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Nonconvex Optimization and Its Applications

Shashi Kant Mishra *Editor*

Topics in Nonconvex Optimization

Theory and Applications

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Topics in Nonconvex Optimization

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I would like to dedicate this volume to my teacher Prof. R. N. Mukherjee, who introduced this wonderful field of mathematics to me. I would also like to dedicate this volume to Prof. B. D. Craven who showed me the path in this research area.

Foreword

It is a great pleasure to learn that the Centre for Interdisciplinary Mathematical Sciences and the Department of Mathematics, Banaras Hindu University organized an Advanced Training Programme on Nonconvex Optimization and Its Applications. This programme was organized to introduce the subject to young researchers and college teachers working in the area of nonconvex optimization.

During the five-day period several eminent professors from all over the country working in the area of optimization gave expository to advanced level lectures covering the following topics.

(i) Quasi-convex optimization
(ii) Vector optimization
(iii) Penalty function methods in nonlinear programming
(iv) Support vector machines and their applications
(v) Portfolio optimization
(vi) Nonsmooth analysis
(vii) Generalized convex optimization

Participants were given copies of the lectures. I understand from Dr. S. K. Mishra, the main organizer of the programme, that the participants thoroughly enjoyed the lectures related to nonconvex programming. I am sure the students will benefit greatly from this kind of training programme and I am confident that Dr. Mishra will conduct a more advanced programme of this kind soon. I also appreciate the efforts taken by him to get these lectures published by Springer. I am sure this volume will serve as excellent lecture notes in optimization for students and researchers working in this area.

Chennai, April 2010

Thiruvenkatachari Parthasarathy
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Preface

Optimization is a multidisciplinary research field that deals with the characterization and computation of minima and/or maxima (local/global) of nonlinear, nonconvex, nonsmooth, discrete, and continuous functions. Optimization problems are frequently encountered in modelling of complex real-world systems for a very broad range of applications including industrial and systems engineering, management science, operational research, mathematical economics, seismic optimization, production planning and scheduling, transportation and logistics, and many other applied areas of science and engineering. In recent years there has been growing interest in optimization theory.

The present volume contains 16 full-length papers that reflect current theoretical studies of generalized convexity and its applications in optimization theory, set-valued optimization, variational inequalities, complementarity problems, cooperative games, and the like. All these papers were refereed and carefully selected from those delivered at the Advanced Training Programme on Nonconvex Optimization and Its Applications held at the DST-Centre for Interdisciplinary Mathematical Sciences, Department of Mathematics, Banaras Hindu University, Varanasi, India, March 22–26, 2010.

I would like to take this opportunity, to thank all the authors whose contributions make up this volume, all the referees whose cooperation helped in ensuring the scientific quality of the papers, and all the people from the DST-CIMS and Department of Mathematics, Banaras Hindu University, whose assistance was indispensable in running the training programme. I would also like to thank to all the participants of the advanced training programme, especially those who travelled a long distance within India in order to participate. Finally, we express our appreciation to Springer for including this volume in their series. We hope that the volume will be useful for students, researchers, and those who are interested in this emerging field of applied mathematics.

Varanasi, April, 2010 *Shashi Kant Mishra*

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Chapter 1
Some Equivalences Among Nonlinear Complementarity Problems, Least-Element Problems, and Variational Inequality Problems in Ordered Spaces

Qamrul Hasan Ansari and Jen-Chih Yao

Abstract In this survey chapter we introduce several Z-type single-valued maps as well as set-valued maps. We present several equivalences among different types of nonlinear programs, different types of least-element problems, and different types of variational inequality problems under certain regularity and growth conditions.

1.1 Introduction

It is well known that the theory of complementarity problems has been become a very effective and powerful tool in the study of a wide class of linear and nonlinear problems in optimization, economics, game theory, mechanics, engineering, and so on, see, for example [9, 15–17], and the references therein. For a long time, a great deal of effort has gone into the study of the equivalence of complementarity problems and other problems. In 1980, Cryer and Dempster [10] studied the equivalence of linear complementarity problems, linear programs, least-element problems, variational inequality problems, and minimization problems in vector lattice Hilbert spaces. In 1981, Riddle [28] established the equivalence of complementarity and least-element problems as well as several related problems. In 1995, Schaible and Yao [30] proved the equivalence of these problems by introducing strictly pseudomonotone Z-maps operating on Banach lattices. In 1999, Ansari et al. [1] extended the results of Schaible and Yao [30] for point-to-set maps and established equivalence among generalized complementarity problems, generalized least-element problems, generalized variational inequality problems,

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and minimization problems. In [34] Yin, Xu, and Zhan established the equivalence of F-complementarity, variational inequality, and least-element problems in the Banach space setting. Very recently, Huang and Fang [14] introduced several classes of strong vector F-complementarity problems and gave their relationships with the least element problems of feasible sets. Furthermore, in [36], Zeng and Yao first gave an equivalence result for variational-like inequality problems and least element problems.

