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Poncelet Porisms and Beyond

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Poncelet Porisms and Beyond

Integrable Billiards, Hyperelliptic Jacobians and Pencils of Quadrics

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Contents

Chapter 1

Introduction to Poncelet Porisms

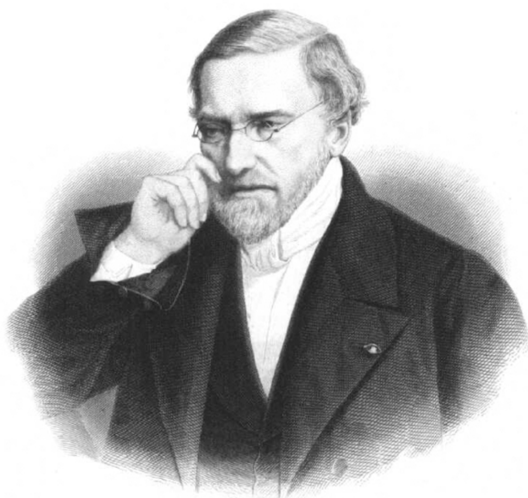

Figure 1.1: Jean Victor Poncelet

"One of the most important and also most beautiful theorems in classical geometry is that of Poncelet (...) His proof was synthetic and somewhat elaborate in what was to become the predominant style in projective geometry of last century. Slightly thereafter, Jacobi gave another argument based on the addition theorem for elliptic functions. In fact, as will be seen below, the Poncelet theorem and addition theorem are essentially equivalent, so that at least in principle Poncelet gave a synthetic derivation of the group law on an elliptic curve. Because of the appeal of the Poncelet theorem it seems reasonable to look for higher-dimensional analogues... Although this has not yet turned out to be the case in the Poncelet-type problems..."

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These introductory words from [GH1977], written by Griffiths and Harris exactly 30 years ago, serve as a motto of the present book.

In a few years, we are going to reach a significant anniversary, the bicentennial of Jean Victor Poncelet's proof of one of the most beautiful and most important theorems of projective geometry. As is well known, he proved it during his captivity in Russia, in Saratov in 1813, after Napoleon's wars against Russia. The first proof was in a sense an analytic one. In 1822, Poncelet published another, purely geometric, synthetic proof in his *Traité des propriétés projectives des figures* [Pon1822]. Suppose that two ellipses are given in the plane, together with a closed polygonal line inscribed in one of them and circumscribed about the other one. Then, Poncelet's theorem states that infinitely many such closed polygonal lines exist – every point of the first ellipse is a vertex of such a polygon. Besides, all these polygons have the same number of sides. Later, using the addition theorem for elliptic functions, Jacobi gave another proof of the theorem in 1828 (see [Jac1884a]). Essentially, Poncelet's theorem is equivalent to the addition theorems for elliptic curves and his proof represents a synthetic way of deriving the group structure on an elliptic curve. Another proof of Poncelet's theorem, in a modern, algebro-geometrical manner, was done quite recently by Griffiths and Harris (see [GH1977]). There, they also presented an interesting generalization of the Poncelet theorem to the three-dimensional case, considering polyhedral surfaces both inscribed and circumscribed about two quadrics.

If we have in mind the geometric interpretation of the group structure on a cubic (see Figure 1.2), then the question of finding an analogous construction of the group structure in higher genera arises.

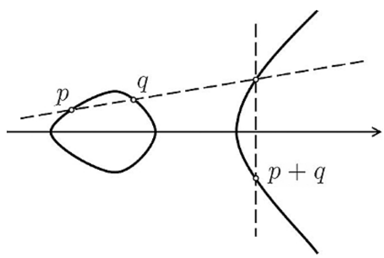

Figure 1.2: The group law on the cubic curve

Thus, thirty years ago, Griffiths and Harris announced a program of understanding higher-dimensional analogues of Poncelet-type problems and a synthetic approach to higher genera addition theorems.

The main aim of the present book is to report on progress made in settling and completing of this program. We will also present in a quite systematic way the most important results and ideas around Poncelet's theorem, both classical and modern, together with their historical origins and natural generalizations.

A natural question connected with Poncelet's theorem is to find an analytical condition determining, for two given conics, if an n-polygon inscribed in one and circumscribed about the second conic exists. In a short paper [Cay1854], Cayley derived such a condition in 1853, using the theory of Abelian integrals. He had dealt with Poncelet's porism in a number of other papers [Cay1853, Cay1855, Cay1857, Cay1858, Cay1861]. Inspired by [Cay1854], Lebesgue translated Cayley's proof to the language of geometry. Lebesgue's proof of Cayley's condition, derived by methods of projective geometry and algebra, can be found in his book *Les coniques* [Leb1942]. In modern settings, Griffiths and Harris derived Cayley's theorem by finding an analytical condition for points of finite order on an elliptic curve [GH1978a].

It is worth emphasizing that Poncelet, in fact, proved a statement that is much more general than the famous Poncelet theorem [Ber1987, Pon1822], then deriving the latter as a corollary. Namely, he considered $n+1$ conics of a pencil in the projective plane. If there exists an n-polygon with vertices lying on the first of these conics and each side touching one of the other n conics, then infinitely many such polygons exist. We shall refer to this statement as the *Full Poncelet theorem* and call such polygons *Poncelet polygons*.

A nice historical overview of the Poncelet theorem, together with modern proofs and remarks is given in [BKOR1987]. Various classical theorems of Poncelet type with short modern proofs are reviewed in [BB1996], while the algebro-geometrical approach to families of Poncelet polygons via modular curves is given in [BM1993, Jak1993].

Figure 1.3: Elliptical billiard table

The Poncelet theorem has an important mechanical interpretation. *An Elliptical billiard* [KT1991, Koz2003] is a dynamical system where a material point of the unit mass is moving under inertia, or in other words, with a constant velocity inside an ellipse and obeying the reflection law at the boundary, i.e., having congruent impact and reflection angles with the tangent line to the ellipse at any bouncing point. It is also assumed that the reflection is absolutely elastic. It is well known that any segment of a given elliptical billiard trajectory is tangent to the same conic, confocal with the boundary [CCS1993]. If a trajectory becomes closed after n reflections, then the Poncelet theorem implies that any trajectory of the billiard system, which shares the same caustic curve, is also periodic with the period n.

