal{C} = \prod C_\alpha/\mathcal{F}$

2) Let $S_n = \{\alpha \mid (V_\alpha : D_\alpha) = n\}$; n = 1, 2, ..., and if some $S_n \in \mathcal{F}$ then $(\mathcal{V}:\mathcal{D}) = n$ (The converse is also true)

3) Let $T_n = \{\alpha \mid (R_\alpha : C_\alpha) = n^2\}$ - and if some $T_n \in \mathcal{F}$ then $(\mathcal{R}:\mathcal{C}) = n^2$. (the converse is also true) .

Proof. Clearly $\prod R_\alpha$ acts on $\prod V_\alpha$ pointwise ; i.e. if $\bar{f} \in \prod R_\alpha$ $\bar{v} \in \prod V_\alpha$ then $(\bar{f}\,\bar{v})(\alpha) = \bar{f}(\alpha)\;\bar{v}(\alpha)$.

Now, if $\bar{f} \equiv 0(\mathcal{F})$ then $\{\alpha \mid (\bar{f}\,\bar{v})(\alpha) = 0\} \supset \{\alpha \mid \bar{f}(\alpha) = 0\}$ Hence, $\bar{f}\bar{v} \equiv 0\ (\mathcal{F})$, and by similar methods one verifies that $\prod R_\alpha/\mathcal{F}$ acts on $\prod V_\alpha/\mathcal{F}$.

Next, $\mathcal{V}$ is $\mathcal{R}$-irreducible for let $v \neq 0$ in $\mathcal{V}$, $u \in \mathcal{V}$ and choose $\bar{v}, \bar{u}$ representatives of v and u respectively then, $\{\alpha \mid \bar{v}(\alpha) = 0\} \notin \mathcal{F}$ since $v \neq 0$, and hence its complement, which is $S = \{\alpha \mid \bar{v}(\alpha) \neq 0\} \in \mathcal{F}$ To each $\alpha \in S$ these exist $r_\alpha \in R_\alpha$ such that

$r_\alpha \bar{v}(\alpha) = \bar{u}(\alpha)$ since V_α is R_α -irreducible. Put $\bar{r} \in \prod R_\alpha$ by setting $\bar{r}(\alpha) = r_\alpha$ for $\alpha \in S$ and zero therwise ; then clearly $rv = u$ where r is the class of $\bar{r}$ mod $\mathcal{F}$, since $\{\alpha \mid \bar{r}\bar{v}(\alpha) = u(\alpha)\} \supseteq S \in \mathcal{F}$.

The rest of (1) is proved similarly.

To prove (2), we can either use the fact that the statement that $(V_\alpha : D_\alpha) = n < \infty$ is elementary, and then use the basic property of the ultra-product; or, continue as follows :

Say $S_n \in \mathcal{F}$, and for every $\alpha \in S_n$ choose a basis $v_{\alpha 1}, \ldots, v_{\alpha n}$ of V_α . Consider the elements $\bar{v}_i \in \prod V_\alpha$, defined by: $\bar{v}_i(\alpha) = v_{\alpha i}$ if $\alpha \in S_n$ and $\bar{v}_i(\alpha) = 0$ for $\alpha \notin S_n$. Then the classes v_i represented by $\bar{v}_i$ are a basis of $\mathcal{V}$ over $\mathcal{D}$; indeed, given $v \in \mathcal{V}$ choose $\bar{v} \in \prod V_\alpha$ representing v, then for every $\alpha \in S$ we can write $\bar{v}(\alpha) = \sum v_{\alpha i} d_{\alpha i}$, $d_{\alpha i} \in D_\alpha$. Put $\bar{d}_i(\alpha) = d_{\alpha i}$ for $\alpha \in S_n$ and zero elsewhere ; then if $d_i \in \mathcal{D}$ are represented by elements $\bar{d}_i \in \prod D_\alpha$ we get $\bar{v} \equiv \sum \bar{v}_i \bar{d}_i \pmod{\mathcal{F}}$ i.e. $v = \sum v_i d_i$.

Next we show that the $\{v_i\}$ are $\mathcal{D}$-independent; indeed if $\sum v_i \delta_i = 0$, $\delta_i \in \mathcal{D}$, let $\bar{\delta}_i$ be representatives of δ_i then $A = \{\alpha \mid \sum v_i(\alpha)\bar{\delta}_i(\alpha) = 0\} \in \mathcal{F}$. But then for $\alpha \in S_n \cap A$ the $\bar{v}_i(\alpha) = v_{\alpha i}$ are D_α -independent , hence $\bar{\delta}_i(\alpha) = 0$ for all $\alpha \in S_n \cap A \in \mathcal{F}$. This proves that $\delta_i = 0$, which completes the proof of (2) . The rest of the proof is similar .

We shall be applying the preceding context in the following form :

<u>Theorem 4.2</u> : Let R be a prime ring which is a subring of $\prod R_\alpha$, then R can be embedded in an ultra-product $\prod R_\alpha / \mathcal{F}$.