In this survey chapter we introduce several Z-type single-valued maps as well as multivalued maps. We present several equivalences among different types of nonlinear programs, least-element problems, complementarity problems, and variational inequality problems under certain regularity and growth conditions.

1.2 Preliminaries

In this section, we introduce some notations and definitions that are used in the sequel.

Let B be a real Banach space with its dual B^*, and let $K \subseteq B$ be a closed convex cone. Let K^* be the dual cone of K; that is,

$$K^* = \{u \in B^* : \langle u,x\rangle \geq 0 \text{ for all } x \in K\},$$

where $\langle u,x\rangle$ denotes the pairing between $u \in B^*$ and $x \in B$.

The vector ordering induced by K on B and induced by K^* on B^* is denoted by $\leq$:

$$\begin{aligned} x \leq y \quad &\text{if and only if } y - x \in K, \quad \text{for all } x,y \in B,\\ u \leq v \quad &\text{if and only if } v - u \in K^*, \quad \text{for all } u,v \in B^*. \end{aligned}$$

Nonzero elements of K^* are said to be *positive*, and $u \in K^*$ is said to be *strictly positive* if

$$\langle u,x\rangle > 0, \quad \text{for all } x \in K,\ x \neq 0.$$

The space B is a vector lattice with respect to $\leq$ if each pair $x,y \in B$ has a unique infimum $x \wedge y$ characterized by the properties

$$x \wedge y \leq x,\ x \wedge y \leq y,\ z \leq x, z \leq y \quad \text{if and only if } z \leq x \wedge y.$$

If B is a vector lattice, so is B^* with respect to the ordering $\leq$ induced by K^*; see, for example, [22].

Proposition 1.1 ([2, pp. 533]). *Let K be a nonempty convex subset of B and let $f : K \to \mathbb{R}$ be a lower semicontinuous and convex functional. Then, f is weakly lower semicontinuous.*

Remark 1.1. From Proposition 1.1, we can see that, if $f : K \to \mathbb{R}$ is upper semicontinuous and concave, then f is weakly upper semicontinuous.

Definition 1.1. Let Ω be an open subset of a real Banach space B. A function $f : \Omega \to \mathbb{R}$ is said to be *Gâteaux differentiable at* $x \in \Omega$ if there exists $\nabla f(x) \in B^*$ such that

$$\lim_{t \to 0^+} \frac{f(x+th) - f(x)}{t} = \langle \nabla f(x), h \rangle, \quad \forall h \in B.$$

$\nabla f(x)$ is called the *Gâteaux derivative of f* at the point x. The function f is Gâteaux differentiable in Ω if it is Gâteaux differentiable at each point of Ω.

Let K be a closed subset of B and $f : K \to \mathbb{R}$. By saying f is Gâteaux differentiable in K we mean that f is Gâteaux differentiable in an open set neighborhood of K.

Definition 1.2 ([3]). Let Ω be an open subset of a real Banach space B and $f : \Omega \to \mathbb{R}$ be Gâteaux differentiable. The function f is said to be

(i) *Pseudoconvex* on Ω if for every pair of points $x, y \in \Omega$, we have

$$\langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow f(y) \geq f(x)$$

(ii) *Strictly pseudoconvex* on Ω if for every pair of distinct points $x, y \in \Omega$, we have

$$\langle \nabla f(x), y - x \rangle \geq 0 \Rightarrow f(y) > f(x)$$

The relation of (strict) pseudoconvexity and (strict) pseudomonotonicity is the following.

Theorem 1.1 ([19, 20]). *Let Ω be an open convex subset of a real Banach space B and $f : \Omega \to \mathbb{R}$ be Gâteaux differentiable. Then f is (strictly) pseudoconvex on Ω if and only if $\nabla f : \Omega \to B^*$ is (strictly) pseudomonotone.*

We note that if $f : \Omega \to \mathbb{R}$ is strictly pseudoconvex, then the solution of $\min_{x \in \Omega} f(x)$ is unique provided a solution exists [3].

Definition 1.3. Let $f : B \to \mathbb{R}$ be a functional. Then an element $u \in B^*$ is called a *subgradient* of f at the point $x \in B$ if $f(x)$ is finite and

$$\langle u, y - x \rangle \leq f(y) - f(x), \qquad \forall y \in B.$$

The set of all subgradients of f at x is called the *subdifferential* of f at x and is denoted by $\partial f(x)$. That is,

$$\partial f(x) = \{u \in B^* : \langle u, y - x \rangle \leq f(y) - f(x)\}, \qquad \forall y \in B,$$

and therefore the subdifferential of f is the point-to-set map $\partial f : x \mapsto \partial f(x)$ from B to B^*.