The Full Poncelet theorem also has a mechanical meaning. The configuration dual to a pencil of conics in the plane is a family of confocal second-order curves [Arn1978]. Let us consider the following, a little bit unusual billiard. Suppose n confocal conics are given. A particle is bouncing on each of these n conics respectively. Any segment of such a trajectory is tangent to the same conic confocal with the given n curves. If the motion becomes closed after n reflections, then, by the Full Poncelet theorem, any such trajectory with the same caustic is also closed.

The statement dual to the Full Poncelet theorem can be generalized to the d-dimensional space [CCS1993] (see also [Pre1999, Pre]). Suppose vertices of the polygon $x_1x_2\dots x_n$ are respectively placed on confocal quadric hypersurfaces $\mathcal{Q}_1$, $\mathcal{Q}_2$, …, $\mathcal{Q}_n$ in the d-dimensional Euclidean space, with consecutive sides obeying the reflection law at the corresponding hypersurface. Then all sides are tangent to some quadrics $\mathcal{Q}^1$, …, $\mathcal{Q}^{d-1}$ confocal with $\{\mathcal{Q}_i\}$; for the hypersurfaces $\{\mathcal{Q}_i, \mathcal{Q}^j\}$, an infinite family of polygons with the same properties exist.

But, more than one century before these quite recent results, in 1870, Darboux proved the generalization of Poncelet's theorem for a billiard within an ellipsoid in the three-dimensional space [Dar1870]. It seems that his work on this topic is completely forgotten nowadays.

Darboux was occupied by Poncelet's theorem for almost 50 years, and many of his results and ideas, in one way or another, are going to be incorporated throughout the book.

Let us mention that in the same year, 1870, appeared another very important work: [Wey1870] of Weyr. It can be treated as the historic origin of the modern Griffits–Harris Space Poncelet Theorem. A few years later, Hurwitz used Weyr's results to get a new proof of the standard Poncelet theorem (see [Hur1879]).

It is natural to search for a Cayley-type condition related to generalizations of the Poncelet theorem. Such conditions for the billiard system inside an ellipsoid in the Eucledean space of arbitrary finite dimension were derived in [DR1998a, DR1998b]. In recent papers [DR2004, DR2005, DR2006b, DR2006a], algebro-geometric conditions for existence of periodical billiard trajectories within k quadrics in d-dimensional Euclidean space were derived. The second important goal of these papers, actually for the present book as well, was to offer a thorough historical overview of the subject with a special attention on the detailed analysis of ideas and contributions of Darboux and Lebesgue. While Lebesgue's work on this subject has been, although rarely, mentioned by experts, on the other hand, it seems to us that relevant Darboux's ideas are practically unknown in contemporary mathematics. We give natural higher-dimensional generalizations of the ideas and results of Darboux and materials presented by Lebesgue, providing the proofs also in the low-dimensional cases if they were omitted in the original works. Besides other results, interesting new properties of pencils of quadrics are established – see Theorems 5.30 and 5.33. The latter gives a nontrivial generalization of the Basic Lemma from Lebesgue's book.

In our presentation of the development connected with the Griffiths–Harris program, we follow the recent paper [DR2008]. We present a geometric construction generalizing a summation procedure on the elliptic curve for the case of hyperelliptic Jacobians. These ideas are continuations of those of Reid, Donagi and Knörrer, see [Rei1972], [Knö1980], [Don1980]. Further development, realization, simplification and visualization of their constructions is obtained by using the ideas of billiard dynamics on pencils of quadrics developed in [DR2004].

The projective geometry nucleus of that billiard dynamics is the Double Reflection Theorem, see Theorem 5.27 below. There are four lines belonging to a certain linear space and forming *the Double reflection configuration*: these four lines reflect to each other according to **the billiard law at some confocal quadrics**.

In higher genera, we construct the corresponding, more general, billiard configuration, again by using the Double Reflection Theorem. This configuration, which we call *s-brush*, is in one of the equivalent formulations, a certain billiard trajectory of length $s \leq g$ and the sum of s elements in the brush is, roughly speaking, the final segment of that billiard trajectory.

The milestones of this presentation are [Knö1980] and [DR2004] and the key observation, from [DR2008], giving a link between them is *that the correspondence* $g \mapsto g'$ *in Lemma* 4.1 *and Corollary* 4.2 *from* [Knö1980] *is* **the billiard map at the quadric** $\mathcal{Q}_\lambda$.

Thus, after observing and understanding the billiard nature behind the constructions of [Rei1972], [Knö1980], [Don1980], we become able to use the billiard tools to construct and study hyperelliptic Jacobians, and particularly their real part. It may be realized as a set T of lines in $\mathbf{R}^d$ simultaneously tangent to given $d-1$ quadrics $\mathcal{Q}_1, \ldots, \mathcal{Q}_{d-1}$ of some confocal family. It is well known that such a set T is invariant under the billiard dynamics determined by quadrics from the confocal family. By using the Double Reflection Theorem and some other billiard constructions we construct a group structure on T, *a billiard algebra*. The usage of billiard dynamics in algebro-geometric considerations appears to be, as usual in such a situation, of a two-way benefit. We derive a fundamental property of T: *any two lines in* T *can be obtained from each other by at most* $d-1$ *billiard reflections at some quadrics from the confocal family.* The last fact opens a possibility to introduce new hierarchies of notions: of *s-skew lines in* T, $s = -1, 0, \ldots, d-2$ and of *s-weak Poncelet trajectories of length* n. The last are natural quasi-periodic generalizations of Poncelet polygons. By using billiard algebra, we obtain complete analytical descriptions of them. These results are further generalizations of our recent description of Cayley's type of Poncelet polygons in arbitrary dimension, see [DR2006b]. Let us emphasize that the method used in [DR2008], based on billiard algebra, differs from the methods exposed in [DR2006b], see also [DR2010]. Both of the methods will be presented in the sequel.

The interrelations between billiard dynamics, subspaces of intersections of quadrics and hyperelliptic Jacobians developed in [DR2008], enable us to obtain higher-dimensional generalizations of several classical results. To demonstrate the power of the methods, generalizations of Weyr's Poncelet theorem (see [Wey1870]) and also the Griffiths–Harris Space Poncelet theorem (see [GH1977]) in arbitrary dimension are derived and presented here. We also give an arbitrary-dimensional generalization of the Darboux theorem [Dar1914].

