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Statistical Properties of Deterministic Systems

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With 4 figures

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Preface

Ergodic theory is a mathematical subject that studies the statistical properties of deterministic dynamical systems. It is a combination of several branches of pure mathematics, such as measure theory, functional analysis, topology, and geometry, and it also has applications in a variety of fields in science and engineering, as a branch of applied mathematics. In the past decades, the ergodic theory of chaotic dynamical systems has found more and more applications in mathematics, physics, engineering, biology and various other fields. For example, its theory and methods have played a major role in such emerging interdisciplinary subjects as computational molecular dynamics, drug designs, and third generation wireless communications in the past decade.

Many problems in science and engineering are often reduced to studying the asymptotic behavior of discrete dynamical systems. We know that in neural networks, condensed matter physics, turbulence in flows, large scale laser arrays, convection-diffusion equations, coupled mapping lattices in phase transition, and molecular dynamics, the asymptotic property of the complicated dynamical system often exhibits chaotic phenomena and is unpredictable. However, if we study chaotic dynamical systems from the statistical point of view, we find that chaos in the deterministic sense usually possesses some kind of regularity in the probabilistic sense. In this textbook, which is written for the upper level undergraduate students and graduate students, we study chaos from the statistical point of view. From this viewpoint, we mainly investigate the evolution process of density functions governed by the underlying deterministic dynamical system. For this purpose, we employ the concept of density functions in the study of the statistical properties of sequences of iterated measurable transformations. These statistical properties often depend on the existence and the properties of those probability measures which are absolutely continuous with respect to the Lebesgue measure and which are invariant under the transformation with respect to time. The existence of absolutely continuous invariant finite measures is equivalent to the existence of nontrivial fixed points of a class of stochastic operators (or Markov operators), called Frobenius-Perron operators by the great mathematician Stanislaw Ulam, who pioneered the exploration of nonlinear science, in his famous book "A Collection of Mathematical Problems" [120] in 1960.

In this book, we mainly study two kinds of problems. The first is the existence of nontrivial fixed points of Frobenius-Perron operators, and the other concerns the computation of such fixed points. They can be viewed as the functional analysis and the numerical analysis of Frobenius-Perron operators, respectively. For the first problem, many excellent books have been written, such as "Probabilistic Properties of Deterministic Systems" and its extended second edition "Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics" by Lasota and Mackey [82], and "Law of Chaos: Invariant Measures and Dynamical Systems in One Dimension" by Boyarsky and Góra [14]. For the second problem, this book might be among the first ones in the form of a textbook on the computational ergodic theory of discrete dynamical systems. One feature that distinguishes this book from the others is that our textbook combines strict mathematical analysis and efficient computational methods as a unified whole. This is the authors' attempt to reduce the gap between pure mathematical theory and practical physical, engineering, and biological applications.

The first famous papers on the existence of nontrivial fixed points of Frobenius-Perron operators include the proof (see, e.g., Theorem 6.8.1 of [82]) of the existence of a unique smooth invariant measure for a second order continuously differentiable expanding transformation on a finite dimensional, compact, connected, smooth Riemann manifold by Krzyzewski and Szlenk [80] in 1969, and the pioneering work [83] on the existence of absolutely continuous invariant measures of piecewise second order differentiable and stretching interval mappings by Lasota and Yorke in 1973. The latter also answered a question posed by Ulam in his above mentioned book. In the same book, Ulam proposed a piecewise constant approximation method which became the first approach to the numerical analysis of Frobenius-Perron operators. A solution to Ulam's conjecture by Tien-Yien Li [86] in 1976 is a fundamental work in the new area of *computational ergodic theory*.

Our book has nine chapters. As an introduction, Chapter 1 leads the reader into a mathematical trip from order to chaos via the iteration of a one-parameter family of quadratic polynomials with the changing values of the parameter, from which the reader enters the new vision of "chaos from the statistical point of view." The fundamental mathematical knowledge used in the book – basic measure theory and functional analysis– constitutes the content of Chapter 2. In Chapter 3, we study the basic concepts and classic results in ergodic theory. The main linear operator studied in this book – the Frobenius-Perron operator – is introduced in Chapter 4, which also presents some general results that have not appeared in other books. Chapter 5 is exclusively devoted to the investigation of the existence problem of absolutely continuous invariant measures, and we shall prove several existence results for various classes of one-dimensional mappings and multi-dimensional transformations. The computational problem is studied in Chapter 6, in which two numerical methods are given for the approximation of Frobenius-Perron operators. One is the classic Ulam's piecewise constant method, and the other is its improvement with higher order approximation accuracy; that is, the piecewise linear Markov method which was mainly developed by the authors. In Chapter 7, we present Keller's result on the stability of Markov operators and its application to the convergence rate analysis of

Ulam's method under the L^1-norm and Murray's work for a more explicit upper bound of the error estimate. We also explore the convergence rate under the variation norm for the piecewise linear Markov method. Chapter 8 gives a simple mathematical description of the related concepts of entropy, in particular the Boltzmann entropy and its relationship with the iteration of Frobenius-Perron operators. Several modern applications of absolutely continuous invariant probability measures will be given in the last chapter.

