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Mathematics and Computing

ICMC 2018, Varanasi, India, January 9–11, Selected Contributions

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Dedicated to
Pandit Madan Mohan Malaviya—The
Founder of Banaras Hindu University

Preface

The Fourth International Conference on Mathematics and Computing (ICMC—2018) was organized in the Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi, India, during January 9–11, 2018, under the dynamic leadership of Dr. Debdas Ghosh along with the support of Prof. R. N. Mohapatra, Prof. D. Giri, Prof. T. Som, Prof. S. Mukhopadhyay, Prof. S. Das, Dr. A. Banerjee, and the faculty members of the Department of Mathematical Sciences, IIT (BHU), India. There was an overwhelming response to the program, and one hundred and twenty papers all over the country and abroad were submitted for the consideration of presentation and later publication in the proceedings. Taking into account the norms of the proceedings, the papers were gone through strict blind reviewing process by at least two referees in the respective areas and only forty-seven papers were selected for the presentation and twenty-nine for inclusion in the Proceeding of Mathematics and Statistics, Springer. The areas covered by the papers are the latest works in the field of cryptography, security, abstract algebra, functional analysis, fluid dynamics, fuzzy modeling and optimization, etc. The ICMC—2018 was attended by several experts of international repute from the nation as well as from USA, UK, Japan, China, Finland, etc., as invited speakers with their high-quality research presentations. Experts were from IIT Madras, ISI Chennai, University of Central Florida, Orlando, USA, Kettering University, USA, University of Surrey, UK, Auburn University, Alabama, USA, Kyushu University, Japan, Tianjin University of Science and Technology, China, Oracle's System of Technology, USA, University of Turku, Finland, Haldia Institute of Technology (HIT), India, Banaras Hindu University (BHU), India, and IIT (BHU), India. Most of the experts have submitted their contributions for the proceeding. The Organizing Committee of ICMC—2018 is truly thankful to all experts and paper presenters for their academic support.

Distinguished Prof. Anthony T. S. Ho of the Tianjin University, of the University of Surrey, also of the Wuhan University of Technology has nicely elaborated and explained the applications of Benford's law for multimedia security and forensics. Professor R. N. Mohapatra, University of Central Florida, has beautifully explored the various aspects of epidemiological models with mutating pathogens with basic SIR model, diffusion equation, the Fisher–Kolmogoroff equation, spatial epidemic models, and his proposed model supported with some nice examples. Professor S. R. Chakravarty of the Kettering University elaborately presented the different aspects of non-preemptive stochastic priority queuing model for two different types of customers and with a new threshold. Professor K. Sakurai of the Kyushu University has discussed non-commutative approach using ring for enhancing the security of cryptosystems. Professor Matti Vuorinen of the University of Turku gave insightful elaboration on computation of condenser capacity. Dr. Srinivas Pyda of Oracle's System of Technology has discussed well the mathematics in machine learning. Professor Dr. Parisa Hariri of the University of Turku has explored the hyperbolic metric of plane domain to a subdomain of R^n ($n \geq 2$), discussed the geometry and topology of metric balls, and compared different hyperbolic type metrics and gave an application to solve Ptolemy–Alhazen problem. Professor S. Ponnusamy of IIT Madras has described the classical Bohr's theorem for bounded functions, bounded n-symmetric functions, half-plane mappings, half-plane n-symmetric mappings, and added some nice examples supporting the theory; Prof. Debasis Giri of HIT has elaborated on authenticated encryption of long messages; Prof. Chris Rodger of the Auburn University has explored the various aspects of graph embedding and construction of Hamilton's decomposition of graphs and elaborated with nice examples having several applications. Professor S. K. Mishra of BHU talked about the properties and relations of strong pseudomonotone and strong quasimonotone operators. Professor T. Som of IIT (BHU) has contributed to convergence of generalized Mann type of iterates to common fixed point though he has dealt with soft relation and fuzzy soft relation with application to decision-making problems in the conference program. The submitted contributions of the experts are included in the proceeding. The organizing committee is truly thankful to all the experts for their valuable contribution to the conference.

I, on behalf of the organizing committee, gratefully acknowledge the financial support to the conference by

- Science and Engineering Research Board, India
- Defence Research and Development Organization, India
- Indian Institute of Technology (BHU), India
- Council of Research and Industrial Research, India
- SCUBE India.