Lemma 1.1 ([52]). *Let $(X, \|\cdot\|)$ be a normed vector space and $\mathscr{H}$ be a Hausdorff metric on the collection $CB(X)$ of all nonempty, closed, and bounded subsets of X, which is defined as*

$$\mathscr{H}(U,V) = \max\left\{\sup_{u\in U}\inf_{v\in V}\|u-v\|, \sup_{v\in V}\inf_{u\in U}\|u-v\|\right\},$$

for U and V in $CB(X)$, where the metric d is induced by $d(u,v) = \|u-v\|$. If U and V are compact sets in X, then for each $u \in U$, there exists $v \in V$ such that

$$\|u-v\| \leq \mathscr{H}(U,V).$$

Let D be a nonempty subset of a topological vector space X. A point-to-set map $G : D \to 2^X$ is called a *KKM map* if for each finite subset $\{x_1,\dots,x_n\} \subseteq D$,

$$\mathrm{co}\{x_1,\dots,x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i),$$

where $\mathrm{co}\{x_1,\dots,x_2\}$ denotes the convex hull of $\{x_1,\dots,x_n\}$.

Lemma 1.2 ([11]). *Let D be an arbitrary nonempty subset of a Hausdorff topological vector space X. Let the point-to-set map $G : D \to 2^X$ be a* KKM *map such that $G(x)$ is closed for all $x \in D$ and is compact for at least one $x \in D$. Then $\cap_{x\in D} G(x) \neq \emptyset$.*

1.3 Equivalence of Nonlinear Complementarity Problems and Least-Element Problems

Given are a closed convex cone $K \subseteq B$, $T : K \to B^*$ and $f : K \to \mathbb{R}$ whose special properties do not concern us for the moment. We denote by $\mathscr{F}$ the *feasible set* of T with respect to K; that is,

$$\mathscr{F} = \{x \in B : x \in K \text{ and } T(x) \in K^*\}.$$

In this section, we consider the following problems.

(I) *Nonlinear program* : For a given $u \in B^*$, find $x \in \mathscr{F}$ such that

$$\langle u,x\rangle = \min_{y\in\mathscr{F}}\langle u,y\rangle.$$

(II) *Least-element problem* : Find $x \in \mathscr{F}$ such that

$$x \leq y, \qquad \forall y \in \mathscr{F}.$$

(III) *Complementarity problem* : Find $x \in \mathscr{F}$ such that

$$\langle u, x \rangle = 0.$$

(IV) *Variational inequality problem* : Find $x \in K$ such that

$$\langle T(x), y - x \rangle \geq 0, \qquad \forall y \in K.$$

(V) *Unilateral minimization problem* : Find $x \in K$ such that

$$f(x) = \min_{y \in K} f(y).$$

The equivalence of (I) and (II) on the one hand, and among (III), (IV), and (V) is well known; see, for example [18, 28]. The purpose of this section is to investigate suitable conditions under which these five problems are equivalent.

Definition 1.4 ([28]). Let B be a Banach space that is also a vector lattice with positive cone K; let $T : K \to B^*$ be a mapping. Then T is called a *Z-map relative to* K if for any $x, y, z \in K$,

$$\langle T(x) - T(y), z \rangle \leq 0, \quad \text{whenever } (x - y) \wedge z = 0.$$

In the case where T is linear, Definition 1.4 reduces to the definition of *condition* Z in [10]. In the case where $B = \mathbb{R}^n$ and K is the nonnegative orthant, T is a Z-map relative to K if and only if it is *off-diagonally antitone* in the sense of [27].

Definition 1.5 ([18, 20, 28]). Let B be a Banach space, K a nonempty convex subset of B, and $T : K \to B^*$ a mapping. Then T is called

(i) *Pseudomonotone* if for any $x, y \in K$,

$$\langle T(y), x - y \rangle \geq 0 \quad \text{implies} \quad \langle T(x), x - y \rangle \geq 0$$

(ii) *Strictly pseudomonotone* if for any distinct points $x, y \in K$

$$\langle T(y), x - y \rangle \geq 0 \quad \text{implies} \quad \langle T(x), x - y \rangle > 0$$

(iii) *Hemicontinuous* if it is continuous on the line segments in K with respect to weak* topology in B^*; that is, for any fixed $x, y, z \in K$, the function

$$t \mapsto \langle T(x + ty), z \rangle, \qquad 0 \leq t \leq 1$$

is continuous

(iv) *Positive at infinity* if for any $x \in K$, there exists a positive real number $\rho(x)$ such that $\langle T(y), y - x \rangle > 0$ for every $y \in K$ such that $\|y\| \geq \rho(x)$.

Lemma 1.3 ([30]). *Let K be a convex cone in a Banach space B and $T : K \to B^*$ be (strictly) pseudomonotone. Then for each fixed $z \in K$, the operator $T_z : K \to 2^{B^*}$ defined by*

$$T_z(x) = T(x+z), \qquad \forall x \in K$$

is also (strictly) pseudomonotone.