This book can be used as a textbook for students of pure mathematics, applied mathematics, and computational mathematics as an introductory course on the ergodic theory of dynamical systems for the purpose of entering the related frontier of interdisciplinary areas. It can also be adopted as a textbook or a reference book for a specialized course for different areas of computational science, such as computational physics, computational chemistry, and computational biology. For students or researchers in engineering subjects such as electrical engineering, who want to study chaos and applied ergodic theory, this book can be used as a tool book. A good background of advanced calculus is sufficient to read and understand this book, except possibly for Section 2.4 on the modern definition of variation and Section 5.4 on the proof of the existence of multi-dimensional absolutely continuous invariant measures which may be omitted at the first reading. Some of the exercises at the end of each chapter complement the main text, so the reader should try to do as many as possible, or at least take a look and read appropriate references if possible. Each main topic of ergodic theory contains matter for huge books, but the purpose of this book is to introduce as many readers as possible with various backgrounds into fascinating new fields having great potential of ever increasing applications. Thus, our presentation is quite concise and elementary and as a result, some important but more specialized topics and results must be omitted, which can be found in other monographs.

Another feature of this textbook is that it contains much of our own joint research in the past fifteen years. In this sense it is like a monograph. Our joint research has been supported by the National Science Foundation of China, the National Basic Research Program of China, the Academy of Mathematics and Systems Science at the Chinese Academy of Sciences, the State Key Laboratory of Scientific and Engineering Computing at the Chinese Academy of Sciences, the Chinese Ministry of Education, the China Bridge Foundation at the University of Connecticut, and the Lucas Endowment for Faculty Excellence at the University of Southern Mississippi, among the others, for which we express our deep gratitude.

Jiu Ding would also like to thank his Ph.D. thesis advisor, University Distinguished Professor Tien-Yien Li of Michigan State University. It is Dr. Li's highly educative graduate course "Ergodic Theory on $[0, 1]$" for the academic year 1988-1989, based on the lecture notes [87] delivered at Kyoto University of Japan one year earlier, that introduced him into the new research field of

computational ergodic theory and led him to write a related Ph.D. dissertation. Aihui Zhou is very grateful to his Ph.D. thesis advisor, Academician Qun Lin, of the Chinese Academy of Sciences, who with a great insight, encouraged him to enter this wide and exciting research area.

The first edition of this book was published in Chinese by the Tsinghua University Press in Beijing, China in January 2006 and reprinted in December in the same year. We thank editors Xiaoyan Liu, Lixia Tong, and Haiyan Wang and five former Ph.D. students of Aihui Zhou, Xiaoying Dai, Congming Jin, Fang Liu, Lihua Shen, and Ying Yang for their diligent editorial work and technical assistance, which made the fast publication of the Chinese edition possible. We thank Lixia Tong for her help during the preparation of this revised and expanded English edition of the book.

Jiu Ding and Aihui Zhou
Beijing, March 2008

Contents

Chapter 1
Introduction

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Abstract Using the famous logistic model $S_r(x) = rx(1-x)$ as an example, we give a brief survey of discrete dynamical systems for the purpose of leading the reader on a mathematical trip from order to chaos, and then we introduce basic ideas behind the statistical study of chaos, which is the main topic of the book.
Keywords Logistic model, period-doubling bifurcation, Li-Yorke chaos, Frobenius-Perron operator, absolutely continuous invariant measure.

In the modern statistical study of discrete deterministic dynamical systems and its applications to physical sciences, there are two important and mutually related problems. On the theoretical part, there is the problem of the *existence* of absolutely continuous invariant measures that give the statistical properties of the dynamics, such as the probability distribution of the orbits for almost all initial points and the speed of the decay of correlations. On the practical part, we encounter the problem of the *computation* of such invariant measures to any prescribed precision in order to numerically explore the chaotic behavior in many physical systems. In this textbook, we try to address these two problems. For this purpose, we need to study a class of positive linear operators, called Frobenius-Perron operators, that describe the density evolution governed by the underlying dynamical system. Density functions, which are the fixed points of Frobenius-Perron operators, define absolutely continuous invariant probability measures associated with the deterministic dynamical system, which can be numerically investigated via structure preserving computational methods that approximate such fixed density functions.

Before we begin to study the statistical properties of discrete dynamical systems, we first review the deterministic properties of one-dimensional mappings in this introductory chapter as a starting point. The well-known logistical model, which has played an important role in the history of the evolution of the concept of chaos in science and mathematics, will be studied in detail from the deterministic point of view. Then, we are naturally led to the statistical study of chaos by introducing the concept of Frobenius-Perron operators with an intuitive approach, which motivates the main topic of this book.

1.1 Discrete Deterministic Systems—from Order to Chaos

In their broad sense, dynamical systems provide rules under which phenomena (states) in the mathematical or physical world evolve with respect to time. Differential equations are widely used to model continuous time dynamical systems in many areas of science, such as classical mechanics, quantum mechanics, neural networks, mathematical biology, etc., as these equations describe mathematically the laws by which they are governed. Transformations on phase spaces not only determine a discrete time dynamical system[23], but also form the basis of investigating continuous time dynamical systems via such mathematical tools as the Poincare' map. Even simple nonlinear transformations may exhibit a quasi-stochastic or unpredictable behavior which is a key feature of the chaotic dynamics. Poincaré deduced this kind of chaotic motion for the three-body problem in celestial mechanics about fifty years before the advent of electronic computers in the 1940s, and about eighty years before Tien-Yien Li and James A. Yorke first coined the term "Chaos" in their seminar paper "Period Three Implies Chaos" [88] in 1975.