Varanasi, India

Prof. Tanmoy Som
Organizing Secretary

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Chapter 1
Constructions and Embeddings of Hamilton Decompositions of Families of Graphs

C. A. Rodger

Abstract In this paper, a discussion of the use of amalgamations in constructing Hamilton decompositions of graphs is presented. Edge-colorings that are fair in various senses are critical to this endeavor, so some discussion of them is also included. Finally, the power of amalgamations is demonstrated in the overview of results in the literature that take a given edge-coloring of a graph and extend it to one of a family of graphs (e.g., a complete graph or a complete multipartite graph) in which each color class is a Hamilton cycle.

Keywords Hamilton cycles · Amalgamations · Fair edge-colorings · Embeddings

1 Introduction

Colorings of graphs are very useful in a variety of settings, especially scheduling problems. In such problems, sharing objects (vertices or edges) out evenly in various ways usually has beneficial effects in the application being considered. For example, the most basic of these fairness notions is to ensure that the coloring is proper (no two adjacent objects receive the same color). But other notions also play a vital role. One could ask for the coloring to be equalized; that is, the number of objects of each color is within one of the number of objects of each other color. Two examples illustrate this.

The first example is the scheduling problem where various companies send representatives to a central location, such as Chicago Airport, where they are to meet other companies for one-on-one discussions. All representatives are in the same industry so, while not every pair of companies' representatives need to meet, there is a lot of congestion to manage. The aim is to schedule the meetings (each is to last 30 min) to minimize the number of time slots needed to satisfy all needs to meet. The number of rooms is also an issue, partly due to availability and partly due to

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expense. This problem can be modeled by a graph G formed by letting each company (representative) be represented by a vertex, two vertices being joined if and only if the corresponding companies need to meet. A proper edge-coloring with k colors provides a schedule using k time slots: Representatives i and j meet at time slot k if the edge $\{i, j\}$ is colored k. Clearly the fact that the edge-coloring is proper ensures that each representative is scheduled to meet at most one other representative at each time. The number of rooms needed is decided by the size of the biggest color class, and this is minimized if the edge-coloring is also equalized. Results in the literature come close to immediately answering this problem: k can be any value at least $\chi'(G)$, which by Vizing's Theorem is either the maximum degree $\Delta = \Delta(G)$ of G, or is $\Delta + 1$, and a result by McDiarmid [1] guarantees that if there exists a proper k-edge-coloring, then there exists an equalized proper k-edge-coloring. Deciding if a schedule with Δ timeslots is possible may be difficult to determine, as this falls in the class of NP-complete problems; but rather than working hard to save just one time period, simply using $\Delta + 1$ timeslots often may not be a problem.

The second example contrasts with the first quite nicely. Various university clubs are to meet one evening to plan their efforts to help Auburn collect enough food to win the Auburn-Alabama Food Fight, designed to help the hungry in Alabama. Ideally, each club would only meet if all its representatives attending that evening are able to be present at the meeting. Again the plan is to schedule the meetings (each is to last 30 min) in a way that minimizes the number of time slots needed for each club to meet, having all members present; as before, the number of rooms is also an issue. In this case, the model is a graph in which each club is represented by a vertex and two vertices are joined by an edge if the corresponding clubs have a member in common. So a proper vertex-coloring with k colors provides a schedule using k time slots: Club i meets at time slot k if vertex i is colored k. The fact that the vertex-coloring is proper ensures that clubs with members in common are scheduled at different times. Minimizing the number of rooms needed again calls for an equalized vertex-coloring (often called an equitable vertex-coloring in the literature). Unfortunately, results in the literature have more trouble solving this problem; both answering the question of how many colors are needed ($\chi(G)$ is not easily determined) and of whether or not an equalized vertex-coloring exists. The number of time slots can be any value at least $\chi(G)$; if it is chosen to be more than $\Delta(G)$, then it is known that the vertex-coloring can be equalized ([2]). Other efforts over the past 40 years to find conditions guaranteeing the existence of equalized vertex-colorings have been found, but much work remains to understand this property.

Many interesting problems associated with fair colorings of various sorts remain open and are of practical use. Several more will be introduced later in the paper as they are needed.