Proof. For any $x, y \in K$, suppose that $\langle T_z(y), x-y\rangle \geq 0$. Then $\langle T(y+z), x-y\rangle \geq 0$, from which it follows that $\langle T(y+z), (x+z)-(y+z)\rangle \geq 0$. Because T is pseudomonotone, we have

$$\langle T(x+z), (x+z)-(y+z)\rangle \geq 0$$

and hence

$$\langle T_z(x), x-y\rangle \geq 0.$$

Therefore, T_z is also pseudomonotone. The case where T is strictly pseudomonotone can be dealt with by a similar argument.

We need the following result to derive the equivalence of problems (I)–(V) under suitable conditions.

Theorem 1.2. *Let K be a nonempty, closed, bounded convex subset of a reflexive Banach space B and let $T : K \to B^*$ be weakly pseudomonotone and hemicontinuous. Then there exist $x \in K$ such that*

$$\langle T(x), y-x\rangle \geq 0, \qquad \forall y \in K.$$

Furthermore, if in addition T is strictly pseudomonotone, the solution is unique.

Theorem 1.2 is an extension of classical existence results for variational inequalities due to [4, 13]. By employing Theorem 1.2, we obtain the following result for perturbed variational inequalities.

Proposition 1.2 ([30]). *Let K be a nonempty, closed, convex cone in a reflexive Banach space B, and $T : K \to B^*$ be pseudomonotone, hemicontinuous, and positive at infinity. Then for each fixed $z \in K$, there exist $x \in K$ such that*

$$\langle T(x+z), y-x\rangle \geq 0, \qquad \forall y \in K. \tag{1.1}$$

If, in addition, T is strictly pseudomonotone, then for each $z \in K$, (1.1) *has a unique solution.*

Proof. For each $z \in K$, we define $T_z : K \to B^*$ by

$$T_z(x) = T(x+z), \qquad \forall x \in K.$$

Then obviously, T_z is hemicontinuous.

By Lemma 1.2, T_z is also pseudomonotone. Let $\rho = \|z\| + \rho(z)$, where $\rho(z)$ is defined as in the definition of positive at infinity. Let

$$D = \{y + z : y \in K, \|y\| \le \rho\},$$

which is a closed, bounded, convex subset of a reflexive Banach space B. Then by Theorem 1.2, there exist $x \in K$ with $\|x\| \le \rho$ such that

$$\langle T_z(x), y - x\rangle \ge 0, \qquad \forall y \in K \qquad \text{with } \|y\| \le \rho. \tag{1.2}$$

We note that $\|x\| < \rho$. Suppose that $\|x\| = \rho$; then

$$\|x + z\| \ge \|x\| - \|z\| = \rho(z).$$

T is positive at infinity, thus we have

$$\langle T(x+z), x\rangle > 0,$$

or

$$\langle T_z(x), x\rangle > 0. \tag{1.3}$$

On the other hand, letting $y = 0$ in (1.2), we have

$$\langle T_z(x), x\rangle \le 0,$$

which is a contradiction of (1.3). Therefore, $\|x\| < \rho$ and by standard technique it can be shown that x is indeed a solution of (1.1).

If, in addition, T is strictly pseudomonotone, then by Lemma 1.3, T_z is also strictly pseudomonotone. Consequently, the solution is unique.

In the remaining part of this section, we assume that B is a real Banach space and K is a closed convex cone of B, and, whenever the ordering induced by K is mentioned, $(B, \le)$ is a vector lattice.

Now we establish the equivalence of problems (I)–(V) under suitable conditions.

Proposition 1.3 ([30]). *Let $T : K \to B^*$ be the Gâteaux derivative of $f : K \to \mathbb{R}$. Then any solution of* (V) *is also a solution of* (IV). *If in addition, T is pseudomonotone, then, conversely, any solution of* (IV) *is also a solution of* (V).

Proposition 1.4 ([18, Lemma 3.1]). *Let $T : K \to B^*$. Then x is a solution of* (III) *if and only if it is a solution of* (IV).

Proposition 1.5 ([30]). *Suppose that $T : K \to B^*$ is strictly pseudomonotone and a Z-map relative to Z. Then any solution of* (IV) *is also a solution of* (II).

Proposition 1.6 ([18, Lemma 3.1]). *Let $T : K \to B^*$ and $u \in K^*$. Then any solution of* (II) *is a solution of* (I).

Proposition 1.7 ([30]). *Let B be a reflexive Banach space. Assume that $T : K \to B^*$ is a Z-map relative to K, strictly pseudomonotone, hemicontinuous, and positive at* *infinity. Then the feasible set* $\mathscr{F} = \{x \in B : x \in K$ *and* $T(x) \in K^*\}$ *is a* $\wedge$*-sublattice; that is,* $x \in \mathscr{F}$ *and* $y \in \mathscr{F}$ *imply* $x \wedge y \in \mathscr{F}$.

