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"THOERY OF GROUP REPRESENTATIONS AND FOURIER ANALYSIS"

Coordinatore: Prog. F. GHERARDELLI

CENTRO INTERNAZIONALE MATEMATICO ESTIVO

(C. I. M. E.)

Alessandro FIGÀ-TALAMANCA

RADOM FOURIER SERIES ON COMPACT GROUPS

Corso tenuto a Montecatini T. dal 25 giugno al 4 luglio 1970

A. Figà-Talamanca

Introduction

In these lectures I shall present some aspects of the theory of random Fourier series on compact topological groups. The results presented here are taken largely from the papers of S. Helgason [8], A. Figà-Talamanca [4], [5], A. Figà-Talamanca and D. Rider [6], [7] and D. Rider [14]. For the classical case these results are contained, in the treatise of A. Zygmund [19] and in the monograph of J. P. Kahane [11] as well as in the original papers which are quoted there. There is some overlap between the exposition given here and that given in the second volume of [9] by E. Hewitt and K. Ross. However the point of view is different and in some cases, aiming at a less complete exposition of the subject matter, I have been able to shorten some of the proofs.

In Chapter I, I give an account, without proofs, of some of the classical results. In Chapter II I begin with a preliminary section in which I state the results from the general theory of compact groups, that I need in the sequel. In the succeeding sections I present extensions to the noncommutative situation of the results stated in Chapter I. Basically very little knowledge of probability theory is needed to understand these lectures. I do not define the notions of probability space, random variable and independence. In order to treat the case of L^∞ in section II.5

some more sophistication in probabilistic reasonings is required. Chapter I of [11] is a convenient, concise reference for the probability methods needed in the theory of Fourier series.

A. Figà-Talamanca

CHAPTER I

I.1. Littlewood's theorem. Many problems in the theory of Fourier series can be formulated roughly as follows: Suppose that f is a function (a measure, a distribution) defined on $[0,2\pi)$ and let

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx, \qquad n = 0, \pm 1, \dots,$$

be its Fourier coefficients. How are the properties of f reflected in properties of the sequence $\{\hat{f}(n)\}_{n=-\infty}^{+\infty}$ and vice versa?

It is well known, for instance, that $f \in L^2$ if and onlyiif $\Sigma |\hat{f}(n)|^2 < \infty$. This theorem (which is called the Riesz-Fisher theorem) constitutes in a sense a model of characterization of a function space, in terms of the Fourier coefficients of its elements.

It is natural to conjecture that some analogue of this theorem should hold for L^p spaces when $p \neq 2$. As a matter of fact this is not the case, that is: No condition on the moduli $|\hat{f}(n)|$ of the Fonrier coefficients of a function f can be necessary and sufficient for f to be a member of L^p, when $p \neq 2$.

This remarkable result is a consequence of a theorem of J. E. Littlewood [12], proved more than 55 years ago:

DHEOREM I.1.1. Let $\sum_{n=-\infty}^{+\infty} |a_n|^2 < \infty$ and $p > 1$, then there *exist complex numbers* ψ_n *of modulus one, such that*

$$(1.1) \qquad \sum_{n=-\infty}^{+\infty} a_n \psi_n e^{inx} ,$$

represents a function in L^p . *Vice versa if* $\sum_{n=-\infty}^{+\infty} |a_n|^2 = \infty$,
then there exist complex numbers ψ_n *of modulus one, such that* (1.1) *is not the Fourier series of a function in* L^1 .

This theorem is probably the first application (although in implicit form) of random Fourier series to harmonic analysis. Littlewood in his proof never mentions random variables or probability, but many concepts relevant to the theory of random Fourier series are developed in this important paper.

The probabilistic content of Littlewood's theorem became apparent five years later, when R.E.A.C. Paley and A. Zygmund in a series of papers [13] investigated not only the series

$$(1.2) \qquad \sum_{n=-\infty}^{+\infty} a_n \psi_n e^{inx} ,$$

(where the ψ_n are now interpreted as independent random variables uniformly distributed on the unit circle), but also the analogous series,

$$(1.3) \qquad \sum_{n=-\infty}^{+\infty} a_n \varphi_n e^{inx} = \sum_{n=-\infty}^{+\infty} \pm a_n e^{inx} ,$$

where the φ_n are independent random variables with values +1 and -1 and mean zero. It is immaterial whether the φ_n or ψ_n are indexed by the integers or the positive integers.