A third practical problem addressed by graph theory is the famous traveling salesman problem. A salesman has to visit a predetermined set of cities, one by one, then return home, following a route that minimizes the distance travelled. It is modeled by a graph, G, in which the vertices represent the cities, edges represent various routes to get from city to city, and all edges are weighted by the distance of the corresponding route. In the unworldly case where all the edges have weight 1, this problem asks whether or not G contains a Hamilton cycle (a cycle in G which includes each vertex in $V(G)$). This too falls into the family of NP-complete problems, so is difficult to solve, even in this seemingly far simpler situation. Related to the Hamiltonicity of a graph is a stronger property. A Hamilton decomposition of G is a partition of $E(G)$, each element of which induces a Hamilton cycle. Since each Hamilton cycle includes exactly two edges incident with each vertex, clearly G needs to be regular in order to have a Hamilton decomposition. Around 125 years ago, Walecki proved that the complete graph K_n has a Hamiltonian decomposition if and only if n is odd [3]. This too has an interpretation in an applied setting, related to the traveling salesman problem. In this case, the salesman wants to visit certain important cities on every trip, but other towns along the way can be visited less often. The Hamilton cycles in a Hamilton decomposition ensure that each time out the salesman visits the important cities (the vertices of the graph), and then since each edge is in exactly one Hamilton cycle, towns along the roads corresponding to the edges will be visited as the road is traversed.

For each of these three problems, interest eventually turned from complete graphs to another natural family of graphs, namely the complete multipartite graphs: The vertices in each such graph are partitioned into p parts, with two vertices being joined by an edge if and only if they are in different parts. The chromatic index of such graphs was settled thirty years ago [4], and the value of the chromatic number is obvious, but finding equalized vertex-colorings is not so straightforward (see [5]). For such graphs to have a Hamilton decomposition, clearly they must be regular; to be regular, clearly all parts must have the same size. So this motivates the following definition: Let $K(n, p)$ denote the complete multipartite graph with p parts in which each part contains n vertices. Deciding whether or not $K(n, p)$ has a Hamilton decomposition was settled by Laskar and Auerbach [6] 40 years ago, showing that it exists if and only if $n(p-1)$ is even.

Much more recently, a third family of graphs has drawn wide interest, motivated by the construction of experimental designs in statistics. A block design with two association classes (BDTAC) can be described graph theoretically as follows. Let $K(P, \lambda_1, \lambda_2)$ be the graph in which P is a partition of the vertices, two vertices being joined by λ_1 edges if they are in the same part of P and by λ_2 edges if they are in different parts. The BDTAC is equivalent to a partition of the edges of $K(P, \lambda_1, \lambda_2)$, each element of which is a copy of K_k for some integer k. In the setting of this paper, the natural question is whether or not there exists a Hamilton decomposition of $K(P, \lambda_1, \lambda_2)$, so of particular interest is the regular graph $K(n, p, \lambda_1, \lambda_2)$ where each of the p parts in $K(P, \lambda_1, \lambda_2)$ contains n vertices. This problem was settled by Bahmanian and Rodger 5 years ago [7]. Their method of proof is the main topic in Sect. 2. Continuing the theme of fairness in colorings, the amalgamation proof technique produces a graph H from a given edge-colored graph G, where G is a graph homomorphism of H, such that the edges in H are shared out among the vertices and among color classes in ways that are fair with respect to several notions of balance. The connectivity of color classes is also addressed.

In Sect. 3, the embedding of edge-colored graphs into edge-colored copies of $K(n, p, \lambda_1, \lambda_2)$ is the main focus. This is a great demonstration of the power of amalgamation proofs, but is also motivated by applications in the following sense. Scheduling problems often require prerequisite conditions to be built into the final schedule. For example, when deciding which teachers should teach which classes at what times, some teachers may not be able to teach early in the morning. Hilton [8] developed the notion of an outline schedule where times are compressed into a small number of groups; say early morning, late morning, early afternoon, and last classes. Similarly, subjects being taught, or classes for the same age students could also form such groups. Once this outline schedule has been developed, reversing the amalgamation approach develops the full schedule. This method also allows prerequisites to be built into about a quarter of the entire schedule. Here, we begin with a given edge-colored copy of $K(n, p_1, \lambda_1, \lambda_2)$ and embed it in a copy of $K(n, p_2, \lambda_1, \lambda_2)$ such that each color class induces a Hamilton cycle. In view of the third problem described above, the given copy can be thought of as the given prerequisites in the final Hamilton decomposition that realizes the schedule of the salesman.

2 Amalgamations and Hamilton Decompositions

In 1984, Hilton [9] made a leap forward in the study of Hamilton decompositions. He had the idea of starting with a single vertex, say α, incident with $n(n-1)/2$ loops, n of each of $(n-1)/2$ colors, and attempted to disentangle n vertices from α, one at a time, to end up with the complete graph K_n in which each color class was a Hamilton cycle. The proof was inductive, so was especially powerful in that it allowed one to start midway through the process rather than with a single vertex. All that was needed was for this midway point to satisfy the conditions described in the inductive hypotheses, conditions which actually turn out to be necessary anyway.