Proposition 1.8 ([30]). *Let B be a reflexive Banach space. Assume that $T : K \to B^*$ is a Z-map relative to K, strictly pseudomonotone, hemicontinuous, and positive at infinity. Let $u \in K^*$ be strictly positive. Then Problem* (I) *corresponding to u has at most one solution, and any solution of* (I) *is also a solution of* (II).

By combining Propositions 1.3 and 1.5–1.7, we have the following main result of this section.

Theorem 1.3. *Let K be a closed convex cone in a reflexive Banach space B such that B is a vector lattice with respect to the order $\leq$ induced by K. Let $T : K \to B^*$ be a Z-map relative to K, strictly pseudomonotone, hemicontinuous, and positive at infinity. If $u \in K^*$ is a strictly positive element, then there exists $x \in \mathscr{F}$ which is a solution of problems* (I)–(IV). *Moreover, the solution x is unique. If T is the Gâteaux derivative of $f : K \to \mathbb{R}$, then x is also a unique solution of problem* (V).

Corollary 1.1. *Let K be a closed convex cone in a reflexive Banach space B such that B is a vector lattice with respect to the order $\leq$ induced by K. Let $T : K \to B^*$ be a Z-map relative to K, strongly pseudomonotone and hemicontinuous. If $u \in K^*$ is a strictly positive element, then there exists $x \in \mathscr{F}$ which is a solution of problems* (I)–(IV). *Moreover, the solution x is unique. If T is the Gâteaux derivative of $f : K \to \mathbb{R}$, then x is also a unique solution of problem* (V).

The following example illustrates that the extension of Riddell's result is not empty.

Example 1.1. Let $B = \mathbb{R}^n$ with the Euclidean norm. Then $B^* = \mathbb{R}^n$. The pairing between $x = (x_1, \ldots, x_n) \in B$ and $u = (u_1, \ldots, u_n) \in B^*$ is given by

$$\langle u, x\rangle = \sum_{i=1}^{n} u_i x_i.$$

Let K be the nonnegative orthant. Then $K^* = K$ and the reduced ordering makes B a vector lattice with

$$x \wedge y = (z_1, \ldots, z_n), \qquad z_i = \min\langle y_i, x_i\rangle.$$

Let $T : [0, \infty) \to \mathbb{R}$ be defined as $T(x) = 2 + (1/10)x + \sin x$ for $x \geq 0$. Then it can be checked that T is strictly pseudomonotone and a Z-map relative to $[0.\infty)$. T is also positive at infinity. Note that T is not monotone because $\langle T(x) - T(y), x - y\rangle < 0$ for $x = (3/2)\pi$ and $y = 0$.

1.4 Equivalence Between Variational-Like Inequality Problem and Least-Element Problem

Let B be a real Banach space with norm $\|\cdot\|$ and dual B^*. Let $K \subset B$ be a nonempty convex subset, $f : K \to B^*$ be a single-valued mapping, and $\varphi : K \to \mathbb{R}$ be a convex functional. For a given mapping $\eta : K \times K \to B$, we consider the following *variational-like inequality problem* of finding $x^* \in K$ such that

$$\langle f(x^*), \eta(x,x^*)\rangle \geq F(x^*) - F(x), \quad \text{for all } x \in K. \tag{1.4}$$

If $B = H$ is a real Hilbert space, $K = H$, $\eta(x,y) = x - y$ for all $x,y \in H$, $f : H \to H$ is a single-valued mapping, and $F : H \to \mathbb{R}$ is a linear continuous functional, then the problem (1.4) reduces to the following *variational inequality problem*. Find $x^* \in K$ such that

$$\langle f(x^*), x - x^*\rangle \geq F(x^*) - F(x), \quad \text{for all } x \in K. \tag{1.5}$$

If $F \equiv 0$, then the problem (1.4) reduces to the following *variation-like inequality problem*: Find $x^* \in K$ such that

$$\langle f(x^*), \eta(x,x^*)\rangle \geq 0, \quad \text{for all } x \in K. \tag{1.6}$$

The problem (1.6) is studied in the setting of finite-dimensional Eucludian space in [26] and infinite-dimensional spaces in [31].

If $K \subset B$ is a closed convex cone, and $\eta(x,y) = x - y$ for all $x,y \in K$, then the problem (1.4) reduces to the *variational inequality problem*: find $x^* \in K$ such that

$$\langle f(x^*), x - x^*\rangle \geq F(x^*) - F(x), \quad \text{for all } x \in K. \tag{1.7}$$

In order to study the F-complementarity problem, Yin, Xu, and Zhang [34] introduced and considered the problem (1.7), and established the equivalence between problem (1.7) and the F-complementarity problem in the case when $F : K \to \mathbb{R}$ is positively homogeneous. More precisely, let B be a real Banach space and B^* the dual space. Let K be a closed convex cone in B, $f : K \to B^*$ and $F : K \to \mathbb{R}$. The *F-complementarity problem* is to find $x^* \in K$ such that

$$\langle x^*, f(x^*)\rangle + F(x^*) = 0 \quad \text{and} \quad \langle x, f(x^*)\rangle + F(x) \geq 0, \quad \text{for all } x \in K.$$

Furthermore, by virtue of the existence of solutions of problem (1.7), they studied the equivalence between the F-complementarity problem and the least element problem.