The discrete time evolution of a dynamical system in the N-dimensional Euclidean space $\mathbb{R}^N$ is usually given by a first order difference equation which is often written as a recurrence relation

$$\boldsymbol{x}_{n+1} = \boldsymbol{S}(\boldsymbol{x}_n), \quad n = 0, 1, \cdots,$$

where $\boldsymbol{S}$ is a transformation from a subset Ω of $\mathbb{R}^N$ into itself. For example, consider a population of organisms for which there is a constant supply of food and limited space, and no predators. In order to model the populations in successive generations, let x_n denote the population of the nth generation, and adjust the numbers so that the capacity of the environment is equal to 1, which means that $0 \leqslant x_n \leqslant 1$ for all n. One popular formula for the dynamics of the population is the so-called *logistic model* , after the differential equation studied by the Belgian mathematician Pierre F. Verhulst about 160 years ago [98]:

$$x_{n+1} = r x_n (1 - x_n), \quad n = 0, 1, \cdots,$$

where $r \in (0, 4]$ is a parameter. In the following, we study the deterministic properties of this logistical model to some extent when the parameter r varies from 0 to 4 and see how the dynamics will change from the regular behavior to the chaotic behavior as r increases toward 4.

First, we introduce some standard terms in discrete dynamical systems . Let X be a set and $S : X \to X$ be a transformation. A point $x \in X$ is called a *fixed point* of S if $S(x) = x$ and an *eventually fixed point* of S if there is a positive integer k such that $S^k(x)$ is a fixed point of S, where $S^k(x) = S(S(\cdots(S(x))\cdots))$ (i.e., S^k is the composition of S with itself $k-1$ times) is the kth iterate of x. A point $x_0 \in X$ is called a *periodic point* of S with period $n \geqslant 1$ or a *period-n point* of S if $S^n(x_0) = x_0$ and if in addition, $x_0,\ S(x_0), S^2(x_0), \cdots, S^{n-1}(x_0)$

are distinct. A fixed point is a periodic point with period 1. An *eventually periodic point* is a point whose kth iterate is a periodic point for some $k > 0$. The *orbit* of an initial point x_0 is the sequence

$$x_0, S(x_0), S^2(x_0), \cdots, S^n(x_0), \cdots$$

of the iterates of x_0 under S. If x_0 is a period-n point, then the orbit

$$x_0, S(x_0), \cdots, S^{n-1}(x_0), \cdots$$

of x_0 is a *periodic orbit* which can be represented by $\{x_0, S(x_0), \cdots, S^{n-1}(x_0)\}$ called an *n-cycle* of S.

From the mean value theorem of calculus, a fixed point x of a differentiable mapping S of an interval is *attracting* or *repelling* if $|S'(x)| < 1$ or $|S'(x)| > 1$, respectively. Similarly, a period-n point x_0 of S is attracting or repelling when $|(S^n)'(x_0)| < 1$ or $|(S^n)'(x_0)| > 1$ respectively, and the corresponding n-cycle is *attracting* or *repelling*. Such information only gives the local dynamical properties of a fixed point or a periodic orbit, not the global ones which need more subtle arguments and more thorough analysis to obtain in general.

Now, we begin to study the iteration of the logistic model. Let

$$S_r(x) = rx(1 - x), \quad \forall\, x \in [0, 1],$$

where the parameter $r \in (0, 4]$ so that S_r maps $[0, 1]$ into itself. It is obvious that S_r has one fixed point 0 when $0 < r \leqslant 1$ and two fixed points 0 and $p_r \equiv 1 - 1/r$ when $r > 1$. Since $S_r'(0) = r$ and $S_r'(p_r) = 2 - r$, one can see that the fixed point 0 is attracting for $r \leqslant 1$ and repelling for $r > 1$, and the fixed point p_r is attracting for $1 < r \leqslant 3$ and repelling for $r > 3$. In the remaining part of this section, we study the global properties of the fixed points and possible periodic points in more detail.

As will be shown below, the dynamics of S_r changes as the parameter r passes through each of the values $1,\ 2,\ 3,\ 1 + \sqrt{6}, \cdots$, called the *bifurcation points* of the one-parameter family $\{S_r\}$ of the quadratic mappings, that is, the number and nature of the fixed points and/or the periodic points change when r passes through each of them. Hence, our discussion below will be split into four cases, from easy to more complicated ones. They are respectively $0 < r \leqslant 1$, $1 < r \leqslant 2$, $2 < r \leqslant 3$, and $3 < r \leqslant 4$. In the analysis, we often use the simple fact that the limit x^* of a convergent sequence $\{x_n\}$ of the iterates of a continuous mapping S must be a fixed point of S if x^* is in the domain of S.

Case 1. $0 < r \leqslant 1$ (see Figure 1.1).