It is helpful to think of the single vertex as originally containing the n vertices that eventually appear in the final graph. As each vertex is disentangled from α, one less vertex is still contained in it, so at the ith step one can naturally define the amalgamation function $f_i(\alpha) = n - i$ to be the number of vertices still in α. Inductively, the setup at the ith step is to have: $f_i(\alpha)(f_i(\alpha) - 1)/2$ loops incident with α; one edge between each pair of disentangled vertices; and $n - f_i(\alpha)$ edges between each disentangled vertex and α. Since we hope to end up with K_n then at the ith step, one end of each of $f_i(\alpha) - 1$ loops is detached from α and joined instead of the new vertex being disentangled from α. Also, from each previously disentangled vertex, one of the $n - f_i(\alpha)$ edges joining it to α is detached from α, its new end becoming the disentangled vertex instead of α. So, with these properties in mind, by the time the $(n-1)$th step is completed, it is easy to see our single vertex with loops has been transformed into K_n.

Advantageously, the method is even more flexible than described so far in that it is possible to start with a graph G having p vertices, each vertex, v, containing $f(v) = n$ vertices (or even setups more general than that). If each of the p vertices, v, has $\lambda_1 f(v)(f(v) - 1)/2 = \lambda_1 n(n-1)/2$ loops on it, and if between each pair of the p vertices, say u and v, there are $\lambda_2 f(u) f(v) = \lambda_2 n^2$ edges, then this graph is the amalgamation (homomorphic image) of $H = K(n, p, \lambda_1, \lambda_2)$: For each of the p parts of $K(n, p, \lambda_1, \lambda_2)$, amalgamate the n vertices into a single vertex to form G. Notice that this includes the classical complete multipartite graphs, when $\lambda_1 = 0$ and $\lambda_2 = 1$.

Of course, the point here is not just to produce K_n or $K(n, p, \lambda_1, \lambda_2)$; we are really trying to produce Hamilton decompositions (or other graph decompositions) of these graphs. The idea is that if the disentangling process can be achieved, then it is much easier to form the amalgamated graph with a suitable edge-coloring (an outline of the final decomposition) than it is to find the final decomposition directly. So attention also needs to be paid to the color of the edges being selected during the disentangling process, both the loops incident with α and the edges joining the previously disentangled vertices to α. It turns out that we now have a lot of control over the disentanglement. Various results appear in the literature, but the following result is a good example of what is possible. Proved in more generality by Bahmanian and Rodger in [7], it ties in nicely with the fairness notions described earlier. Informally, it says that if $D(v)$ is the set of vertices in H disentangled from v in G, then each vertex u in $D(v)$ receives its fair share of the edge ends in G incident with v, and each vertex u in $D(v)$ receives its fair share of the edge ends in G incident with v colored j. That is, $d_H(v) \in \{\lfloor d_G(v)/n \rfloor, \lceil d_G(v)/n \rceil\}$ and $d_{H(j)}(v) \in \{\lfloor d_{G(j)}(v)/n \rfloor, \lceil d_{G(j)}(v)/n \rceil\}$, where $G(j)$ is the subgraph of G induced by the edges colored j. In Theorem 1, ψ plays the role of the amalgamation function, $\ell_G(u)$ is the number of loops in G incident with u, and $m_G(u, v)$ is the number of edges in G joining u to v.

Theorem 1 [7] *Let G be a k-edge-colored graph and let ψ be a function from $V(G)$ into the positive integers such that for each $u \in V(G)$,*

(1) $\psi(u) = 1$ *implies* $\ell_G(u) = 0$*,*
(2) $d_{G(j)}(u)/\psi(u)$ *is an even integer for all* $1 \le j \le k$*,*
(3) $\binom{\psi(u)}{2}$ *divides* $\ell_G(u)$*,*
(4) $\psi(u)\psi(v)$ *divides* $m_G(u, v)$ *for each* $v \in V(G) \setminus \{u\}$*, and*
(5) $G(j)$ *is connected for* $1 \le j \le k$*.*

Then, there exists a detachment H of G in which each $u \in V(G)$ is disentangled into vertices $u_1, \ldots, u_{\psi(u)}$, such that for all $u \in G$:

(i) $m_H(u_i, u_{i'}) = \ell_G(u)/\binom{\psi(u)}{2}$ *for all* $1 \le i < i' \le \psi(u)$ *if* $\psi(u) \ge 2$*,*
(ii) $m_H(u_i, v_{i'}) = m_G(u, v)/\psi(u)\psi(v)$ *for* $v \in V(G) \setminus \{u\}$*,* $1 \le i \le \psi(u)$*, and* $1 \le i' \le \psi(v)$*,*
(iii) $d_{H(j)}(u_i) = d_{H(j)}(u)/\psi(u)$ *for* $1 \le i \le \psi(u)$ *and* $1 \le j \le k$*, and*
(iv) Each color class $H(j)$ *is connected for* $1 \le j \le k$*.*

Condition (2) is critical for proving that connected color classes in G can remain connected during the disentangling process, thus guaranteeing that condition (iv) is satisfied by H. Since we aim to have each color class disentangled into a Hamilton cycle, clearly each vertex v in the amalgamated graph we construct needs to be incident with $2\psi(v)$ edges colored j, for each color j, since each of the $\psi(v)$ vertices inside v needs to be incident with exactly two edges colored j in the disentangled graph.

Not only does this approach give a new proof of Walecki's [3] result, but it also lends itself beautifully to other families of graphs than complete graphs.

Theorem 2 [6, 10] *There exists a Hamilton decomposition of* $\lambda K(n, p)$ *if and only if* $\lambda n(p-1)$ *is even.*

To see how Theorem 1 is of use in proving Theorem 2, start with p vertices, each joined to each other with λn^2 edges. The edges are then colored with $\lambda n(p-1)/2$ colors so that each color class is connected and $2n$-regular (the details of how the edge-coloring is accomplished are not included here, but one natural approach is to add Hamilton cycles of K_p, each containing edges of just one color, to complete most of the task). It is easy to see that this edge-colored graph satisfies conditions (1–5) of Theorem 1 with $\psi(v) = n$ for all vertices. So the disentangled graph, H: by condition (ii) H is simple, so it must be that $H = \lambda K(n, p)$; by conditions (iii–iv), each color class of H is 2-regular and connected, so is a Hamilton cycle. This completes the proof.

More recently, the existence of Hamilton decompositions of $K(n, p, \lambda_1, \lambda_2)$ was completely settled in the following theorem.

Theorem 3 [7] *Let* $p > 1$*,* $\lambda_1 \geq 0$*, and* $\lambda_2 \geq 1$*, with* $\lambda_1 \neq \lambda_2$ *be integers. Then, there exists a Hamilton decomposition of* $K(n, p, \lambda_1, \lambda_2)$ *if and only if*

(ii) $\lambda_1(n-1) + \lambda_2 n(p-1)$ *is even, and*
(iii) $\lambda_1 \leq \lambda_2 n(p-1)$.

It is hopefully not surprising now that the proof of the sufficiency follows the above approaches closely, starting with p vertices, each joined to each other with λn^2 edges, but this time each vertex is also incident with $\lambda_1 n(n-1)/2$ loops. The edges are then colored so that each color class is connected and $2n$-regular. Once this is done, the result follows essentially immediately from Theorem 1.

The proof of the necessity of Theorem 3 is not included here, but it is worth giving some feel for why condition (iii) is necessary. First note that every Hamilton cycle in $K(n, p, \lambda_1, \lambda_2)$ must use at least p edges joining vertices in different parts in order to be connected. So if we allow λ_1 to grow while holding all other parameters constant, we will eventually run out of the edges joining vertices in different parts. For this reason, an upper bound on λ_1 is to be expected.

3 Embeddings of Edge-Colorings into Hamilton Decompositions

The embedding interest followed the same line as the construction results described in Sect. 2: First studied was embeddings of edge-colored graphs into Hamilton decompositions of K_n (see Theorem 4), then of complete multipartite graphs (see Theorem 5), and then of $K(n, p, \lambda_1, \lambda_2)$ (see Theorems 6 and 7). We now survey this progress, one by one.

In the previous section, Hilton's paper [9] introducing amalgamations as a means of producing graph decompositions was described. One of the great applications he developed was the idea of building prerequisites into the final Hamilton decomposition. In his paper, he proved the following result which completely describes when it is possible to start with a given edge-coloring of K_n and embed it in a Hamilton decomposition of K_m; that is, add $m - n$ new vertices to the given edge-colored K_n, and edges to form a K_m, then color all the added edges so that each color class is a Hamilton cycle. This was truly an amazing result, since typically the given edge-coloring would seemingly need to have much postulated structure or symmetry to make such a result provable. But the amalgamation method is so flexible that he completely solved the problem with the following result.