Let us mention at the end of a brief outline of main results which are going to be presented here, that the line we are going to establish and follow, is to demonstrate the deep intimate relationship between on one hand general hyperelliptic Jacobians and integrable billiard systems generated by pencils of quadrics on the other hand. This can be seen as a very simple and specialized level of general ideology of integrable systems which culminated with the so-called Novikov's conjecture, solved by Shiota in 1985.

Let us recall that Novikov's conjecture demonstrates the deepest relationship between the theory of integrable dynamical systems and theory of algebraic curves. It solved a century old, general and important Riemann–Schottky problem of description of period matrices of Jacobians among Riemannian matrices through the solutions of the Kadomtsev–Petviashvili integrable hierarchy.

There is another, very important connection of our subject with some of the most prominent parts of contemporary mathematics.

The Euler–Chasles correspondences, or symmetric (2-2)-correspondences play one of the main roles in our exposition. They were used by Jacobi, then by Trudi [Tru1853, Tru1863] and finally, Darboux extended their use in the theory of Poncelet porisms essentially.

One of the central objects in mathematical physics in the last 25 years is the R-matrix, or the solution $R(t, h)$ of the quantum Yang–Baxter equation

$$R^{12}(t_1 - t_2, h)R^{13}(t_1, h)R^{'23}(t_2, h) = R^{23}(t_2, h)R^{13}(t_1, h)R^{12}(t_1 - t_2, h),$$

as a paradigm of modern understanding of the addition relation. Here t is a so-called *spectral parameter* and h is the *Planck constant*. If the h dependence satisfies the quasi-classical property $R = I + hr + O(h^2)$, the classical r-matrix r satisfies the classical Yang–Baxter equation. Classification of the solutions of the classical Yang–Baxter equation was done by Belavin and Drinfeld in 1982 [BD1982]. The problem of classification of the quantum R-matrices is still open. However, some important results of classification have been obtained in the basic 4×4 case by Krichever in [Kri1981], and following his ideas in [Dra1992a, Dra1993].

Krichever in [Kri1981] applied the idea of "finite-gap" integration to the theory of the Yang equation:

$$R^{12}L^{13}L^{'23} = L^{'23}L^{13}R^{12}.$$

The principal objects that are considered are $2n \times 2n$ matrices L, understood as 2×2 matrices whose elements are $n \times n$ matrices; $L = L^{i\alpha}_{j\beta}$ is considered as a linear operator in the tensor product $\mathbf{C}^n \otimes \mathbf{C}^2$. The theorem from [Kri1981] uniquely characterizes them by the following spectral data:

1. the vacuum vectors, i.e., vectors of the form $X \otimes U$, which L maps to vectors of the same form $Y \otimes V$, where $X, Y \in \mathbf{C}^n$ and $U, V \in \mathbf{C}^2$;
2. the vacuum curve $\Gamma : P(u, v) = \det L = 0$, where $L^i_j = V^\beta L^{i\alpha}_{j\beta} U_\alpha$, $(V^\beta) = (1, -v)$, $X_n = Y_n = U_2 = V_2 = 1$; $U_1 = u$, $V_1 = v$;
3. the divisors of the vector-valued functions $X(u, v)$, $Y(u, v)$, $U(u, v)$, $V(u, v)$, which are meromorphic on the curve Γ.

It appeared that vacuum curves in 4×4 case are exactly Euler–Chasles correspondences. The Yang–Baxter equation itself provides the condition of commutation of the two Euler–Chasles correspondences. The classification follows by application of the Euler theorem in the general case, and by studying possible degenerations.

This is practically the same picture we meet in the study of the Poncelet theorem. The hope is that our study of higher-dimensional analogues of the Poncelet theorem could provide us the intuition that will help us in classification of higher-dimensional solutions of the Yang–Baxter equation.

Thus, we include the story about Krichever's algebro-geometric approach to 4×4 solutions of the Quantum Yang–Baxter equation in the last chapter. We explained there the relationship between the Poncelet theorem for a triangle and the Darboux theorem from one side and Krichever's commuting relation of vacuum curves from another side (see Theorem 10.12). We underline connection of classification results for 4×4 R-matrices to the classification of pencils of conics, see Theorem 10.12 and Proposition 10.13. Pencils of conics and their classification played a crucial role in previous chapters. Finally, we point out a sort of billiard construction within the Algebraic Bethe Ansatz associated to four-dimensional R-matrices, see Lemma 10.14 and Theorem 10.15.

The Poncelet theorem is usually called the Poncelet *porism*. Let us give some explanation of the meaning of the word **porism**. It has roots in ancient Greek mathematics, and it is usually translated in two ways. The first one is *lemma* or *corollary*. The second one goes deeper into the philosophy of ancient Greek mathematics. Scientists of that time used to divide mathematical statements into two categories:

- *Theorems* – where something has to be proven, and
- *Problems* – where something needs to be constructed.

Nevertheless, they recognized the third, intermediate, class as well, called *Porisms*, directed to finding what is proposed. The most famous collection of porisms of ancient times was the book *The Porisms* of Euclid. Unfortunately, this work is lost, and the trace which survived leads through *The Collection* of Pappus of

Alexandria. Even then, there was much discussion about the definition of the notion of porism as well as about Euclid's porisms. These discussions continue today. In the XVII century, important contributions were made by Albert Girard and Pierre Fermat. In the XVIII century, we can mention Robert Simson and John Playfair. Here is Simson's definition of a porism.

> *"Porisma est propositio in qua proponitur demonstrate rem aliquam vel plures Batas ease, cui vel quibus, ut et cuilibet ex rebus innumeris non quidem datis, sed quae ad ea quae data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in propositione descriptam. Porisma etiam in forma problematis enuntiari potest, si nimirum ex quibus data demonstranda aunt, invenienda proponantur."*

Playfair, continuing the work of Simson, tried to understand the probable origin of porisms, to find out what led the ancient geometers to the discovery of them. He remarked that the careful investigation of all possible particular cases of a proposition would show that:

(1) under certain conditions a problem becomes impossible;

(2) under certain other conditions, indeterminate or capable of an infinite number of solutions.

For more details see [1911, E.B.].