Since $0 < S_r(x) = rx(1 - x) < x$ for $0 < x < 1$, the iteration sequence $\{S_r^n(x)\}$ is positive and monotonically decreasing, and so it converges to the

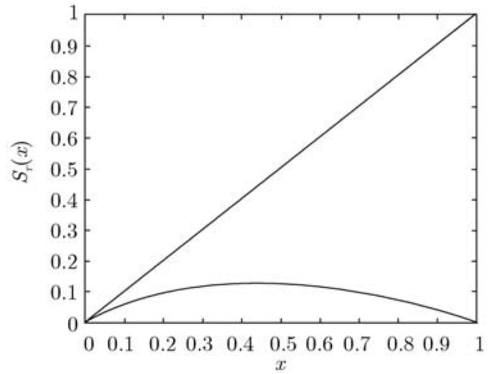

Figure 1.1 S_r at $r = 0.5$

unique fixed point 0 of S_r as n approaches infinity. It follows that the *basin of attraction* of 0, which is the set of all the initial points whose orbit converges to the fixed point 0 by definition, is the closed interval $[0, 1]$. So there are no periodic points except for the unique fixed point 0.

Case 2. $1 < r \leqslant 2$ (see Figure 1.2).

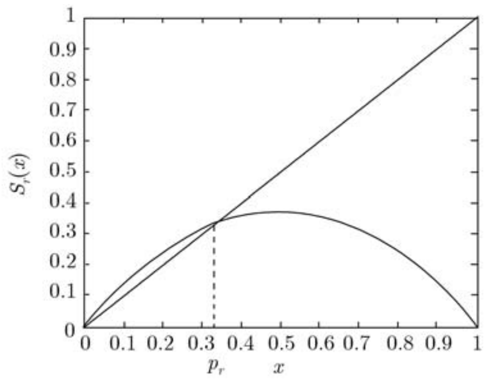

Figure 1.2 S_r at $r = 1.5$

Now, S_r has two fixed points, 0 and $p_r = 1 - 1/r$. We know that the fixed point 0 is repelling and the fixed point p_r is attracting. Let $0 < x < p_r$. Then, $1/r < 1 - x$, so $x < rx(1 - x) = S_r(x)$. By induction we see that $x < S_r(x) < \cdots < S_r^n(x) < \cdots$. On the other hand, since S_r is strictly increasing on $[0, p_r]$,

$$S_r(x) < S_r(p_r) = p_r,$$

which implies that $S_r^n(x) < p_r$ for all n. Thus, the sequence $\{S_r^n(x)\}$ is strictly increasing, bounded above by p_r, and hence it converges to the fixed point p_r. Similarly, if $p_r < x \leqslant 1/2$, then $\{S_r^n(x)\}$ is a monotonically decreasing sequence bounded below by p_r, so it also converges to p_r. Finally, if $1/2 < x < 1$, then $0 < S_r(x) \leqslant 1/2$, so by the above argument, $\{S_r^n(x)\}$ converges to p_r. Therefore, when $1 < r \leqslant 2$, the basin of attraction of the fixed point p_r is the open interval $(0, 1)$, the basin of attraction of the fixed point 0 is the 2-point set $\{0, 1\}$, and there are no other periodic points besides the two fixed points.

Case 3. $2 < r \leqslant 3$. (see Figure 1.3).

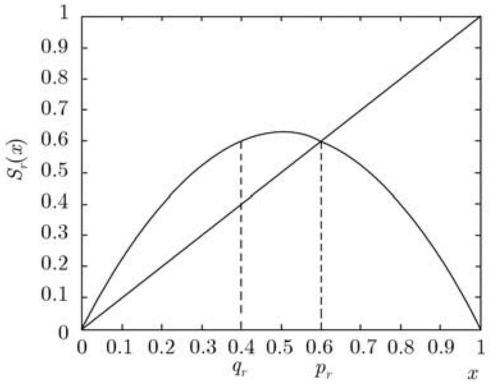

Figure 1.3 S_r at $r = 2.5$

When $r > 2$, the fixed point $p_r > 1/2$. Assume that $r < 3$ and let q_r be the unique number in $(0, 1/2)$, which is symmetric to p_r about $1/2$, such that $S_r(q_r) = S_r(p_r) = p_r$. Then, using the geometry of the graph of S_r and the fact that $q_r \leqslant S_r(r/4)$, one can show that (see Exercise 1.1):

(i) if $x \in (0, q_r)$, then x has an iterate $> q_r$;

(ii) if $q_r < x \leqslant p_r$, then $p_r \leqslant S_r(x) \leqslant r/4$;

(iii) if $p_r < x \leqslant r/4$, then $q_r \leqslant S_r(x) < p_r$;

(iv) if $r/4 < x < 1$, then $0 < S_r(x) < p_r$.

From (i)-(iv) it follows that if $0 < x < 1$, then x has an iterate in the interval $(q_r, p_r]$. Moreover, (ii) and (iii) imply that the iterates of x oscillate between the intervals $(q_r, p_r]$ and $[p_r, r/4]$. Thus,

(v) if x is in $(q_r, p_r]$, then so is the sequence $\{S_r^{2n}(x)\}$;

(vi) if x is in $[p_r, r/4]$, then so is the sequence $\{S_r^{2n}(x)\}$.