Theorem 4 [9] *A k-edge-colored K_n (some colors may appear on no edges) can be embedded into a Hamiltonian decomposition of K_m if and only if*

1. *m is odd,*
2. *$k = \lfloor m/2 \rfloor$, and*
3. *Each color class of the given edge-coloring of K_n has at most $m - n$ components, each of which is a path (isolated vertices are considered to be paths of length 0).*

Proof The necessity of these conditions is quite clear: (1–2) follow since in K_m each vertex is incident with exactly two edges of each color; (3) follows because each one of the $m - n$ added vertices can be used to connect just two components in each color class.

Proving the sufficiency clearly demonstrates the power of amalgamations. At first sight, it is not clear at all how to color all the added edges. But we immediately know how to color them in the graph formed by taking any solution to the embedding and amalgamating the added vertices to form a single vertex (in the notation of Sect. 2, the amalgamated vertex is like α, with $f(\alpha) = m - n$). The following shows how to form the amalgamated graph G, even though we do not have a solution (i.e., a Hamilton decomposition of $H = K_m$) in hand.

1. Join each vertex in K_n to the added vertex α with $m - n$ edges.
2. Color the added edges so that each vertex in K_n has degree 2 in each color class. (This is possible since then vertices in K_n would have degree $2k = (n - 1) + (m - n)$.)
3. Add $(m - 1)(m - n - 1)/2$ loops incident with α.
4. To complete the edge-coloring of G, color the loops so that α is incident with exactly $2(m - n)$ edge ends of each color; each loop contributes two edge ends. (This is possible since condition (3) guarantees the number of loops to be added is nonnegative and the number of edges of each color added in the second step is even.)

We can now immediately form a Hamilton decomposition of H from G using Theorem 1 with $\psi(u) = 1$ for all vertices in K_n and $\psi(\alpha) = m - n$. To see this, refer to the various parts of Theorem 1 in turn as follows.

(i) Shows that once the $m-n$ vertices in α are disentangled, the $((m-n)(m-n-1)/2$ loops on α induce a simple graph, which must be K_{m-n}.
(ii) Shows that once the $m-n$ vertices in α are disentangled, the $m-n$ edges joining each vertex u in K_n to α become single edges joining u to each of the $m-n$ disentangled vertices. So at this stage we know that H is K_m.
(iii) Shows that each vertex in H has degree 2 in each color class.
(iv) Shows, together with what was just shown in (iii), that each color class is a Hamilton cycle.

□

Hilton and Rodger [10] extended Theorem 4 to the complete multipartite graphs. They proved the following result as a corollary of a much more general amalgamation theorem.

Theorem 5 [10] *Suppose that $2t \leq s$. Then, a k-edge-coloring of the complete t-partite graph $K_{a_1,\ldots,a_t}$ can be embedded into a Hamiltonian decomposition of the complete p-partite graph $K(n, p, 0, 1)$ if and only if*

(i) Each color class is a set of vertex-disjoint paths,
(i) $a_i \leq n$ for $1 \leq i \leq t$, and
(i) $p(n-1)$ is even.

The proof of Theorem 5, while more complicated, follows the approach outlined above for proving Theorem 4. In this case, the given t-partite graph is first embedded greedily into an edge-colored $K(n, t, 0, 1)$ in which each color class is still a set of vertex-disjoint paths; this can be done since we are assuming that $2t \leq s$. The second step introduces one new vertex, an amalgamated vertex playing the role of α in the outline of the proof of Theorem 4 above, but in this case the technique calls for all vertices within the same part to be disentangled before moving on to vertices from other parts still contained in α.

The embedding of edge-colored copies of $K(n, t, \lambda_1, \lambda_2)$ into Hamilton decompositions of $K(n, p, \lambda_1, \lambda_2)$ is really very interesting. Reasonably obvious numerical conditions are sufficient when p is somewhat larger than t (see Theorem 6), but at this stage it appears that there are conditions which depend upon the existence of certain components in a companion bipartite graph to the given edge-colored graph which are necessary for the embedding to exist (see Theorem 7). This structural property is reminiscent of the long-standing unsolved embedding problem for partial idempotent latin squares of order n into idempotent latin squares of order $n+t$ when t is small: When $t \geq n$ numerical conditions do prove to be sufficient (see [11–13]), but for smaller values of t the existence of certain components in a closely related graph can prevent such an embedding (see [11, 14]).

As in other results mentioned so far, the following is a consequence of a more general amalgamation result in [15] which requires some postulations that are unlikely to be necessary in a complete solution to the problem. Nevertheless, the result is sufficiently general to allow the embedding problem to be solved whenever the number of parts, r, being added to the given edge-colored copy of $K(n, t, \lambda_1, \lambda_2)$

is large enough. It is always a little worrying when a result is described in terms of some parameter being sufficiently large. Often that necessary size for the result to work is really very large. However, the good news in this case is that in fact the lower bound on r for the result to be applicable is not really so large, as the following result indicates.