In this section, we establish the existence results for solutions of variational-like inequality problems in the case when $K \subset B$ is a nonempty closed convex subset containing zero. Furthermore, we prove that the feasible sets of problem (1.4) are $\wedge$-sublattices in the vector lattice. Moreover, we investigate the equivalence between problem (1.4) and the least element problems. The results of this section improve and generalize the results of Yin et al. [34] by extending the variational inequality problem (1.7) in [34] to the variational-like inequality problem (1.4). In addition, these results also generalize and extend the corresponding results in [26, 28, 30].

We give some notations and definitions that are used in the rest of this section.

Definition 1.6. Let $f : K \to B^*$ and $\eta : K \times K \to B$. f is said to be η-*hemicontinuous* on K if for every fixed $x, y \in K$, the function

$$t \mapsto \langle f(x + t(y - x)), \eta(y, x)\rangle$$

is continuous at 0^+. In particular, if $\eta(x, y) = x - y$ for all $x, y \in K$, then f is said to be hemicontinuous on K.

Definition 1.7. Let $f : K \to B^*$ and $\eta : K \times K \to B$. Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative function and $F : K \to \mathbb{R}$ be a convex functional.

(i) f is said to be *strictly* η – α*-monotone* on K if for each $x, y \in K$ and $x \neq y$,

$$\langle f(x) - f(y), \eta(x, y)\rangle > \alpha(\|x - y\|).$$

In the case where $\alpha(t) = 0$, f is said to be *strictly* η*-monotone* on K. In particular, if $\eta(x, y) = x - y$ for all $x, y \in K$, then f is said to be *strictly* α*-monotone* on K.

(ii) f is said to be η – F*-pseudomonotone* on K if for each $x, y \in K$ and $x \neq y$,

$$\langle f(y), \eta(x, y)\rangle \geq F(y) - F(x) \Longrightarrow \langle f(x), \eta(x, y)\rangle \geq F(y) - F(x).$$

In particular, if $\eta(x, y) = x - y$ for all $x, y \in K$, then f is said to be F*-pseudomonotone* on K.

(iii) f is said to be *strictly* η – F*-pseudomonotone* on K if for each $x, y \in K$,

$$\langle f(y), \eta(x, y)\rangle \geq F(y) - F(x) \Longrightarrow \langle f(x), \eta(x, y)\rangle > F(y) - F(x).$$

In particular, if $\eta(x, y) = x - y$ for all $x, y \in K$, then f is said to be *strictly* F*-pseudomonotone* on K.

(iv) f is said to satisfy the η*-coercive condition* with respect to F if for any given $y \in K$, there exists a positive number $\rho(y)$ such that

$$\langle f(x + y), \eta(x, 0)\rangle + F(x) > F(0)$$

for all $x \in K$ with $\|x\| = \rho(y)$. In particular, if $\eta(x, y) = x - y$ for all $x, y \in K$, then f is said to satisfy the *coercive condition* with respect to F.

It is clear that strictly η – α-monotone $\Rightarrow$ strictly η-monotone $\Rightarrow$ strictly η – F-pseudomonotone $\Rightarrow$ $\eta - F$-pseudomonotone.

Remark 1.2. If $\eta(x, y) = x - y$ for all $x, y \in K$, then Definitions 1.6 and 1.7 reduce to Definitions 2.1 and 2.2 in Yin, Xu, and Zhang [34], respectively. Definition 1.6 with $\eta(x, y) = x - y$ was previously introduced by Riddell [28].

Definition 1.8 ([34]). Let $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative function and $F : K \to \mathbb{R}$ a functional, where $K + K \subset K$. F is said to be *α-bounded* on K if for each $x, y \in K$,

$$F(x) + F(y) - F(x+y) \leq \min\{\alpha(\|x\|), \alpha(\|y\|)\}.$$

Throughout this section, unless otherwise specified, we assume that B is a real Banach space and that $K \subset B$ is a nonempty, closed, convex subset containing zero.

Theorem 1.4 ([36]). *Let B be a reflexive Banach space, and $F : K \to \mathbb{R}$ a lower semicontinuous and convex functional. Let $f : K \to B^*$ be an η-hemicontinuous and $\eta - F$-pseudomonotone mapping, where $\eta : K \times K \to B$ has the properties:*

(i) $\eta(x,y) + \eta(y,x) = 0$ *for all* $x, y \in K$.
(ii) $\eta(\cdot,\cdot)$ *is affine in the first variable.*
(iii) For each fixed $y \in K$, $x \mapsto \eta(y,x)$ is sequentially continuous from the strong topology to the weak topology.