In the next sections we shall discuss some of the properties of these series which were discovered by Paley and Zygmund and which are in a sense related to the theorem of Littlewood. In section 3 I will present an application of the results of Paley, Zygmund and Littlewood to lacunary series, finally in section 4 I will present the results of P. Billard [1] and [2], on bounded and continuous random Fourier series.

I.2. The case of L^p $(p < \infty)$. Paley and Zygmund give the the following much stronger version of Littlewood's theorem (Theorem I.1.1.)

THEOREM I.2.1. The series (1.2) or (1.3) represent almost surely a function in L^1 if and only if they represent almost surely a function in $\bigcap_{p<\infty} L^p$.

In order to prove this theorem they establish two basic lemmas concerning the independent random variables φ_n and ψ_n . It is convenient to realize φ_n and ψ_n as functions defined on the interval [0,1] , this can be done quite simply as in [11, Chapter I]. For the sake of convenience I will only state the two lemmas of Paley and Zygmund for the functions φ_n . Entirely analogous results hold for the functions ψ_n .

LEMMA I.2.2. _Let_ s _be a positive integer_, $s > 1$, _then_

$$\int_0^1 |\Sigma a_n \varphi_n|^{2s} dt \leq 4^s s^s (\Sigma |a_n|^2)^s .$$

LEMMA I.2.3. _Let_ $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, _let_ $M \subseteq [0,1]$ _be a set of positive measure and_ $\epsilon > 0$. _Then there exists a finite set of indices_ F , _such that_

$$m(M) \sum_{n \notin F} |a_n|^2 \leq (1 + \epsilon) \int_M |\sum_{n \notin F} a_n \varphi_n(t)|^2 dt$$

The first lemma is basically already contained in the work of Littlewood [12], at least for the functions ψ_n . It implies that for series of the type $\Sigma a_n \varphi_n$, all the L^p norms, for $p > 2$, are equivalent to the L^2 norm and specifies a very precise bound for the norm of the embedding of this subspace of L^p into L^2 . From the lemma it also follows easily that actually all the L^p norms $(1 \leq p < \infty)$ are equivalent for the series in question.

The second lemma originates in the work of Zygmund on lacunary series [18]. It says essentially that if a series $\Sigma a_n \varphi_n$ is square summable on a set of positive measure, then it is square summable on $[0,1]$. This lemma is essential to show that if the random Fourier series (1.2) and (1.3) are in L^1 with positive probability, they are actually in L^2 with probability one. It is also used in the proof of Billard's theorems on randomly continuous Fourier series (see section 4).

I will not give proofs of these lemmas here, nor will I show how to obtain Theorem I.2.1 from them. The proofs can be found in Zygmund's treatise [19, v.I, p. 213]. The results which will be proved in the second chapter will actually imply both the lemmas and the theorem.

Lemma I.2.2 can also be used to prove interesting sufficient conditions for the series (1.2) and (1.3) to represent almost surely continuous functions. (The link between L^p norms and continuous functions is provided by the fact that if $f \in L^p$ and $g \in L^q$ with $1/p + 1/g = 1$, then $f * g$, the convolution of f and g, is continuous and $\|f*g\|_\infty \leq \|f\|_p \|g\|_q$). The theorem on randomly continuous functions, also due to Paley and Zygmund is the following:

THEOREM I.2.4. <u>Let</u> $\sum_{n=-\infty}^{+\infty} |a_n|^2 (\log|n|)^{1+\epsilon} < \infty$ <u>for some</u> $\epsilon > 0$ <u>then the series</u>

$$\sum_{n=-\infty}^{+\infty} a_n \varphi_n e^{inx},$$

<u>represents almost surely a continuous function with uniformly convergent Fourier series.</u>

This theorem is a consequence of the following lemma, also due to Paley and Zygmund [13].