Theorem 6 [15] *Let* $n > 1$, $\lambda_1 \geq 0$, $\lambda_2 \geq 1$, $\lambda_1 \neq \lambda_2$, $p = t + r$ *and*

$$r \geq \frac{\lambda_1(n-1) + \lambda_2 n(t-1)}{\lambda_2 n(n-1)}. \tag{1}$$

Then, a k*-edge-coloring of* $K(n, t, \lambda_1, \lambda_2)$ *can be embedded into a Hamiltonian decomposition of* $K(n, p, \lambda_1, \lambda_2)$ *if and only if*

1. $k = \big(\lambda_1(n-1) + \lambda_2 n(p-1)\big)/2$,
2. $\lambda_1 \leq \lambda_2 n(p-1)$,
3. *Every component of* $G(j)$ *is a path (possibly of length* 0*) for* $1 \leq j \leq k$*, and*
4. $G(j)$ *has at most* nr *components for* $1 \leq j \leq k$.

In the same paper, using the same general amalgamation theorem, it turns out that the case where $r = 1$ (so just one part is being added) can also be completely solved. So now we need to explore the values of r between 1 and $(\lambda_1(n-1) + \lambda_2 n(t-1))/\lambda_2 n(n-1)$. Starting with the smallest values seems enticing! It turns out that even just considering the case where $r = 2$ is particularly challenging. We appear to enter a different world where the structure can play a deciding role in determining whether or not the embedding of the k-edge-coloring of $G = K(n, t, \lambda_1, \lambda_2)$ into a Hamiltonian decomposition of $H = K(n, p = t + 2, \lambda_1, \lambda_2)$ is possible. To see this, it is best to describe the issue in terms of a related bipartite graph, B. Its vertex set is of course partitioned into two sets: $V(G)$ and $C = \{c_j \mid 1 \leq j \leq k\}$. Each $v \in V(G)$ is joined to c_j in B with x edges if and only if $d_{G(j)}(v) = 2 - x$. Recall that in H each color class is a Hamilton cycle, so each vertex has degree 2 in each color class. So B is keeping a track of how many more edges of each color, j, that v needs added during the embedding process. Connectivity is also a critical aspect of the embedding: The added vertices in the $r = 2$ new parts need to be used to connect up all the paths in $G(j)$ for each color j (so $1 \leq j \leq k$). For various reasons, it seems likely, possibly even necessary, that if $d_B(c_j) \equiv 2 \pmod 4$ then at least one of the components (paths) in $G(j)$ must have its end vertices in G, say $v_{j,1}$ and $v_{j,2}$, joined to different new parts in H. Reproducing this during a proof of the sufficiency is managed by forming B^*, a modification of B constructed by disentangling such c_j into two vertices, one having degree 2 being adjacent to $v_{j,1}$ and $v_{j,2}$. As the embedding proceeds, choosing the path for each color, j, which determines $v_{j,1}$ and $v_{j,2}$ seems to be critical, as is described in condition $(*)$ of Theorem 7 below. Let $\Pi = \{\{v_{j,1}, v_{j,2}\} \mid 1 \leq j \leq k\}$ describe this choice. It is conceivable that condition $(*)$ is also a necessary condition. Let C_2 denote the set of vertices in C of degree 2 (mod 4).

Theorem 7 [16] *Let $n > 1$, $\lambda_1 \geq 0$, $\lambda_2 \geq 1$ and $\lambda_1 \neq \lambda_2$. Suppose we are given a k-edge-coloring of $G = K(n, t, \lambda_1, \lambda_2)$, and that*

(*) *Π can be chosen such that in the detached graph, B^*, the number of components having an odd number of color vertices of degree divisible by 4 is at most $\lambda_2 n^2$.*

Then, the k-edge-coloring of G can be embedded into a Hamiltonian decomposition of $K(n, p = t + 2, \lambda_1, \lambda_2)$ if and only if

(i) *Conditions (1–4) of Theorem 6 with $r = 2$ are satisfied, and*
(v) $|C_2| \leq 2\lambda_1 \binom{n}{2} + \lambda_2 n^2$.