This is exactly the situation we recognize in the Poncelet theorem. For two given conics, there are two possibilities. Either, a polygon inscribed in one of them and circumscribed about the other has an infinite number of sides, or the number of sides is finite. If it is finite, then the number of sides does not depend on an initial point. We want to stress here that the idea of porism of Poncelet type, in a very special case, existed almost 70 years before Poncelet. This case of Poncelet's theorem is the one with two circles, inscribed and circumscribed about the same triangle. We come to such a situation starting from an arbitrary triangle, and considering its inscribed and circumscribed circle. Denote by r and R their radii respectively, and by d the distance between the centers of the circles. The formula connecting these three values, sometimes referred as "Euler's formula" is well known:

$$d^2 = R^2 - 2rR.$$

However, this relation was discovered by English mathematician Chapple in 1746, and he caught sight of the *poristic* nature of the problem: if there are two circles satisfying the last *Chapple formula*, **then there are infinitely many triangles inscribed in one and circumscribed about the other circle**. Probably, this is the first known appearance of porisms of Poncelet type.

The Euler school was also interested in that subject. Nicolas Fuss, one of Euler's personal secretaries, and after Euler's death the secretary of St. Petersburg

Academy of Sciences, published several works on study of bicentric polygons. In 1797 he published the formula for bicentric quadrilaterals:

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2).$$

But, although it was 50 years after Chapple, Fuss did not understand the poristic nature of the problem.

It was Jacobi in 1828 who understood the relationship between Poncelet porism in general and study of bicentric polygons of Fuss, Steiner and others.

Some parts of the material presented here were used by the authors for graduate courses they taught: V. D. in 2002/2003 in the International School of Advanced Studies in Trieste [Dra2003], and M. R. in 2006 in the Weizmann Institute of Science in Rehovot. Both authors read mini-courses on the subject, M. R. in the Weizmann Institute of Sciences in 2005 and V. D. at the University of Lisbon in 2007. Also, both authors gave several lectures on seminars and conferences in Italy, France, Germany, Serbia, Spain, Portugal, Montenegro, Israel, Czechia, Poland, Hungary, Great Britain, Austria, Russia, Brazil, USA, Canada, and Bulgaria. One of our observations was that there was a visible division between the communities of Algebraic and Projective Geometry, although some 50 years ago these fields were quite a unified subject. Having this experience in mind, we decided to include introductions to both subjects in order to make the book self-contained as much as possible and usable for both communities and for the mathematical community at large.

Acknowledgement

For many years we felt support and constant care about our work from Professor Boris Dubrovin and he was the one who suggested us to write this book. Enthusiastic discussions about the subject and presentations by some of the leading world experts in the fields such as Philip Griffiths, Marcel Berger, and Valery Kozlov were very encouraging and stimulating for us. We learned a lot from numerous discussions with our distinguished colleagues: Alexander Veselov, Alexey Bolsinov, Victor Buchstaber, Igor Krichever, Yuri Fedorov, Emma Previato, Borislav Gajić, Božidar Jovanović, Rade Živaljević, Gojko Kalajdžić, Vered Rom-Kedar, Jean-Claude Zambrini, Simonetta Abenda, Alexey Borisov, Armando Treibich, Nikola Burić… It is our great pleasure to thank them all.

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Chapter 2

Billiards – First Examples

2.1 Introduction to billiards

Let us start from the following well-known problem.

Suppose that a railway is passing near two neighbouring villages, and that a new railway station, common for both of them, is about to be built. Where to place the station, in order to minimize the length of the road connecting the villages with it? (See Figure 2.1.)

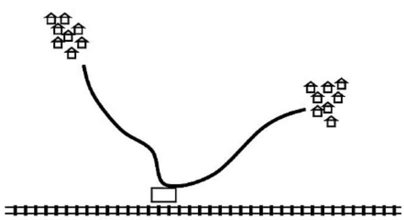

Figure 2.1.

In other words, on a given line (i.e., the railway), we need to find a point such that the sum of its distances from two fixed points is smallest possible.

Denote the given points (villages) by A and B, and by r the line (railway). Let B' be the point symmetric to B with respect to r. The intersection point S of r with AB' has the requested properties. Indeed, notice that $AS+SB = AS+SB' = AB'$, while for any other point $S' \in r$, $AS'+S'B = AS'+S'B' > AB'$. (See Figure 2.2.)

It is easy to see that segments AS and BS form the same angles with the line r, i.e., the segment BS is the billiard reflection of AS on the line r. In other words, *the minimal trajectory from A to B that meets the line r is exactly the billiard trajectory, with the reflection point on r.*

Exercise 2.1. Let M be a point inside a convex angle α. Find the points K, L on the sides of α such that the triangle KLM has the minimal perimeter. Prove that

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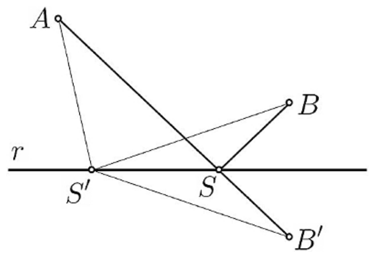

Figure 2.2.

segments MK,KL and KL,LM satisfy the billiard reflection law on the sides of the angle.

2.2 Triangular billiards

Now, we are going to investigate the billiards within a triangle in the Euclidean plane. A trajectory of such a billiard is a polygonal line, finite or infinite, with vertices on the sides of the triangle, such that consecutive edges of the trajectory satisfy the billiard law: i.e., they form the same angle with the side of the triangle on which their common vertex lies. The reflection is not defined only at the vertices of the triangle – thus we omit from our consideration trajectories falling at a vertex of a triangle.

Let us try to find out if closed trajectories of a billiard within a triangle exist. Denote by A, B, C the vertices of the triangle. It is clear, from Section 2.1, that the edges of the triangle with minimal perimeter, whose vertices are inner points of the sides $\triangle ABC$, will represent a billiard trajectory.

Theorem 2.2. *Let $\triangle ABC$ be an acute angled triangle. If $\triangle KLM$ is the triangle with minimal perimeter inscribed in $\triangle ABC$, then its vertices are the feet of the altitudes of $\triangle ABC$. Moreover, inside $\triangle ABC$, KLM is a unique closed billiard trajectory with* 3 *bounces.*

Proof. Let M be a fixed point on the edge AB. We want to find points $K \in BC$, $L \in AC$ such that the triangle KLM has the minimal perimeter. Denote by M', M'' points symmetric to M with respect to the sides BC, AC. It is easy to see that K,L are intersection points of $M'M''$ with BC,AC respectively (see Figure 2.3).