LEMMA I.2.5. <u>Let</u> $S_N(x,t) = \sum_{-N}^{N} a_n \varphi_n(t) e^{inx}$. <u>Define</u>

$$M_N(t) = \sup_x |S_N(x,t)| ,$$

Then there exists an absolute constant c such that,

$$\int_0^1 M_N(t)dt \leq c(\log N)^{\frac{1}{2}} \left(\sum_{-N}^{N} |a_n|^2\right)^{\frac{1}{2}} .$$

A proof of Lemma I.2.5 can be obtained directly from Lemma I.2.2. In fact in Chapter II we will give a proof, due to D. Rider [14] of an extension of I.2.5 based on a generalization of I.2.2.

I.3. Applications to lacunary series. A sequence of positive integers $\{n_k\}$ is said to be Hadamard lacunary if, for some $\lambda > 1$, $n_{k+1}/n_k \geq \lambda$. A Fourier series $\Sigma a_n e^{inx}$ is called Hadamard lacunary, or simply lacunary if $a_n = 0$ except for $n = \pm n_k$, where n_k is a Hadamard lacunary sequence. Lacunary series have a number of interesting properties; we are concerned here only with two of these properties:

I) Let $E = \{\pm n_k\}$ and $n_{k+1}/n_k \geq \lambda > 1$. Then there exists a constant B, which only depends on λ, such that

$$\sum_{n\in E} |a_n| \leq B \sup_x \left|\sum_{n\in E} a_n e^{inx}\right| ,$$

for every trigonometric polynomial of the form $\sum_{n\in E} a_n e^{inx}$.

II) Let $E = \{\pm n_k\}$, $n_{k+1}/n_k \geq \lambda > 1$ and $1 \leq p < \infty$. Then there exist positive constants B_p and C_p which depend only on λ and p, such that

$$C_p\left(\sum_{n\in E} |a_n|^2\right)^{\frac{1}{2}} \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|\sum_{n\in E} a_n e^{inx}\right|^p dx\right\}^{1/p} \leq B_p\left(\sum_{n\in E} |a_n|^2\right)^{\frac{1}{2}} .$$

Property II is the analogue, with $\{e^{inx}\}_{n \in E}$ instead of $\{\gamma_n\}$ or $\{\psi_n\}$, of Lemma I.2.2. Also I) expresses a property which is shared by $\{\gamma_n\}$ and $\{\psi_n\}$. Indeed

$$\Sigma|a_n| = \sup_t |\Sigma a_n \psi_n(t)| \leq 2 \sup_t |\Sigma a_n \gamma_n(t)|$$

for all finite sequence of complex numbers $\{a_n\}$.

A proof of II) can be obtained mimicking the proof of Lemma I.2.2. However II) can also be deduced from I) using random Fourier series or more specifically using Lemma I.2.2 and this indirect proof yields more precise estimates of the constants B_p and C_p. This was first noticed by Zygmund who gave the indirect proof in the second volume of his treatise. His reasoning was generalized by W. Rudin [15] and it is Rudin's treatment of this subject that I shall present (cfr. also [16]).

DEFINITION. _Let_ E _be a subset of the integers. Then_ E _is called a Sidon set if every trigonometric polynomial of the form_ $\sum_{n \in E} a_n e^{inx}$ _satisfies, for some_ $B > 0$ _which only depends on_ E ,

$$\sum_{n \in E}|a_n| \leq B \sup_x |\sum_{n \in E} a_n e^{inx}| . \tag{3.1}$$

THEOREM I.3.1. _Let_ E _be a Sidon set and_ B _the constant appearing in the inequality_ (3.1). _Then for trigonometric polynomials of the form_ $\sum_{n \in E} a_n e^{inx}$,

$$(3.2)\quad \Big\| \sum_{n\in E} a_n e^{inx} \Big\|_p \leq B\sqrt{p}\,\Big(\sum_{n\in E} |a_n|^2 \Big)^{\frac{1}{2}}, \quad \text{for } 2 < p < \infty .$$

The proof of this theorem is based on an application of the theory of random Fourier series. Later on in Chapter II, I shall present a proof of an extension of I.3.1. to general compact groups.

One interesting aspect of Rudin's theorem is that no subset of the integers is known which satisfies the conclusion of the theorem and is not a Sidon set. It might very well be that the property expressed by (3.2) characterizes Sidon sets.