Since 0 and p_r are the fixed points of S_r, a simple calculation shows that

$$S_r^2(x) - x = rx(x - p_r)\left[-r^2x^2 + \left(r^2 + r\right)x - r - 1\right]. \tag{1.1}$$

The expression inside the brackets has no real roots when $2 < r < 3$. Therefore, if $2 < r < 3$, then the only fixed points of S_r^2 are 0 and p_r. Since $S_r^2(x) - x$ has no roots in (q_r, p_r), it has the same sign as $S_r^2(1/2) - 1/2$ which is positive. Consequently $x < S_r^2(x)$ for all $x \in (q_r, p_r)$, and by (v) the sequence $\{S_r^{2n}(x)\}$

is monotonically increasing, lies in $(q_r, p_r]$, and converges to the only positive fixed point p_r of S_r^2. Using the continuity of S_r, we find that

$$S_r^{2n+1}(x) = S_r(S_r^{2n}(x)) \to S_r(p_r) = p_r$$

as n increases without bound. Therefore, $S_r^n(x) \to p_r$ whenever $x \in (q_r, p_r]$. Since every x in $(0,1)$ has an iterate in $(q_r, p_r]$, we conclude that $S_r^n(x) \to p_r$ as n increases without bound, for all $x \in (0,1)$. In other words, the basin of attraction of p_r is $(0,1)$, so the basin of attraction of 0 is $\{0,1\}$. A consequence of this result is that there are no periodic points for S_r other than the fixed points. The same conclusion can be proven for $r = 3$ with a more careful analysis.

Case 4. $3 < r \leqslant 4$ (see Figure 1.4).

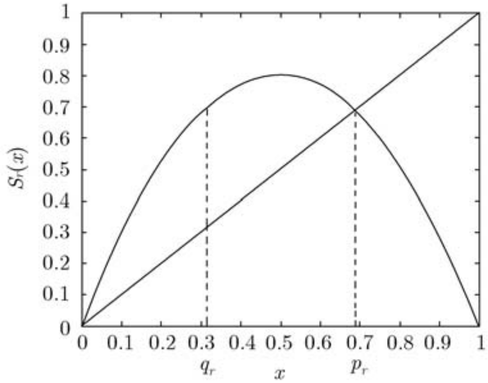

Figure 1.4 S_r at $r = 3.2$

We have learned that the dynamics of S_r is regular when $0 < r \leqslant 3$, and in particular the only periodic points are fixed points. When $3 < r \leqslant 4$, both 0 and $p_r = 1 - 1/r$ are repelling fixed points. Do the iterates of other points in $(0,1)$ converge, or oscillate, or have no pattern at all? Are there periodic points other than 0 and p_r? The analysis of the dynamics of S_r becomes more and more complicated as r increases from 3 to 4. We only study the case $3 < r < 1 + \sqrt{6}$ in detail and list the main results that follow.

For our purpose, we need to study the dynamics of S_r^2. When $r = 3$, the graph of S_r^2 is tangent to the diagonal $y = x$ at the point (p_r, p_r). From (1.1), the other two fixed points of S_r^2 besides 0 and p_r are the real roots of the quadratic equation

$$-r^2x^2 + (r^2 + r)x - r - 1 = 0,$$

which are

$$s_r = \frac{1}{2} + \frac{1}{2r} - \frac{1}{2r}\sqrt{(r-3)(r+1)} \text{ and } t_r = \frac{1}{2} + \frac{1}{2r} + \frac{1}{2r}\sqrt{(r-3)(r+1)}.$$

Since 0 and p_r are the only fixed points of S_r for $r > 1$, it is obvious that $\{s_r, t_r\}$ is a 2-cycle for $r > 3$. After a simple computation, we find that

$$(S_r^2)'(s_r) = S_r'(s_r)S_r'(t_r) = (r - 2rs_r)(r - 2rt_r) = -r^2 + 2r + 4.$$

Since $|-r^2 + 2r + 4| < 1$ if and only if $3 < r < 1 + \sqrt{6}$, the 2-cycle $\{s_r, t_r\}$ is attracting if $3 < r < 1 + \sqrt{6}$. It can further be shown that the basin of attraction of the 2-cycle $\{s_r, t_r\}$ consists of all $x \in (0, 1)$ except for the fixed point p_r and the points whose iterates are eventually p_r.

When $r > 1 + \sqrt{6}$, the 2-cycle $\{s_r, t_r\}$ becomes repelling. As we may expect, an attracting 4-cycle is born. Actually, there exists a sequence $\{r_n\}$ of the so-called *period-doubling* bifurcation values for the parameter r, with $r_0 = 3$ and $r_1 = 1 + \sqrt{6}$, such that

- if $r_0 < r \leqslant r_1$, then S_r has two repelling fixed points and one attracting 2-cycle;
- if $r_1 < r \leqslant r_2$, then S_r has two repelling fixed points, one repelling 2-cycle, and one attracting 2^2-cycle;
- if $r_2 < r \leqslant r_3$, then S_r has two repelling fixed points, one repelling 2-cycle, one repelling 2^2-cycle, and one attracting 2^3-cycle;

In general, for $n = 1, 2, \cdots$,

- if $r_{n-1} < r \leqslant r_n$, then S_r has two repelling fixed points, one repelling 2^k-cycle for $k = 1, 2, \cdots, n-1$, and one attracting 2^n-cycle.

