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Lectures given at the
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PROBLEMI DI GEOMETRIA DIFFERENZIALE IN GRANDE

CHAPTER I

DIFFERENTIABLE MANIFOLDS AND THEIR IMBEDDING

1. DIFFERENTIABLE MANIFOLDS. A differentiable manifold, X^n, is an astract object having the following properties :

(1) It is a topological manifold, covered with open sets U_i. It is usually assumed to be paracompact In most of these lectures we assume it to be compact

(2) There is a map : $\phi_i : U_i \to E^n$ for each U_i These establish coordinates in U_i.

(3) In overlapping open sets, i.e. in $U_i \cap U_j$, the corresponding coordinates are related by differentiable functions.

X^n is $C^{(r)}$ if these functions have r continous derivatives; C^∞ if all derivatives exist; C^ω if the functions are real analytic

2. IMBEDDINGS By virtue of a theorem of Whitney (Annals of Mathematics - 1936) X^n can be considered to be a subspace of a Euclidean space of sufficently high dimension. The theorem is :

THEOREM. Let X^n be a $C^{(r)}$ manifold ($1 \leqslant r \leqslant \infty$, not $r = \omega$). Then X^n is $C^{(r)}$ homeomorphic to an analytic submanifold of E^{2n+1}

If X^n carries a Riemann metric : $ds^2 = g_{ij}dx^i dx^j$, there are additional results for the case of C^ω manifolds. These are :

Bochner (Duke Journal 1937): If X^n is C^ω and compact and has an analytic Riemann metric, then X^n is C^ω homeomorphic with an analytic submanifold in E^{2n+1}.

Malgrange (Bull. Soc Math. France 1957): Bochner's result for non-compact case.

Morrey (unpublished, 1958): If X^n is C^ω and compact, X^n is C^ω homeomorphic with an analytic submanifold in E^{2n+1}. The proof is based on the lemma :

Lemma (Morrey). With each point P of X^n are associated n functions ϕ_i $(i = 1,\ldots,n)$ wich are C^ω over X^n and have linearly indipendent gradients at P . This lemma is an important result in its own right.

Then $\phi_i(P)$ have independent gradients in N(P). Cover X^n with $N(P_i)$ $i = 1\ldots q$. This gives $\phi_{i\alpha}(i = 1\ldots n,\ \alpha = 1\ldots q)$. Take these as coordinates in E^{nq}. This is an imbedding which is C^ω and locally one-to one. Hence it induces a C^ω Riemann metric. The result now follows from the above theorem of Bochner.

3. ISOMETRIC IMBEDDING. When X^n has a Riemann metric, we may further require that the given metric coincide with that induced by the imbedding, i.e. that the imbedding be isometric. The results are :

Janet (1926) If X^n is C^ω, it can be locally imbedded with preservation of the metric in $E^{n(n+1)/2}$.

Nash-Kuiper (1955 - Annals of Mathematics) : If X^n is C^1 and compact, and if it can be differentiably imbedded in E^N $(N \geqslant n+1)$, then it has a C^1 isometric imbedding in E^N. This result is efficient regarding dimension, but is true only for C^1; the case of the torus in E^3 shows it to be false for C^2.

Nash (1956 - Annals of Mathematics). If X^n is $C^{(h)}$ $(3 \leqslant h \leqslant \infty)$ and is compact , it has an isometric $C^{(h)}$ imbedding in an Euclidean space of dimension $(n/2)\cdot(3n+11)$. When X^n is non-compact, the dimension required is $3n^3/2 + 7n^2 + 11n/2$. The cases of C^2 and C^ω are open.

4. RIGID IMBEDDING. If an isometric imbedding is unique to within motion in the euclidean space, it is said to be "rigid". Sufficient

conditions for rigid imbedding will be given later in this series of lectures.

5. NOTATIONS FOR IMBEDDED MANIFOLDS. Let X^n be imbedded in E^{n+N}

Local coordinates in E^{n+N} : y^α $(\alpha, \beta, \gamma = 1 \ldots n+N)$

Local coordinates in X^n . x^i $(i, j, k = 1 \ldots n)$

Also : $\rho, \sigma, \tau = n+1 \ldots n+N$.

The imbedding is given locally by the functions :

$$y^\alpha = f^\alpha(x^i) .$$

Then

(1) $$dy^\alpha = (\partial f^\alpha / \partial x^i) dx^i .$$

These are a base for the tangent vectors to X^n, and so any tangent vector is a linear combination of the dx^i.

It will be convenient to choose an orthonormal base for the tangent vectors, e_i^α , such that

$$\sum_\alpha e_i^\alpha e_j^\alpha = \delta_{ij} .$$

In this notation α represents the Euclidean component of the vector, and i enumerates the vector. Then

(2) $$dy^\alpha = \phi^i e_i^\alpha ,$$

where

$$\phi^i = \Sigma_\alpha dy^\alpha e_i^\alpha = \Sigma_\alpha (\partial f^\alpha / \partial x^i) dx^j e_i^\alpha .$$

Thus ϕ^i is a linear differential form

In particular

(3) $$ds^2 = \Sigma_\alpha dy^\alpha dy^\alpha = \Sigma \phi^i \phi^i$$

We also introduce an orthonormal frame of normal vectors e_σ^α such that

$$\Sigma_a \, e^a_i \, e^a_\sigma = 0 \quad , \qquad \Sigma_a \, e^a_\sigma \, e^a_\rho = \delta_{\sigma\rho} \; .$$

It follows at once that :

$$(4) \qquad \begin{cases} de_i = \omega^j_i \, e_j + \omega^\sigma_i \, e_\sigma \\ de_\sigma = \omega^j_\sigma \, e_j + \omega^\rho_\sigma \, e_\rho \; , \end{cases}$$

where we have suppressed the upper index a ; and ω^j_i, ω^j_σ, and ω^ρ_σ are linear differential forms.

From the orthogonality of the chosen frames, we have seen that $\omega^j_i = -\omega^i_j$; $\omega^\sigma_i = -\omega^i_\sigma$; $\omega^\sigma_\rho = -\omega^\rho_\sigma$.

6. EQUATIONS OF STRUCTURE. These are the basic equations of ówn geometry. From (2) we derive

$$0 = ddy^a = d\phi^i \, e_i + de_i \wedge \phi^i$$

$$= d\phi^j \, e_j = \omega^j_i \wedge \phi^i e_j = \omega^\sigma_i \wedge \phi^i e_\sigma$$

$$= (d\phi^j = \omega^j_i \wedge \phi^i) \, e_j = (\omega^\sigma_i \wedge \phi^i) e_\sigma$$

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$$(5) \qquad \begin{aligned} d\phi^j + \omega^j_i \wedge \phi^i &= 0 \\ \omega^\sigma_i \wedge \phi^i &= 0 \; . \end{aligned}$$

By differentiating (4) and substituting back for de_i and de_σ from (4), we further derive :

$$(6) \qquad \begin{cases} d\omega^k_i + \omega^k_j \wedge \omega^j_i + \omega^k_\sigma \wedge \omega^\sigma_i = 0 \\ d\omega^\sigma_i + \omega^\sigma_j \wedge \omega^j_i + \omega^\sigma_\rho \wedge \omega^\rho_i = 0 \\ d\omega^\sigma_\rho + \omega^\sigma_j \wedge \omega^j_\rho + \omega^\sigma_\tau \wedge \omega^\tau_\rho = 0 \; . \end{cases}$$