I.4. The case of L^∞. Although we spoke already of randomly continuous Fourier series, up to now we have only considered direct applications of the L^p inequalities $(p < \infty)$, expressed by I.2.2 and I.2.3. We shall consider now results which are essentially different from those discussed in the previous sections. These are Billard's results on randomly continuous and randomly bounded Fourier series [1] and [2].

THEOREM I.4.1. *Suppose that*

$$(4.1)\qquad \sum_{n=-\infty}^{+\infty} a_n \varphi_n(t) e^{inx} ,$$

represents with probability one, a function in L^∞. *Then, almost surely, (4.1) represents a continuous function with uniformly convergent Fourier series.*

This result is more recent than those stated in section 2. It contains the solution of a problem posed in the second edition of Zygmund's book [19, v. I, p.220] and makes use of techniques described in the previous section. In a sense Billard's theorem is a "natural" result not for the series (4.1), but for the series of the type

(4.2) $$\sum_{n=-\infty}^{+\infty} a_n \psi_n(t) e^{inx} ,$$

and it is only with considerable effort that one can deduce a result for the series (4.1) from the corresponding result for (4.2).

I.5. <u>More general random Fourier series.</u> It is natural to consider besides the series $\Sigma \varphi_n a_n e^{inx}$ and $\Sigma \psi_n a_n e^{inx}$, also more general series such as

(5.1) $$\sum_{n=-\infty}^{+\infty} X_n e^{inx} ,$$

where X_n are <u>independent complex valued random variables</u>, defined on a probability space Ω . It turns out that a seemingly mild condition on the X_n , namely that X_n and $-X_n$ have the same probability distribution (i.e. the X_n are symmetric), for each n , is sufficient to insure that the substance of the results of the previous sections remain true. This interesting observation was first made by Kahane [10].

His argument coupled with Theorem I.2.1 and Theorem I.4.1 yields the following:

THEOREM I.5.1. Let X_n be independent symmetric random variables, then

1) if (5.1) represents with probability one an integrable function, it represents with probability one a function in $\bigcap_{p<\infty} L^p$

2) if (5.1) represents with probability one an element of L^∞, then it represents with probability one, a continuous function.

I will present a proof of 1) only, based on Theorem I.2.1. Let (5.1) represent an integrable function with probability one, then because of the symmetry of the X_n, the series

$$\sum_{n=-\infty}^{+\infty} \pm X_n e^{inx} ,$$

represents an integrable function with probability one, for every given choice of signs + or - . Thus if we define the random variables $\varphi_n(\omega')$ on a space Ω', to be independent random variables with mean zero and values ± 1, the series

$$\Sigma \varphi_n(\omega')X_n(\omega)e^{inx} \tag{5.2}$$

is in L^1 with Ω-probability one, for every $\omega' \in \Omega$. Let M be the subset of $\Omega \times \Omega'$ consisting of all pairs (ω,ω') for which (5.2) is in L^1. By Fubini's theorem M has measure one therefore the set $P_{\omega} = \{\omega' : (\omega, \omega') \in M\}$ has measure one for a set of $\omega \in \Omega$ which has measure one. Thus by Theorem I.2.1 for a set of $\omega \in \Omega$ of measure one (5.2) is in $\bigcap_{p<\infty} L^p$ with Ω'-probability one. Applying again Fubini's theorem we obtain that at least for one fixed $\omega' \in \Omega'$, (5.2) is in $\bigcap_{p<\infty} L^p$ with Ω-probability one. Finally, since the X_n are symmetric and $\varphi_n(\omega')X_n(\omega) = \pm X_n(\omega)$, we conclude that (5.1) is in $\bigcap_{p<\infty} L^p$, with probability one.

CHAPTER II

II.1. Preliminaries. In this section we will review briefly the general theory of Fourier series on compact groups. For a very thorough treatment of harmonic analysis on compact group the reader is referred to [9].