It is well-known that $\lim_{n\to\infty} r_n = r_\infty = 3.561547\cdots$. This number r_∞ is called the *Feigenbaum number* for the quadratic family $\{S_r\}$. Moreover, the sequence $\{c_n\}$ of the ratios

$$c_n = \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

converges to a number $c_\infty = 4.669202\cdots$, which is called the *universal constant* since for many other families of one-humped mappings, the bifurcations occur in such a regular fashion that the ratios of the distances between successive pairs of the bifurcation points approach the very same constant c_∞! This universal constant c_∞ is also referred to as the *Feigenbaum constant* because the physicist Michael Feigenbaum first found it and its universal property in 1978.

So far the dynamics of the quadratic family $\{S_r\}$ is still regular for $0 < r < r_\infty$ since every point $x \in (0, 1)$ is periodic, eventually periodic, or attracted to a fixed point or a periodic orbit. So, the eventual behavior of the orbits is *predictable.* When $r \geqslant r_\infty$, there could exhibit a complicated irregular or chaotic behavior for the dynamics of S_r. For example, if $3.829 \leqslant r \leqslant 3.840$, then S_r has period-3 points. The celebrated *Li-Yorke theorem* [88] says that if a continuous mapping S from an interval I into itself has a period-3 point, then it has a period-k point for any natural number k, and there is an uncountable set $\Lambda \subset I$, containing no periodic points, which satisfies the following conditions:

(i) For every pair of distinct numbers $x,\ y \in \Lambda$,

$$\limsup_{n\to\infty} |S^n(x) - S^n(y)| > 0 \quad \text{and} \quad \liminf_{n\to\infty} |S^n(x) - S^n(y)| = 0.$$

(ii) For every $x \in \Lambda$ and each periodic point $p \in I$,

$$\limsup_{n\to\infty} |S^n(x) - S^n(p)| > 0.$$

Thus, from the Li-Yorke theorem, the eventual behavior of the iterates of S_r with $3.829 \leqslant r \leqslant 3.840$ is *unpredictable.*

The case $r = 4$ is worth a special attention. It is well-known [7] that S_4 is *topologically conjugate* to the *tent function*

$$T(x) = \begin{cases} 2x, & \text{if } x \in \left[0, \dfrac{1}{2}\right], \\[2ex] 2(1-x), & \text{if } x \in \left[\dfrac{1}{2}, 1\right]. \end{cases} \tag{1.2}$$

That is, there is a homeomorphism $h : [0,1] \to [0,1]$ such that $S_4 \circ h = h \circ T$. Since T has a 3-cycle $\{2/7, 4/7, 6/7\}$, there is a period-3 orbit for S_4. By the Li-Yorke theorem, S_4 is chaotic. As a matter of fact, if we randomly pick an initial point $x_0 \in [0,1]$, then the *limit set* of the sequence $\{x_n\}$ with $x_n = S_4^n(x_0)$ is the whole interval $[0,1]$, that is, for each $x \in [0,1]$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x$.

Chaotic dynamical systems are now very popular in science and engineering. Besides the original definition of Li-Yorke chaos in [88], there have been various definitions for "chaos" in the literature, and the most often used one is given by Devaney in [27]. Although there is no universal definition for chaos, the essential feature of chaos is *sensitive dependence on initial conditions* so that the eventual behavior of the dynamics is unpredictable. The theory and methods of chaotic dynamical systems have been of fundamental importance not only in mathematical sciences [22, 23, 27], but also in physical, engineering, biological, and even economic sciences [7, 18, 94, 98].

We have examined a family of discrete dynamical systems from the deterministic point of view and have observed the passage from order to chaos as the parameter value of the mappings changes. In the next section, we study chaos from another point of view, that is, from the probabilistic viewpoint.

1.2 Statistical Study of Chaos

Although a chaotic dynamical system exhibits unpredictability concerning the asymptotic behavior of the orbit starting from a generic point, it often behaves regularly as far as the statistical properties are concerned. In other words, a chaotic dynamical system in the deterministic sense may not be chaotic in the probabilistic sense.

In physical measurements, we often consider a probabilistic distribution of a physical quantity. Let $S : X \to X$ be a dynamical system on a phase space X of finite measure $\mu(X) < \infty$, and let A be a subset of X. Instead of observing the deterministic properties of individual orbits, let us consider the probabilistic properties by observing the *frequencies* of the first n terms of the orbit $\{S^n(x)\}$ of an initial point x that enter A for all natural numbers n. To calculate the frequency, let χ_A be the *characteristic function* of A, that is,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases} \tag{1.3}$$

Then, the frequency for a given n is exactly $n^{-1} \sum_{i=0}^{n-1} \chi_A(S^i(x))$. The *time average* or the *time mean*, which is the *asymptotic frequency* of all the terms of an orbit starting at $x \in X$ that enter A, is given by the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(S^i(x))$$

if it exists, which measures how frequently the orbit stays in A. The classical ergodic theory deals with the existence of the time average, their metric properties, and their close relationships with other mathematical concepts and quantities, which originated from *Boltzmann's ergodic hypothesis* in statistical mechanics. In our context, this hypothesis concerns the following question: given a *measure preserving* transformation $S : X \to X$, i.e., $\mu(S^{-1}(A)) = \mu(A)$ for all measurable subsets A of X, and an integrable function $f : X \to \mathbb{R}$, find the conditions under which the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(x)) \tag{1.4}$$

exists and is constant for $x \in X$ almost everywhere (a.e.).