Assume that there exists a positive number $r > 0$ such that

$$\langle f(x), \eta(x,0)\rangle + F(x) > F(0), \quad \textit{for all } x \in K \textit{ with } \|x\| = r. \tag{1.8}$$

Then the variational-like inequality problem (1.4) *has a solution in K. In particular, if f is strictly $\eta - F$-pseudomonotone, then the solution is unique.*

As consequences of Theorem 1.4, we immediately obtain the following corollaries.

Corollary 1.2 ([34, Theorem 3.1]). *Let B be a reflexive Banach space, and $F : K \to \mathbb{R}$ a lower semicontinuous and convex functional. Let $f : K \to B^*$ be a hemicontinuous and F-pseudomonotone mapping. If there exists a positive number $r > 0$ such that*

$$\langle f(x), x\rangle + F(x) > F(0), \quad \textit{for all } x \in K \textit{ with } \|x\| = r,$$

then the variational inequality problem (1.7) *has a solution in K. In particular, if f is strictly F-pseudomonotone on K, then the solution is unique.*

Corollary 1.3 ([34, Corollary 3.2]). *Let B be a reflexive Banach space, and $F : K \to \mathbb{R}$ a lower semicontinuous and convex functional. Let $f : K \to B^*$ be a hemicontinuous and strictly monotone mapping. If f satisfies the coercive condition with respect to F, then for any given $z \in K$, there exists a unique element $x^* \in K$ such that*

$$\langle x - x^*, f(x^* + z)\rangle \geq F(x^*) - F(x), \quad \textit{for all } x \in K.$$

Following the idea of Yin, Xu, and Zhang [34], we define the feasible set of the variational-like inequality problem (1.4) as follows,

$$\mathscr{D} = \{w \in K : \langle f(w), \eta(u, u \wedge w)\rangle + F(u - u \wedge w) \geq 0 \text{ for all } u \in K\}.$$

In particular, if $\eta(x,y) = x - y$ for all $x,y \in K$, then the feasible set of the variational-like inequality problem (1.4) reduces to that of the variational inequality problem (1.7); that is,

$$\mathscr{D} = \{x \in K : \langle f(x), y - y \wedge x\rangle + F(y - y \wedge x) \geq 0 \text{ for all } u \in K\}.$$

Definition 1.9. Let $(B, \leq)$ be a vector lattice. A function $f : K \to B^*$ is said to be an $\eta - Z$*-mapping* on K if for each $u, v, w \in K$,

$$v \wedge (w - u) = 0 \Rightarrow \langle f(w) - f(u), \eta(u + v, u)\rangle \leq 0.$$

In particular, if $\eta(x,y) = x - y$ for all $x,y \in K$, then f is said to be a *Z-mapping* on K.

Theorem 1.5 ([36]). *Let B be a reflexive Banach space, and $(B, \leq)$ a vector lattice. Let $F : K \to \mathbb{R}$ be a functional and $f : K \to B^*$ an η – Z-mapping, where $\eta : K \times K \to B$ is a mapping such that $\eta(x,y) + \eta(y,x) = 0$ for all $x,y \in K$. Assume that the following conditions are satisfied.*

(i) There exists a nonnegative function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(a) f is strictly η – α-monotone on K.
(b) F is α-bounded on K.

(ii) For any given $z \in K$, there exists $x^* \in K$ such that

$$\langle f(x^* + z), \eta(u, u \wedge z + x^*)\rangle \geq F(x^*) - F(u - u \wedge z) \quad \text{for all } u \in K.$$

If the feasible set $\mathscr{D}$ of the variational-like inequality problem (1.4) *is nonempty, then $\mathscr{D}$ is a $\wedge$-sublattice of B.*

Corollary 1.4 ([36]). *Let B be a reflexive Banach space, and $(B, \leq)$ a vector lattice. Let $F : K \to \mathbb{R}$ be a lower semicontinuous and convex functional, $f : K \to B^*$ a hemicontinuous Z-mapping, and f satisfies the coercive condition with respect to F. Assume that there exists a nonnegative function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

(i) f is strictly α-monotone on K.
(ii) F is α-bounded on K.

If the feasible set $\mathscr{D}$ of the variational inequality problem (1.7) *is nonempty, then $\mathscr{D}$ is a $\wedge$-sublattice of B.*

Theorem 1.6 ([36]). *Let B be a reflexive Banach space and $(B, \leq)$ be a vector lattice. Let $F : K \to \mathbb{R}$ be a functional and $f : K \to B^*$ an η – Z-mapping, where $\eta : K \times K \to B$ is a mapping such that $\eta(x,y) + \eta(y,x) = 0$ for all $x,y \in K$. Assume that there exists a nonnegative function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that the condition* (i) *in Theorem* 10.20 *is satisfied. If the variational-like inequality problem* (1.4) *has a solution x^* in the feasible set $\mathscr{D}$, then x^* is the least element of $\mathscr{D}$.*

Corollary 1.5 ([36]). *Let $(B,\leq)$ be a vector lattice. Let $F : K \to \mathbb{R}$ be a positively homogeneous and convex functional, and $f : K \to B^*$ a Z-mapping. Assume that there exists a nonnegative function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that the conditions* (i) *and* (ii) *in Corollary* 10.5 *are satisfied. If the variational inequality problem* (1.7) *has a solution x^* in K, then x^* is the least element of $\mathscr{D}$.*

Now, from Theorems 1.4–1.6 we immediately obtain the following result.