Let G be a compact topological group. We let m be the unique normalized Haar measure of G, that is a positive Borel measure such that $m(G) = 1$ and $m(Ex) = m(E)$, for every Borel set $E \subseteq G$ and $x \in G$. The integral with respect to the Haar measure will be denoted by

$$\int_G f dx .$$

A unitary representation of G, is a homomorphism U, mapping G into the group of unitary operators of some Hilbert space $\mathfrak{H}$. We will only consider representations which are strongly continuous, that is $U(x_\alpha)u \to U(x)u$ for every $u \in \mathfrak{H}$, if $x_\alpha \to x$. A unitary representation is called irreducible if there exists no proper subspace $\mathfrak{M}$ of $\mathfrak{H}$ such that $U(x)\mathfrak{M} \subseteq \mathfrak{M}$ for every $x \in G$. The following result is basic in the theory of compact groups: *if U is an irreducible unitary representation of G into the group of unitary operators of a Hilbert space $\mathfrak{H}$, then the dimension of $\mathfrak{H}$ is a finite number d.* This number is called the degree of the representation U.

Let U_1 and U_2 be two unitary representations of G and let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be their associated Hilbert spaces. If there exists a unitary map V of $\mathfrak{H}_1$ into $\mathfrak{H}_2$ such that $VU(x) = U(x)V$ for all x, then U_1 and U_2 are called <u>unitarily equivalent</u>. The relation of unitary equivalence is an equivalence relation. We denote by Σ the set of equivalence classes of irreducible (hence finite dimensional) representations of G, defined by the relation of unitary equivalence. If $\sigma \in \Sigma$ and U^{σ} is a representative member of σ, then the degree d_{σ} of U^{σ} only depends on σ. Similarly if tr denotes the ordinary trace of operators on finite dimensional Hilbert spaces, the function $tr(U^{\sigma}(x)) = \chi_{\sigma}(x)$, only depends on σ. The functions $tr(AU^{\sigma}(x))$, where A is an operator on the d_{σ}-dimensional space $\mathfrak{H}_{\sigma}$, where $U^{\sigma}(x)$ operates, are called the <u>coordinate functions</u> of U^{σ}. Since tr is a unitary invariant the set of coordinate functions does not change if we consider a representation U_1^{σ} equivalent to U^{σ}. We may speak therefore of the coordinate functions of σ.

Another important result is the following: <u>The set of coordinate functions of irreducible unitary representations of G separate the points of G.</u> In other words if $x,y \in G$ and $x \neq y$, there exists a unitary irreducible representation U, such that $tr(AU(x)) \neq tr(AU(y))$ for some opeaator A acting on the space where $U(x)$ and $U(y)$ are defined.

A consequence of this result is that the algebra generated by the functions of the type $tr(AU^{\sigma}(x))$ with $\sigma \in \Sigma$ and A an operator on $\mathcal{H}_{\sigma}$, is dense in $C(G)$. We shall see now that the vector space spanned by the coordinate functions is already an algebra. For this we need the definition of tensor product of two finite dimensional representations.

Let U_1 and U_2 be two finite dimensional representations of G and $\mathcal{H}_1$ and $\mathcal{H}_2$ their associated Hilbert spaces, of dimension d_1 and d_2 respectively. Let $u_1, \dots, u_{d_1}$ be a basis for $\mathcal{H}_1$ and $v_1, \dots, v_{d_2}$ a basis for $\mathcal{H}_2$. Let $\mathcal{H}_1 \otimes \mathcal{H}_2$ be the Hilbert space of dimension $d_1 \times d_2$. We denote the elements of an orthonormal basis in $\mathcal{H}_1 \otimes \mathcal{H}_2$ by $u_i \otimes v_j$, $i = 1, \dots, d_1$; $j = 1, \dots, d_2$. Define for $x \in G$ the unitary operator $U_1 \otimes U_2(x)$ with the formula:

$$<U_1 \otimes U_2(x) u_i \otimes v_j, u_k \otimes v_\ell> = <U_1(x)u_i, u_k> <U_2(x)v_j, v_\ell>$$

where $<,>$ denotes the inner product in each of the Hilbert spaces considered. It is easy to see that, for each $x \in G$, $U_1 \otimes U_2(x)$ is a unitary operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Similarly if A and B are operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, we define $A \otimes B$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ as follows:

$$< A \otimes B \; u_i \otimes v_j , u_k \otimes v_e > = < Au_i, u_n > < Bv_j, v_e >$$