In 1931, George D. Birkhoff proved that for any S and f the limit (1.4) exists for $x \in X$ almost everywhere, and furthermore, if S is *ergodic*, that is, $S^{-1}(A) = A$ implies that $A = \varnothing$ or X a.e., then the time average coincides with the *space average* or the *space mean*

$$\frac{\mu(A)}{\mu(X)} = \frac{1}{\mu(X)} \int_X \chi_A \mathrm{d}\mu$$

for $x \in X$ a.e. More specifically, the celebrated Birkhoff pointwise ergodic theorem [123] can be stated as follows (c.f. Theorem 3.3.1 in Chapter 3):

Theorem 1.2.1 (**Birkhoff's pointwise ergodic theorem**) *Let μ be a probability measure on X which is invariant under $S : X \to X$. Then, for any integrable function f defined on X and almost all $x \in X$, the time average*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(S^i(x))$$

exists and is denoted as $\tilde{f}(x)$. Moreover,

$$\tilde{f}(S(x)) = \tilde{f}(x), \ \ \forall\, x \in X \ \mu - \text{a.e.}$$

If in addition S is ergodic, then $\tilde{f}$ is the constant function $\int_X f \mathrm{d}\mu$.

Now, another question arises naturally: given a transformation $S : X \to X$, what measure μ on X is invariant under S? If we do not impose more requirements for μ, the answer may be trivial or of no physical importance. For example, for the logistic model $S(x) = 4x(1-x)$, since 0 is a fixed point of S, it is easy to see that the *Dirac measure* δ_0 concentrated at 0 is invariant, where $\delta_0(A) = 1$ if $0 \in A$ and $\delta_0(A) = 0$ if $0 \notin A$. In general, any fixed point a of S gives rise to an invariant measure δ_a, the Dirac measure concentrated at a. Note that the Dirac measure δ_a with $a \in [0, 1]$ is *not* absolutely continuous with respect to the Lebesgue measure of the unit interval. In other words, it cannot be represented as the integral of an integrable function on $[0, 1]$.

The existence of an invariant measure for a continuous transformation on a compact metric space has been established by the following theorem [123], which will be proved in Section 3.4.

Theorem 1.2.2 (**Krylov-Bogolioubov**) *Let X be a compact metric space and let $S : X \to X$ be a continuous transformation. Then, there is an invariant probability measure μ under S.*

In many applications, we are more interested in the existence and computation of invariant probability measures which are *absolutely continuous* with respect to a given measure. In other words, we want to find invariant measures that can be expressed as integrals of *density functions* with respect to the given measure. In this textbook, we intend to study this problem. Here the concept of Frobenius-Perron operators, which gives the corresponding way the density functions change under the deterministic dynamical system, plays an important role. Considering the iteration of the Frobenius-Perron operator leads us to the following observation: *chaos in the deterministic sense may not be so in the probabilistic sense.*

Let (X, Σ, μ) be a σ-finite measure space, let $S : X \to X$ be a nonsingular transformation, i.e., $\mu(A) = 0$ implies $\mu(S^{-1}(A)) = 0$ for all $A \in \Sigma$, and let

$P : L^1(X) \to L^1(X)$ be the Frobenius-Perron operator associated with S which is defined *implicitly* by the relation

$$\int_A Pf\mathrm{d}\mu = \int_{S^{-1}(A)} f\mathrm{d}\mu, \ \ \forall\, A \in \Sigma, \tag{1.5}$$

where $L^1(X)$ is the space of all integrable functions defined on X with respect to the measure μ (see Chapters 2 and 4 for their precise definitions). In Chapter 4, it will be proved that any fixed point f of P, which is also a density function, gives an absolutely continuous S-invariant probability measure μ_f on X defined by $\mu_f(A) = \int_A f\mathrm{d}\mu, \ \forall\, A \in \Sigma$.

The *existence* problem of fixed density functions of Frobenius-Perron operators is one of the main topics in modern ergodic theory. On the other hand, in physical sciences, one often needs to *compute* one or higher dimensional absolutely continuous invariant finite measures [7]. For example, in neural networks, condensed matter physics, turbulence in fluid flow, arrays of Josephson junctions, large-scale laser arrays, reaction-diffusion systems, etc., "coupled map lattices" often appear as models for phase transition, in which the evolution and convergence of density functions under the action of the Frobenius-Perron operator are examined. Understanding the statistical properties of these systems will become possible if we are able to calculate such global statistical quantities as invariant measures, entropy, Lyarpulov exponents, and moments. Thus, in many applied areas of physical sciences, not only the existence but also the computation of fixed density functions of Frobenius-Perron operators is essential for the investigation of the complicated dynamics.