Theorem 1.7 ([36]). *Let B be a reflexive Banach space, and $(B,\leq)$ a vector lattice. Assume that the following conditions are satisfied.*

(i) $F : K \to \mathbb{R}$ is a lower semicontinuous and convex functional.
(ii) $f : K \to B^*$ is an η-semicontinuous η – Z-mapping, where $\eta : K \times K \to B$ has the following properties.

(a) $\eta(x,y)+\eta(y,x) = 0$ for all $x,y \in K$.
(b) $\eta(\cdot,\cdot)$ is affine in the first variable.
(c) For each fixed $y \in K$, $x \mapsto \eta(y,x)$ is sequentially continuous from the strong topology to the weak topology.

(iii) There exists a positive number $r > 0$ such that

$$\langle \eta(x,0), f(x)\rangle + F(x) > F(0), \quad \textit{for all } x \in K \textit{ with } \|x\| = r.$$

(iv) There exists a nonnegative function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(a) f is strictly η – α-monotone on K.
(b) F is α-bounded on K.

(v) For any given $z \in K$, there exists $x^* \in K$ satisfying the following inequality.

$$\langle f(x^*+z), \eta(u, u\wedge z + x^*)\rangle \geq F(x^*) - F(u - u\wedge z), \quad \textit{for all } u \in K.$$

Then the variational-like inequality problem (1.4) *has a unique solution x^* in K. In particular, if this solution x^* lies in $\mathscr{D}$, then $\mathscr{D}$ is a $\wedge$-sublattice of B, and x^* is the least element of $\mathscr{D}$.*

Finally, from Corollaries 1.3, 1.4, and 1.5 we immediately have the following corollary.

Corollary 1.6 ([36]). *Let B be a reflexive Banach space, and $(B,\leq)$ a vector lattice. Assume that the following conditions are satisfied.*

(i) $F : K \to \mathbb{R}$ is a lower semicontinuous, positively homogeneous and convex functional.
(ii) $f : K \to B^*$ is a semicontinuous Z-mapping.
(iii) f satisfies the coercive condition with respect to F.
(iv) There exists a nonnegative function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(a) f is strictly α-monotone on K.
(b) F is α-bounded on K.

Then the variational inequality problem (1.7) *has a unique solution x^* in the feasible set $\mathscr{D}$ of itself, $\mathscr{D}$ is a $\wedge$-sublattice of B, and x^* is the least element of $\mathscr{D}$.*

1.5 Equivalence Between Extended Generalized Complementarity Problems and Generalized Least-Element Problem

In this section, we extend the formulations and results of Section 1.3 for set-valued maps.

Given is a closed convex cone $K \subseteq B$ and $T : K \to 2^{B^*}$, where 2^B is the family of all nonempty subsets of B. We denote by $\mathscr{F}$, the *feasible set* of T with respect to K; that is,

$$\mathscr{F} = \{x \in B : x \in K \text{ and } T(x) \cap K^* \neq \emptyset\}.$$

We consider the following problems.

(I) *Generalized nonlinear program*: For a given $u \in B^*$, find $x \in \mathscr{F}$ such that

$$\langle u, x\rangle = \min_{y \in \mathscr{F}} \langle u, y\rangle.$$

(II) *Generalized least-element problem*: Find $x \in \mathscr{F}$ such that

$$x \leq y, \quad \forall y \in \mathscr{F}.$$

(III) *Extended generalized complementarity problem*: Find $x \in K$ and $u \in T(x) \cap K^*$ such that

$$\langle u, x\rangle = 0.$$

(IV) *Generalized variational inequality problem*: Find $x \in K$ and $u \in T(x)$ such that

$$\langle u, y - x\rangle \geq 0, \qquad \forall y \in K.$$

The equivalence of (III) and (IV) has been studied by Saigal [29]. The main object of this section is to investigate suitable conditions under which these four problems are equivalent.

Definition 1.10. Let B be a Banach space that is also a vector lattice with positive cone K; let $T : K \to 2^{B^*}$ be a point-to-set map. Then T is called

(i) *Z-map relative to K* if for any $x, y, z \in K$,

$$\langle u - v, z\rangle \leq 0, \qquad \forall u \in T(x) \quad \text{and} \quad v \in T(y), \quad \text{whenever } (x - y) \wedge z = 0$$

(ii) *Monotone* if for any $x, y \in K$,

$$\langle u - v, x - y\rangle \geq 0, \qquad \forall u \in T(x) \quad \text{and} \quad v \in T(y)$$