A computation shows that

$$\operatorname{tr}(AU_1(x))\operatorname{tr}(BU_2(x) = \operatorname{tr}(A \otimes BU_1 \otimes U_2(x)) .$$

In general the representation $U_1 \otimes U_2$ is not irreducible, even if both U_1 and U_2 are irreducible. However $U_1 \otimes U_2$ is finite dimensional and therefore it is _equivalent to the direct sum of irreducible representations_ in the following sense: there exist mutually orthogonal subspaces $\mathfrak{M}_1, \ldots, \mathfrak{M}_p$ of $\mathcal{H}_1 \otimes \mathcal{H}_2$, whose sum is $\mathcal{H}_1 \otimes \mathcal{H}_2$ and which are invariant under the action of $U_1 \otimes U_2(x)$, $x \in G$ and such that the restriction representations U'_j, $j = 1, \ldots, p$ defined by $U'_j(x) = U_1 \otimes U_2(x)|_{\mathfrak{M}_j}$ are irreducible. This means in particular that, with an appropriate choice of basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$

$$< U_1 \otimes U_2(x)u_i \otimes u_j,\ u_h \otimes u_k > = \sum_{\ell=1}^{p} < U'_\ell u_i \otimes u_j,\ u_h \otimes u_k > .$$

This implies that if A is an operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$, then $\operatorname{tr}(AU_1 \otimes U_2(x))$ is a linear combination of functions of the type $\operatorname{tr}(BU'_j(x))$, with U'_j irreducible representations.

Now denote by $\mathcal{T}$ the space of all linear combinations of functions such as $\operatorname{tr}(AU^{\sigma}(x))$ where $\sigma \in \Sigma$. In other words $\mathcal{T}$ is the linear space of all coordinate functions. Then the remarks just made imply that $\mathcal{T}$ is an algebra under pointwise multiplication. An application of the Stone-Weierstrass theorem yields that $\mathcal{T}$ is dense in $C(G)$ and therefore in $L^p(G)$ for $p < \infty$. The elements of $\mathcal{T}$ are often called trigonometric polynomials.

For $\sigma \in \Sigma$ we have already introduced the notation $\chi_\sigma(x) = tr(U^\sigma(x))$. The function χ_σ is called the character of σ. We have that: an element $\sigma \in \Sigma$ is completely determined by its character. That is if $tr(U_1(x)) = tr(U_2(x))$ for every $x \in G$, and U_1 and U_2 are irreducible representations of G, then U_1 is equivalent to U_2.

Finally we mention the so-called orthogonality relations for irreducible representations:

I) Let $\sigma, \tau \in \Sigma$ and $\sigma \neq \tau$, let A and B be operators on $\mathcal{H}_\sigma$ and $\mathcal{H}_\tau$ respectively, then

$$\int_G tr(AU^\sigma(x))\overline{tr(BU^\tau(x))}dx = 0$$

II) Let U be an irreducible representation of G and $\mathcal{H}$ its associated Hilbert space. Let $u_1, \ldots, u_d$ be an orthonormal basis in $\mathcal{H}$. Then

$$\int_G < U(x)u_i, u_j > \overline{< U(x)u_h, u_k >} \, dx = \frac{1}{d} \delta_{ih}\delta_{jk}$$

I) and II) together imply that if $u_1^\sigma, \ldots, u_{d_\sigma}^\sigma$ are orthonormal basis in the spaces $\mathcal{H}_\sigma$, then the set $\{d_\sigma^{\frac{1}{2}} < U^\sigma(x)u_i^\sigma, u_j^\sigma >: \sigma \in \Sigma, 1 \leq i,j \leq d_\sigma\}$ forms an orthonormal set of continuous functions in $L^2(G)$. Furthermore this set is complete because it spans $\mathcal{T}$ which is dense in $C(G)$.