However, the following two main difficulties make solving the above problems a challenge. First, the underlying space $L^1(X)$ is not reflexive in general, and second, the Frobenius-Perron operator P is usually not compact on $L^1(X)$. Thus, we can only apply some special techniques and the structure analysis to prove the existence and to develop convergent computational algorithms.

We use the following probabilistic argument to motivate the definition (1.5) of Frobenius-Perron operators, before we formally define this operator in Chapter 4. Consider again the dynamical system $S(x) = 4x(1-x)$. Instead of studying the eventual behavior of individual orbits, we investigate the asymptotic distribution of the iterates on $[0, 1]$ under S. In other words, we examine the flow of density functions of these iterates' distributions if the density function of the initial distribution is known. Here, we give an intuitive description of this approach. Pick a large positive integer n and apply S to each of the n initial states

$$x_1^0, \ x_2^0, \ \cdots, \ x_n^0,$$

and then we have n new states

$$x_1^1 = S(x_1^0), \ x_2^1 = S(x_2^0), \ \cdots, \ x_n^1 = S(x_n^0).$$

The initial states can be represented by a function f_0 in the sense that the integral of f_0 over any interval I (not too small) is roughly the fraction of the number of the states in the interval, that is,

$$\int_I f_0(x)\mathrm{d}x \simeq \frac{1}{n}\sum_{i=1}^{n} \chi_I(x_i^0).$$

f_0 is called the *density function* of the initial states. Similarly, the density function f_1 for the states $x_1^1, x_2^1, \cdots, x_n^1$ satisfies

$$\int_I f_1(x)\mathrm{d}x \simeq \frac{1}{n}\sum_{i=1}^{n} \chi_I(x_i^1).$$

Our purpose is to find a relation between f_1 and f_0.

For the given $I \subset [0,1]$,

$$x_i^1 \in I \quad \text{if and only if} \quad x_i^0 \in S^{-1}(I).$$

Thus, from the equality $\chi_I(S(x)) = \chi_{S^{-1}(I)}(x)$, we have

$$\int_I f_1(x)\mathrm{d}x \simeq \frac{1}{n}\sum_{i=1}^{n} \chi_{S^{-1}(I)}(x_i^0),$$

which implies that

$$\int_I f_1(x)\mathrm{d}x = \int_{S^{-1}(I)} f_0(x)\mathrm{d}x.$$

If we write f_1 as Pf_0, then the above relationship between f_1 and f_0 is

$$\int_I Pf_0(x)\mathrm{d}x = \int_{S^{-1}(I)} f_0(x)\mathrm{d}x.$$

The operator P that maps the density function f of the initial states to the density function Pf of the next states is actually the Frobenius-Perron operator corresponding to the transformation S, as defined by (1.5).

Let $I = [0,x]$. Then, differentiating both sides of the equality

$$\int_0^x Pf(t)\mathrm{d}t = \int_{S^{-1}([0,x])} f(t)\mathrm{d}t$$

with respect to x gives

$$Pf(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_{S^{-1}([0,x])} f(t)\mathrm{d}t.$$

Since

$$S^{-1}([0,x]) = \left[0, \frac{1}{2} - \frac{1}{2}\sqrt{1-x}\right] \cup \left[\frac{1}{2} + \frac{1}{2}\sqrt{1-x}, 1\right],$$

after carrying out the indicated differentiation, we obtain

$$Pf(x) = \frac{1}{4\sqrt{1-x}}\left[f\left(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}\right) + f\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-x}\right)\right].$$

which is an explicit formula for the Frobenius-Perron operator corresponding to the quadratic mapping S. This formula tells us how S *transforms* a given density function f into a new density function Pf. In particular, if the initial density function $f(x) \equiv 1$, that is, if the initial distribution of the states is uniform, then the distribution of the new states under S is given by the density function

$$Pf(x) = \frac{1}{2\sqrt{1-x}}.$$

If we keep iterating, we can see that the density function sequence $\{P^n f(x)\}$ will approach the density function

$$f^*(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

as $n \to \infty$, which satisfies $Pf^* = f^*$. This fixed density function for the logistic model $S(x) = 4x(1-x)$ was found by Ulam and von Neumann [121] in 1947.

It turns out that the probability measure μ^* defined by

$$\mu^*(A) = \int_A f^*(x)\mathrm{d}x, \quad \forall \text{ measurable } A \subset [0,1],$$

which is absolutely continuous with respect to the Lebesgue measure on $[0,1]$, is invariant under the quadratic polynomial S. Thus, the chaotic dynamical system in the deterministic sense is *stable* in the probabilistic sense, that is, the probability distribution of the states of the iterates of S will approach eventually the stationary probability distribution given by f^*.

In this book, we shall mainly study Frobenius-Perron operators and the related concept of absolutely continuous invariant finite measures. There are two main issues that we would like to discuss: the existence of fixed density functions of Frobenius-Perron operators and their numerical computation. The main mathematical foundation for achieving our goals is integration theory and functional analysis, and a useful analytic tool is the concept of variation. So, in the next chapter we introduce the basis of measure theory and functional analysis as preliminaries for the subsequent chapters on the theoretical and numerical analysis of Frobenius-Perron operators.