The perimeter of $\triangle KLM$ is equal to the segment $M'M''$. Notice that $M'M''$ is a side of the isosceles triangle $CM'M''$, with $CM' \cong CM'' \cong CM$ and $\angle M'CM'' = 2\angle BCA$. It follows that $M'M''$ will be the shortest for CM being an altitude of the triangle ABC, i.e., M being its foot. Similarly, we prove K, L are also feet of the corresponding altitudes (see Figure 2.4). $\square$

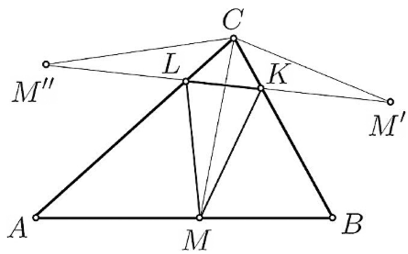

Figure 2.3.

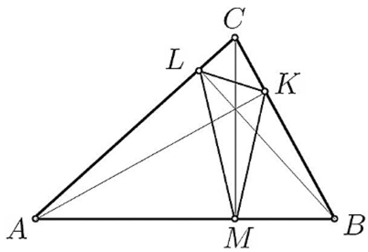

Figure 2.4.

After having this periodic trajectory inside an acute triangle, it is easy to see that there is an infinity of other closed billiard trajectories (see Figure 2.5).

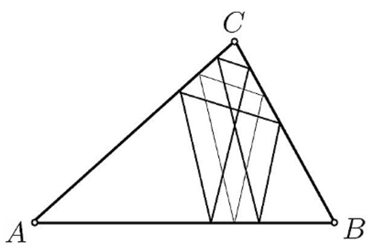

Figure 2.5.

There are also closed billiard trajectories inside a right triangle. One of them, the polygonal line $KLMNMLK$, is shown on Figure 2.6.

For obtuse triangles in general, the existence of periodic trajectories is not proved. There are only examples for some special cases.

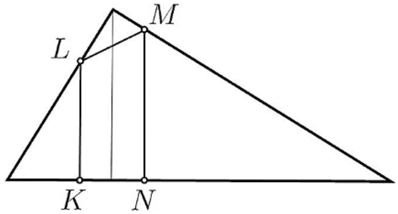

Figure 2.6.

2.3 Billiards within an ellipse

In this section, we are going to discuss in an elementary way the most important aspects of billiards within an ellipse in the plane.

A billiard trajectory within an ellipse is a polygonal line with the vertices lying on the ellipse and with consecutive edges satisfying *the billiard law*, i.e., forming the same angles with the tangent line to the ellipse at the joint vertex of the edges (see Figure 2.7).

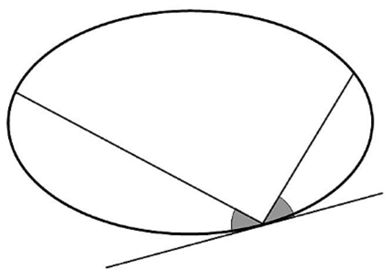

Figure 2.7.

Proposition 2.3 (Focal property of the ellipse). *Let $\mathcal{E}$ be an ellipse with foci F_1, F_2 and $A \in \mathcal{E}$ an arbitrary point. Then segments AF_1, AF_2 satisfy the billiard law on $\mathcal{E}$. (See Figure 2.8.)*

Proof. It is enough to prove that for any point C on the tangent, the sum $CF_1 + CF_2$ is greater than $AF_1 + AF_2$. Let B be the intersection of the segment CF_1 with the ellipse. Then $AF_1 + AF_2 = BF_1 + BF_2 < BF_1 + BC + CF_2 = CF_1 + CF_2$. (See Figure 2.9.) □

As an immediate consequence of this proposition, we have: if one segment of a billiard trajectory within ellipse $\mathcal{E}$ contains a focus of $\mathcal{E}$, then all segments of the trajectory contain one or the other focus, alternately.

Now, we are going to prove the following important property of the billiard within an ellipse.

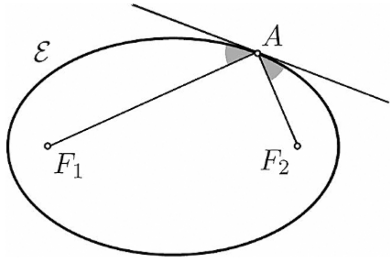

Figure 2.8: Focal property of the ellipse

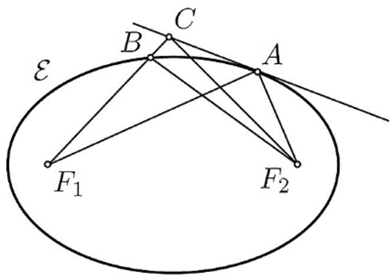

Figure 2.9.

Proposition 2.4. *Let two lines satisfy the billiard law on the ellipse $\mathcal{E}$. If one of the lines is tangent to the ellipse $\mathcal{E}'$ that is confocal with $\mathcal{E}$, then the other one is also tangent to $\mathcal{E}'$.* (*See Figure* 2.10.)

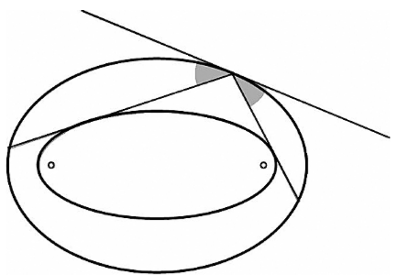

Figure 2.10.