It follows then that if $f \in L^2(G)$ and we define

$$C^{\sigma}_{ij}(x) = < U^{\sigma}(x)u^{\sigma}_i , u^{\sigma}_j >$$

and

$$\hat{f}(\sigma)_{ji} = \int_G f(x)\overline{C^{\sigma}_{ij}(x)}dx = \int_G f(x)\ C^{\sigma}_{ji}(x^{-1})dx$$

then

$$f = \sum_{\sigma \in \Sigma} \sum_{i,j=1}^{d_\sigma} d_\sigma \hat{f}(\sigma)_{ji} C^{\sigma}_{ij}(x) ,$$

in the sense that the series converges to f in L^2. Moreover

$$\int_G |f(x)|^2 dx = \sum_{\sigma \in \Sigma} \sum_{i,j=1}^{d_\sigma} d_\sigma |\hat{f}(\sigma)_{ij}|^2$$

We now define $\hat{f}(\sigma)$ to be the operator on $\mathcal{H}_\sigma$ given by the formula

(1.1) $$\hat{f}(\sigma) = \int_G f(x) U^{\sigma}(x^{-1})dx .$$

Thus with respect to the basis $u^{\sigma}_1, \ldots, u^{\sigma}_{1_\sigma}$, the operator $\hat{f}(\sigma)$ is represented by the matrix

$A_\sigma = (\hat{f}(\sigma)_{ij})$, and

$$\sum_{i,j=1}^{d_\sigma} \hat{f}(\sigma)_{ji} C^{\sigma}_{ij}(x) = \sum_{j=1}^{d_\sigma} \sum_{i=1}^{d_\sigma} \hat{f}(\sigma)_{ji} C^{\sigma}_{ij}(x) = tr(A_\sigma U^{\sigma}(x)) .$$

Furthermore

$$\sum_{i,j=1}^{d_\sigma} |\hat{f}(\sigma)_{ij}|^2 = tr(A_\sigma A^*_\sigma) .$$

In conclusion we have that

(1.2) $$f(x) = \sum_{\sigma \in \Sigma} d_\sigma \operatorname{tr}(A_\sigma U^\sigma(x))$$

and

(1.3) $$\int_G |f(x)|^2 dx = \sum_{\sigma \in \Sigma} d_\sigma \operatorname{tr}(AA_\sigma^*) .$$

One should notice that (1.2) and (1.3) imply that *a unitary representation* U *is irreducible if and only if*

$$\int_G |\operatorname{tr}(U(x))|^2 dx = 1 .$$

Define now $\hat{f}(\sigma)$ to be the *Fourier coefficient* of f at σ. Of course $\hat{f}(\sigma)$ is defined up to unitary equivalence. We will also call the series in (1.2) the *Fourier series* of f.

Since (1.1) makes sense also if $f \in L^1(G)$, we will say that a series

(1.4) $$\sum_{\sigma \in \Sigma} d_\sigma \operatorname{tr}(A_\sigma U^\sigma(x)) ,$$

represents a function $f \in L^1(G)$, if

$$A_\sigma = \int_G f(x)\, U^\sigma(x^{-1}) dx .$$

For convenience we shall suppose from now on that a fixed representative U^σ has been chosen for each $\sigma \in \Sigma$ and that a fixed basis $u_1^\sigma, \dots, u_{d_\sigma}^\sigma$, has been chosen in each $\mathcal{H}_\sigma$. Thus we will identify the operators $\hat{f}(\sigma)$ and $U^\sigma(x)$ with the matrices which represent them.

We recall now the definition of convolution between two integrable functions on G. If $f, g \in L^1(G)$, then, by definition

$$f*g(x) = \int_G f(xy^{-1})g(y)dy .$$

The value $f*g(x)$ is defined and finite almost everywhere, $f*g \in L^1$ and $\|f*g\|_1 \leq \|f\|_1 \|g\|_1$. If $f \in L^P$ and $g \in L^{P'}(1/P + 1/P' = 1)$, then $f*g \in C$ and $\|f*g\|_\infty \leq \|f\|_p \|g\|_{p'}$. Finally if $f \in L^1$ and $g \in L^P$, then $f*g \in L^P$ and $\|f*g\|_p \in \|f\|_1 \|g\|_p$.