Proof. For every $r \neq 0$ in R we consider the set $S_r = \{\alpha \mid r(\alpha) \neq 0\}$. By definition $r \neq 0$ implies $S_r \neq \emptyset$. The sets S_r satisfy the finite intersection property, since if $r \neq 0$ then $r R t \neq 0$ as R is prime, and if we choose $rxt \neq 0$ then clearly $S_r \cap S_t \supseteq S_{rxt}$.

This proves the existence of an ultra-filter containing all the sets S_r. Let $\mathcal{F}$ be such an ultra filter and consider the composite map : $R \to \prod R_\alpha \to \prod R_\alpha/\mathcal{F}$, where the first map is the injection of R and the second is the canonical epimorphism. The composite is again a monomorphism and gives the required embedding ; indeed let $r \in R$ and $r \equiv 0 \pmod{\mathcal{F}}$ then $B = \{\alpha \mid r(\alpha) = 0\} \in \mathcal{F}$ and if $r \neq 0$ then $S_r = \{\alpha \mid r(\alpha) \neq 0\} \in \mathcal{F}$ but this will yield that $\phi = B \cap S_r$ which is impossible for a filter.

A simple application of the last result, and of the basic properties of polynomial identities in the following .

Theorem 4.3. The free ring $C[x_1, x_2, \dots]$ can be embedded in a division ring; and if C is ordered then the embedding can be made into an ordered division ring.

Proof: We quote, without proof, the fact that given C there exists a division ring D infinite over its center which contains C, and if C is ordered then D can be found to be also ordered.

Let $\mathcal{X} = \{\varphi\}$ the set of al (C-) homomorphisms of $C[x_1, x_2, \dots]$ into D. Each φ is uniquely determined by the set of values $\varphi(x_i) = a_i$. We can map $C[x_1, x_2, \dots]$ into the functions $D^{\mathcal{X}}$ by setting $p[x](\varphi) = \varphi(p[x]) = p[\varphi(x)]$; namely, we replace the formal polynomial by polynomial functions and this map is a monomorphism, for if $p[x](\varphi) = 0$ for all φ will mean that $p[x] = 0$ vanishes for all $x_i = a_i$, i.e. it is an identity for D. But by the previous section, it follows that infinite dimensional division algebras over their center do not have non trivial identities -hence $C[x]$ can be considered as a subring of $D^{\mathcal{X}}$ and, therefore, by theorem 4.2, it follows that $C[x]$ can be embedded in an ultra product $D^{\mathcal{X}}/\mathcal{F}$ which is a (ordered) division ring, since these properties are invariant under ultra products.

5. Prime and semi-prime rings.

Recall a ring R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$; and R is semi-prime if it does not have nilpotent ideals.

The previous results will be used to prove the following properties of prime rings which satisfy polynomial identities.

Theorem 5.1. Let R be a prime ring which satisfy a 'proper' identity of minimal degree d then :

0) R satisfies a strong ore condition (to be stated in the proof)

1) R has a right and left ring of quotients Q.

2) Q is a finite dimensional central simple algebra with a center C and $RC = Q$

3) $d = 2n$ where $(Q : C) = n^2$

4) Q satisfies the same identities as R, and if $R \neq [GF(q)]_m$ then, R, $Q \otimes K$, $R \otimes K$ satisfy the same identities, for any commutative K.

5) The identities of R are the same as those of C_n; and hence its minimal multilinear identity is $S_{2n}[x_1, x_2, \dots x_{2n}] = 0$.

6) If R is semi-simple, and R_α is a primitive image of R then R_α is a central simple algebra of dim $n_\alpha^2 \leq n^2$ over its center C_α ; furthermore $n = \text{Max } n_\alpha$ where the maximum ranges over all primitive images of R.

Remarks 1) Note that (2) implies that R can actually be embedded in a finite dimensional matrix ring F_n over a field F. Since $Q \otimes_C F \cong F_n$ if F is a splitting field of Q. Thus, the prime rings with identities are subring of matrix rings over fields.

2) The result stated in (6) is almost immediate if R is an algebra over Ω and the coefficient domain Ω is a field. But the proof is slightly more complicated for arbitrary coefficients as it may happen that the given identity $p=0$ for R will become a trivial identity in R_α .

3) The basic results,(1), (4)for prime rings which are algebras over fields and for identities which are multilinear are due to Posner who has proved them by showing that prime rings which satisfy identities satisfy the Goldie-conditions and, therefore, have a ring of quotients. We shall pursue a different way.

First we need the following interesting result :

Lemma 5.2. The exists a (multilinear) identity which holds in every proper subalgebra of a matrix ring F_n, but does not hold in F_n itself. This result is true also for every central simple algebra of dim n^2 its center.

Proof. Let $(x_1^i, \dots, x_{2n-2}^i), y_i$; $i = 1, 2, \dots, n$ be non commutative indeterminates. The identity we need is ;