Proof. Let A, B, C be points on $\mathcal{E}$ such that segments AB and BC satisfy the reflection law at B. Suppose that AB is tangent to $\mathcal{E}'$. Denote by F_1, F_2 the foci of the two ellipses and by T_1, T_2 points on the tangent to $\mathcal{E}$ in B such that angles $\angle F_1BT_1$ and $\angle F_2BT_2$ are acute. By Proposition 2.3, these angles are congruent. Notice that then AB is placed inside one of these angles, say $\angle F_1BT_1$. Since, by the billiard law, $\angle ABT_1 \cong \angle CBT_1$, the segment BC is placed inside angle $\angle F_2BT_2$, as shown on Figure 2.11. Let D_1, D_2 be points symmetric to F_1, F_2 with

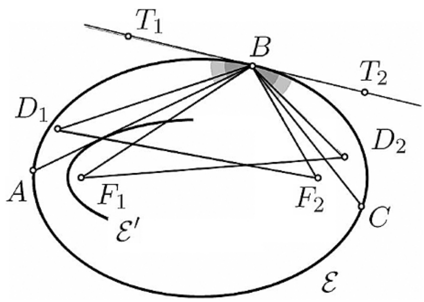

Figure 2.11.

respect to lines AB, BC respectively.

We are going to show that $\triangle D_1BF_2 \cong \triangle F_1BD_2$. We have that $D_1B \cong F_1B$, $F_2B \cong D_2F$, since the corresponding segments are symmetric with respect to AB, BC. Also, $\angle D_1BF_2 = \angle D_1BF_1 + \angle F_1BF_2$ and $\angle F_1BD_2 = \angle F_2BD_2 + \angle F_1BF_2$. Since $\angle D_1BF_1 = 2\angle F_1BA = 2(\angle F_1BT_1 - \angle ABT_1) = 2(\angle F_2BT_2 - \angle CBT_2) = 2\angle F_2BC = \angle D_2BT_2)$, we have that $\angle D_1BF_2 \cong \angle F_1BD_2$, which proves the congruence of triangles $\triangle D_1BF_2$ and $\triangle F_1BD_2$. Hence, $D_1F_2 \cong F_1D_2$.

The segment D_1F_2 is equal to the minimal sum of distances from F_1 and F_2 of a point on line AB, i.e., to the sum of distances of an arbitrary point on ellipse $\mathcal{E}'$ from its foci, since this line is touching $\mathcal{E}'$. Similarly, F_1D_2 is equal to the minimal sum of distances from F_1 and F_2 of a point on line BC, thus BC is also tangent to $\mathcal{E}'$. $\square$

We leave to the reader to prove the following

Exercise 2.5. Let two lines satisfy the billiard law on the ellipse $\mathcal{E}$. If one of the lines is tangent to the hyperbola $\mathcal{H}$ that is confocal with $\mathcal{E}$, then the other one is also tangent to $\mathcal{H}$.

From Propositions 2.4 and Exercise 2.5, immediately we derive

Corollary 2.6. *Let $\mathcal{T}$ be a billiard trajectory within ellipse $\mathcal{T}$. If $\mathcal{C}$ is a conic confocal to $\mathcal{E}$ such that one segment of $\mathcal{T}$ is tangent to $\mathcal{C}$, then all segments of $\mathcal{T}$ are tangent to $\mathcal{C}$.*

Corollary 2.7. *Let B be a point outside the ellipse $\mathcal{E}'$ with focal points F_1 and F_2. Denote tangents to the ellipse from the point B as BB_1 and BB_2, where B_i are points of contact with the ellipse. Then the angles B_1BF_1 and B_2BF_2 are equal.*

2.4 Periodic orbits of billiards and Birkhoff's theorem

We saw in Section 2.3 that billiards within an ellipse have remarkable properties. It would be interesting to examine closer periodical trajectories of these billiards. As a first step, let us present a classical result due to Birkhoff on periodic trajectories of a more general class of billiards.

Consider a billiard table bounded by a closed convex curve in the plane. Assume that the length of the curve is equal to 1 and introduce a natural parameter φ. Suppose that $\varphi_1, \dots, \varphi_n$ represents the sequence of bouncing points of an n-periodic trajectory. Additionally, we may choose the values $\varphi_1, \dots, \varphi_n, \varphi_{n+1}$ such that the differences $\varphi_2 - \varphi_1, \varphi_3 - \varphi_2, \dots, \varphi_n - \varphi_{n-1}, \varphi_{n+1} - \varphi_n$ are between 0 and 1, and $\varphi_{n+1} \equiv \varphi_1 \mod 1$.

The integer $k = \varphi_{n+1} - \varphi_1$ is called the *rotation number* of the periodic trajectory.

Theorem 2.8 (Birkhoff). *Suppose there is given a smooth, closed, convex plane curve having nonzero curvature at any point. Then for any numbers $n, k \in \mathbf{N}$, $n > k$, there exist at least two geometrically different periodic trajectories, with the rotational number k and n bounces, of the billiard within the given curve. For one of these trajectories, the corresponding polygonal line has maximal length among all nearby closed polygonal lines inscribed in the curve. If this maximum is an isolated critical point of the length function on the set of inscribed polygonal lines with n vertices, then the polygonal line corresponding to the other trajectory is not an isolated maximum of the length function.*

2.5 Bicentric polygons

We have shown in Section 2.3 that billiard trajectories within an ellipse have a caustic, which is a conic confocal to the boundary. Notice that any pair of conics in a plane can be, by a projective mapping, transformed into a confocal pair. Thus, it is natural to consider two general conics and polygonal lines inscribed in one and circumscribed about the other one.

The case when these two conics are circles can be analyzed in an elementary way.

Triangles

It is easy to prove, even to students of elementary schools, that it is possible to inscribe a circle in a triangle and also to circumscribe another one about it. Harder is to find out, for two given circles, if they are inscribed and circumscribed about some triangle. In the next proposition, a sufficient and necessary condition on two circles is given.

Theorem 2.9 (Chapple–Euler formula). *Let k and K be two given circles with radii r and R. Denote by d the distance between their centers. Then there exists a triangle inscribed in K and circumscribed about k if and only if*

$$d^2 = R^2 - 2Rr. \tag{2.1}$$

Moreover, if condition (2.1) is satisfied, then every point of K is a vertex of such a triangle.

Proof. Let ABC be a triangle inscribed in K and circumscribed about k and denote by O, S the centers of these circles. Let N, M be the second intersection points of lines AS, NO with K, as shown on Figure 2.12. It is easy to see that

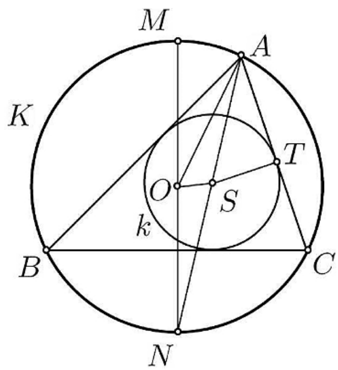

Figure 2.12.