An important property of convolution is the following:
if

$$A_\sigma = \int_G f(x)U^\sigma(x^{-1})dx, \tag{1.5}$$

and

$$B_\sigma = \int_G g(x)U^\sigma(x^{-1})dx ,$$

then

$$A_\sigma B_\sigma = \int_G f*g(x)\ U^\sigma(x^{-1})dx$$

In particular, if $\chi_\sigma(x) = tr(U^\sigma(x))$ and $f \in L^1(G)$, then $\chi_\sigma * f(x) = tr(A_\sigma U^\sigma(x) = f*\chi_\sigma(x)$, where A_σ is defined by (1.5). Thus if $e_\sigma = d_\sigma \chi_\sigma$, (1.2) can be written as $f(x) = \Sigma\, f*e_\sigma(x)$.

A <u>central function</u> in $L^1(G)$ is an integrable function g , such that $f*g(x) = g*f(x)$ for every $f \in L^1(G)$. Characters χ_σ are central functions. Conversely every central function g has a Fourier series development of the type

$$g(x) \sim \sum_{\sigma\in\Sigma} g*e_\sigma(x) = \sum_{\sigma\in\Sigma} a_\sigma e_\sigma(x) = \sum_{\sigma\in\Sigma} a_\sigma d_\sigma tr(U^\sigma(x)),$$

where a_σ are complex numbers.

An approximate identity in $L^1(G)$ is a net of integrable functions $\{h_\alpha\}$, such that $\lim_\alpha h_\alpha * f = f$ (in the L^1 norm), and $\|h_\alpha\|_1 \leq M$. We shall use in the sequel the fact that there exists an approximate identity in $L^1(G)$, consisting of central trigonometric polynomials, that is consisting of finite linear combinations of characters χ_σ.

II.2. Random Fourier series. Let Ω be a probability space and let $X_\sigma(\omega)$, $\omega \in \Omega$, $\sigma \in \Sigma$, be a set of independent random variables with values in the algebra of linear operators on $\mathcal{H}_\sigma$. A natural analogue of the series considered in I.5 are the following series

$$\sum_{\sigma\in\Sigma} d_\sigma tr(X_\sigma U_\sigma(x)), \tag{2.1}$$

where x varies in the compact group G. (Recall that Σ is the space of (equivalence classes of) irreducible representations of G)).

We would like to find conditions on the random variables X_σ which will allow us to extend the results stated in Chapter I. This means that we would like to find conditions on the X_σ which will allow us to conclude that

I) if $\Sigma d_\sigma tr(X_\sigma U^\sigma(x)) \in L^1$ with probability one, then, almost surely it belongs to $\bigcap_{p<\infty} L^p$.

II) if $\Sigma d_\sigma \operatorname{tr}(X_\sigma U^\sigma(x)) \in L^\infty$ with probability one, then it represents almost surely a continuous function.

It is possible to translate Kahane's argument, given in I.5, in the following way. Let G_σ be compact subgroups of $\mathcal{U}(d_\sigma)$, the group of unitary operators on h_σ. Let $\mathcal{G} = \prod_{\sigma \in \Sigma} G_\sigma$, then $\mathcal{G}$ is a probability space with respect to its Haar measure which is the product of the Haar measures of G_σ. Let A_σ be a fixed $d_\sigma \times d_\sigma$ matrix, for each σ, and define, for $V = \{V_\sigma\} \in \mathcal{G}$, $X_\sigma(V) = A_\sigma V_\sigma$. The series (2.1) becomes then

(2.2) $$\sum_{\sigma \in \Sigma} d_\sigma \operatorname{tr}(A_\sigma V_\sigma U^\sigma(x)) .$$

The following theorem can be proved using the argument of Theorem I.5.1:

THEOREM II.2.1. Suppose that the statements I) and II) above are true for random Fourier series of the type (2.2). Then I) and II) are true for the random Fourier series (2.1), provided that each X_σ is symmetric with respect to the group G_σ. That is provided that for each $V_\sigma \in G_\sigma$, X_σ and $U_\sigma X_\sigma$ have the same probability distribution.

In view of Theorem II.2.1 we are led to investigate the series (2.2). We restrict our problem then to the problem of finding closed subgroups G_σ of $\mathcal{U}(d_\sigma)$ for which the following statements are true