ON is a bisector of BC.

The power of S with respect to circle K equals $d^2 - R^2 = -SA \cdot SN$.

From triangle SNC, we see that $SN = NC$. Namely, $\angle SCN = \angle SCB + \angle BCN = \dfrac{\angle BCA}{2} + \angle BAN = \dfrac{\angle BCA + \angle CAB}{2}$ and $\angle CSN = \angle SCA + \angle SAC = \dfrac{\angle BCA + \angle CAB}{2}$.

Thus, $SO^2 - R^2 = -SA \cdot SN = -SA \cdot NC$.

Notice that $\triangle AST \sim \triangle MNC$, where T is a common point of k and AC. Therefore, $AS \cdot NC = ST \cdot MN = r \cdot 2R$.

Thus, the relation (2.1) follows.

Now, suppose the equality (2.1) holds. Let A be an arbitrary point on K.

Denote by N the intersection of AS with K, and with B, C points on K such that $NB = NC = NS$. $\triangle ABC$ is inscribed in K and its inscribed circle k' is concentric with k. Then, $SO^2 = R^2 - 2Rr'$, where r' is the radius of k'. Since we also have $SO^2 = R^2 - 2Rr$, it is $r = r'$, i.e., $k = k'$. □

Quadrilaterals

In this section, we are going to prove the following

Theorem 2.10. *Let k and K be two given circles with radii r and R. Denote by d the distance between their centers. Then there exists a quadrilateral inscribed in K and circumscribed about k if and only if*

$$d^2 = r^2 + R^2 - r\sqrt{r^2 + 4R^2}. \tag{2.2}$$

Moreover, if condition (2.2) *is satisfied, then every point of K is a vertex of such a quadrilateral.*

First, let us prove several lemmata.

Lemma 2.11. *Let $ABCD$ be a cyclic quadrilateral. If K, L, M, N are normal projections of the intersection of the diagonals to the sides of $ABCD$, then quadrilateral $KLMN$ is circumscribed about a circle.*

Proof. Let $P = AC \cap BD$. Since $ABCD$, $AKPN$ and $KBLP$ are cyclic quadrilaterals, we have

$$\angle PKN \cong \angle PAN = \angle CAD \cong \angle CBD = \angle LBP \cong \angle LKP,$$

i.e., $\angle PKN \cong \angle LKP$. Thus KP is a bisector of $\angle NKL$. Similarly, LP, MP, NP

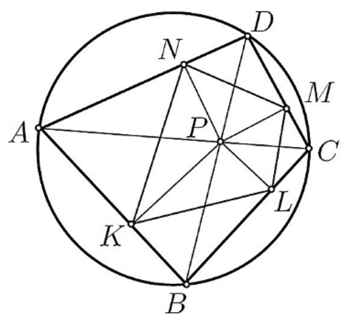

Figure 2.13.

are bisectors of corresponding angles, hence P is the center of the circle inscribed in $KLMN$. □

Lemma 2.12. *Let $ABCD$ be a cyclic quadrilateral and K, L, M, N as in Lemma* 2.11*. Additionally, suppose that AC is perpendicular to BD, $P = AC \cap BD$. Let O be the circumcenter and R_1 the circumradius of $ABCD$, r the inradius of $KLMN$, and $d_1 = OP$. Then*

$$r = \frac{R_1^2 - d_1^2}{2R_1}. \tag{2.3}$$

Proof. Let $\alpha = \angle DAC$, $\beta = \angle CAB$. Then we have

$$\begin{aligned} r &= PK \cdot \sin\alpha \\ &= PB \cdot \sin(90^\circ - \beta) \cdot \sin\alpha \\ &= PB \cdot PD \cdot \frac{\sin(90^\circ - \beta) \cdot \sin\alpha}{PD} \\ &= |p_{P,k}| \cdot \frac{\sin(90^\circ - \beta) \cdot \sin\alpha}{PD} \\ &= (R_1^2 - d_1^2) \cdot \frac{\sin(90^\circ - \beta) \cdot \sin\alpha}{AD \cdot \sin\alpha} \\ &= (R_1^2 - d_1^2) \cdot \frac{\sin(90^\circ - \beta)}{2R_1 \cdot \sin(90^\circ - \beta)} \\ &= \frac{R_1^2 - d_1^2}{2R_1}. \end{aligned}$$ □

Lemma 2.13. *Let $ABCD$ be a cyclic quadrilateral with $AC \perp BD$, P, K, L, M, N, R_1, O, d_1 as in Lemmata 2.11 and 2.12, and S_1, S_2, S_3, S_4 midpoints of AB, BC, CD, AD (see Figure 2.14). Then $K, L, M, N, S_1, S_2, S_3, S_4$ belong to the*

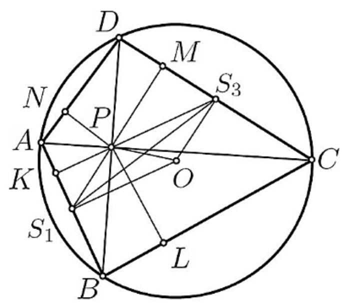

Figure 2.14.

same circle. If R is its radius, then

$$R = \frac{1}{2}\sqrt{2R_1^2 - d_1^2}. \tag{2.4}$$

Proof. First, let us prove that $PS_3 \perp AB$, i.e., S_3, P, K are collinear. Namely,

$$\begin{aligned} \angle S_3PK &= \angle KPA + \angle APD + \angle DPS_3 \\ &= 90^\circ - \angle PAK + 90^\circ + \angle DPS_3 \\ &= 180^\circ - \angle PDC + \angle DPS_3. \end{aligned}$$

Since S_3 is the midpoint of CD, we have $\angle DPS_3 = \angle PDS_3$. Thus $\angle S_3PK = 180^\circ$.

It follows that $S_3P \parallel OS_1$, because both lines OS_1, S_3P are perpendicular to AB. Similarly, $OS_3 \parallel PS_1$. It follows that OS_3PS_1 is a parallelogram. Thus

$$\begin{aligned} S_1S_3^2 + OP^2 &= 2 \cdot \left(PS_3^2 + PS_1^2\right) \\ &= 2 \cdot \left(\left(\frac{CD}{2}\right)^2 + \left(\frac{AB}{2}\right)^2\right) \\ &= \frac{1}{2}\left((2R_1 \sin\alpha)^2 + (2R_1 \sin(90^\circ - \alpha))^2\right) \\ &= 2R_1^2, \end{aligned}$$

with $\alpha = \angle DAC$. Hence, $S_1S_3^2 = 2R_1^2 - OP^2 = 2R_1^2 - d_1^2$.

Now, we are going to prove that all points K, L, M, N, S_1, S_2, S_3, S_4 belong to the circle with radius $S_1S_3/2$ and center at the midpoint of OP. Clearly, S_1S_3 is a diameter of this circle, and K, M belong to it because $\angle S_3MS_1 = \angle S_3KS_1 = 90^\circ$. We get the same for S_2, S_4, L, N.

Finally, we have $R = \frac{1}{2}S_1S_3 = \frac{1}{2}\sqrt{2R_1^2 - d_1^2}$. □

Now, let us return to the bicentric quadrilateral from Theorem 2.10 – denote it by $KLMN$.

Construct lines perpendicular to the bisectors of angles of $KLMN$ at the vertices. These lines determine a quadrilateral $ABCD$. Let P be the incenter of $KLMN$.

It is easy to prove that $ABCD$ is inscribed in a circle, that P is the intersection of its diagonals and that the diagonals are perpendicular to each other.

Denote as in previous lemmata: O – the center of the circle circumscribed about $ABCD$; R_1 – its radius; r, R – the radii of inscribed and circumscribed circle of $KLMN$; $d_1 = PO$; S – the midpoint of PO; $d = SO = d_1/2$.

By Lemmata 2.12 and 2.13,

$$r = \frac{R_1^2 - 4d^2}{2R_1} \quad \text{and} \quad R = \frac{1}{2}\sqrt{2R_1^2 - 4d^2}. \tag{2.5}$$

Eliminating R_1 from (2.5), we get

$$\frac{1}{r^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}. \tag{2.6}$$

From (2.6) it is possible to express d and obtain equation (2.2).

Now, let us prove the opposite part of Theorem 2.10. Suppose that $k(P, r)$ and $K(S, R)$ are given circles, such that equation (2.2) holds, $d = PS$.

Construct the point O such that S is the midpoint of OP and the circle $K_1(O, R_1)$, where R_1 satisfies (2.3) with $d_1 = 2d$. Notice that equation (2.3) is quadratic with respect to R_1, but only one of its solutions is positive.

Take A to be an arbitrary point on K_1 and construct $B, C, D \in K_1$ such that $AC \cap BD = P$, $AC \perp BD$. Denote by K, L, M, N the normal projections of P to the sides of $ABCD$. By Lemmata 2.11 and 2.12, $KLMN$ is circumscribed about k. By Lemma 2.13, $KLMN$ is also inscribed in a circle with center O and radius equal to $\frac{1}{2}\sqrt{2R_1^2 - d_1^2}$. Eliminating r from (2.2) and (2.3), we obtain that this is equal to R.

Thus, we constructed a quadrilateral $KLMN$ inscribed in K and circumscribed about k. In the construction, choosing arbitrarily the initial point A on K_1, we can get that any point on K can be a vertex of such a quadrilateral.

2.6 Poncelet theorem

In this section, we are going to state the Poncelet theorem and give its mechanical interpretation. The proof of the theorem will be given at the end of Chapter 4 and in Chapter 5.

Poncelet theorem

As we already mentioned, the Poncelet theorem is one of the most beautiful and deepest theorems of geometry, with numerous consequences and interrelations in a wide range of areas of mathematics. It was proved by Jean Victor Poncelet, while he was imprisoned in Russia, in 1813. He published another proof in 1822 in [Pon1822].

Theorem 2.14 (Poncelet Theorem). *Let $\mathcal{C}$ and $\mathcal{D}$ be two conics in the plane. Suppose that there is a polygon inscribed in $\mathcal{C}$ and circumscribed about $\mathcal{D}$. Then there are infinitely many such polygons and all of them have the same number of sides. Moreover, each point of $\mathcal{C}$ is a vertex of such a polygonal line.*

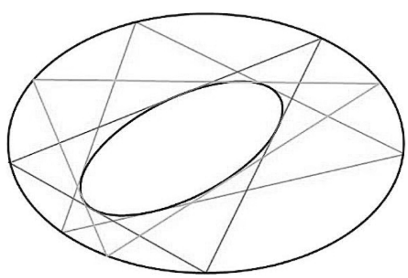

Figure 2.15: Three triangles inscribed in an ellipse and circumscribed about the other one

Mechanical interpretation of the Poncelet theorem

Figure 2.16: Elliptical billiard table

The Poncelet theorem obtains a natural and beautiful mechanical interpretation, if we take that $\mathcal{C}$ is an ellipse and $\mathcal{D}$ a conic confocal to $\mathcal{C}$. Then, as was shown in Section 2.3, the polygonal lines inscribed in $\mathcal{C}$ and circumscribed about $\mathcal{D}$ are trajectories of the billiard motion within $\mathcal{C}$.

In other words, consider a billiard trajectory within ellipse $\mathcal{C}$. Suppose that a line containing one segment of the trajectory is tangent to a conic $\mathcal{D}$, confocal with $\mathcal{C}$. Then, all segments of the trajectory are also tangent to $\mathcal{D}$ (see Figure 2.17).

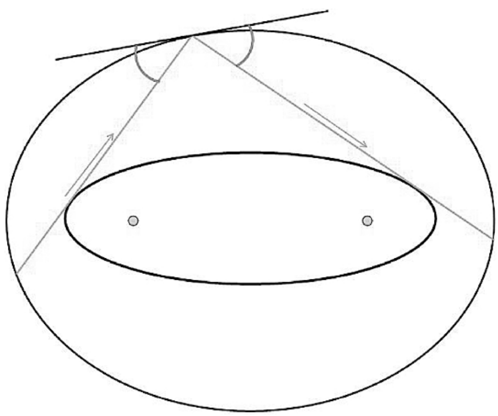

Figure 2.17: Billiard system and confocal conics

If a billiard trajectory is not periodic, then it will densely wind in the region bounded with the caustic and the billiard boundary as is shown on Figure 2.18.