Apart from amalgamations, there is another aspect of the proof of this result which is of interest here since a 2-edge-coloring of B^* is required that has the colors fairly divided in two ways. An edge-coloring of a graph is said to be equitable at vertex v if, for all colors i and j, the number of edges incident with v colored i is within 1 of the number of edges colored j. An edge-coloring of a graph is said to be evenly equitable at vertex v if, for all colors i and j, the number of edges incident with v colored i is even and is within 2 of the number of edges colored j. Hilton [17] proved that evenly equitable edge-colorings (i.e., evenly equitable at all vertices) exist whenever all vertices have even degree. Equitable edge-colorings (i.e., equitable at all vertices) are much more problematic (see [18] for example), but de Werra [19] has shown that they always exist for bipartite graphs. To prove Theorem 7, it was critical that these two results of Hilton and de Werra be generalized to require some vertices to be evenly equitably colored and others to be equitably colored. We end with this crucial lemma, which is of interest in its own right.

Lemma 1 [16] Let B be a finite even bipartite graph with bipartition $\{V, C\}$ of its vertex set. For any subset $X \subseteq C$, there exists a 2-edge-coloring $\sigma : E(B) \to \{1, 2\}$ such that

(i) $d_{B(1)}(v) = d_{B(2)}(v)$ for all $v \in V$,
(ii) $d_{B(1)}(c) = d_{B(2)}(c)$ for all $c \in X$, and
(iii) $|d_{B(1)}(c) - d_{B(2)}(c)| = 2$ for all $c \in C \setminus X$

if and only if

(iv) $|V(D) \cap (C \setminus X)|$ is even for each component D of B.

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Chapter 2
On Strong Pseudomonotone and Strong Quasimonotone Maps

Sanjeev Kumar Singh, Avanish Shahi and S. K. Mishra

Abstract We introduce strong pseudomonotone and strong quasimonotone maps of higher order and establish their relationships with strong pseudoconvexity and strong quasiconvexity of higher order, respectively, which yields first-order characterizations of strong pseudoconvex and strong quasiconvex functions of higher order. Moreover, we answer the open problem (converse part of Proposition 6.2) of Karamardian and Schaible (J. Optim. Theory Appl. 66:37–46,1990), for even more generalized functions, namely strongly pseudoconvex functions of higher order.

Keywords Generalized monotone maps · Generalized convexity · First-order conditions

1 Introduction

Minty [9] introduced the concept of monotone maps. Further, in addition to that Karamardian [5] discussed strict monotone and strongly monotone maps. It is well known that every differentiable function is convex if and only if its gradient map is monotone (see [2, 10]). Karamardian [5] stated the relationship between strongly convex functions and strongly monotone maps. In 1976, Karamardian [4] introduced the concept of pseudomonotone maps and showed that a differentiable pseudoconvex function (see [3, 8]) is characterized by pseudomonotonicity of its gradient map and used monotonicity/pseudomonotonicity in establishing several existence theorems for complementarity problems. Further, Karamardian and Schaible [6] introduced strictly pseudomonotone, quasimonotone, strongly monotone, and strongly

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pseudomonotone maps and showed that for gradient maps, these generalized monotonicity properties are related to generalized convexity properties of the underlying functions.

Lin and Fukushima [7] along with other results for nonlinear programs and mathematical programs with equilibrium constraints introduced strong convexity of order σ and strong monotone maps of order σ. Lin and Fukushima [7] showed that the strong monotonicity of order σ of the gradient map is related to strong convexity of order σ of the function. Arora et al. [1] introduced strongly pseudoconvex functions of order σ and its generalization to characterize solution sets and optimality conditions for optimization problems. Arora et al. [1] have also introduced strongly quasiconvex function of order σ.

It is very natural to see that the concept of strongly monotone maps of order σ due to Lin and Fukushima [7] can be extended to strongly pseudomonotone maps of order σ and strongly quasimonotone maps of order σ can be studied, as Karamardian and Schaible [6] extended the concept of monotone maps to pseudomonotone maps.

In 1990, Karamardian and Schaible [6] left an open problem as the converse of Proposition 6.2 [6], and we have answered that open question positively for a more general function, namely strongly pseudoconvex of order σ, which is also an extension of strongly convex function of order σ given by Lin and Fukushima [7].

2 Preliminaries

2.1 Pseudoconvexity and Quasiconvexity

Definition 1 [2, 6] A differentiable function f on an open convex subset X of $\mathbb{R}^n$ is pseudoconvex on X if, for every pair of distinct points $x, y \in X$, we have

$$\langle \nabla f(y), x - y \rangle \geq 0 \Rightarrow f(x) \geq f(y).$$

Definition 2 [2, 6] A function f is quasiconvex on a convex set X of $\mathbb{R}^n$ if, for all $x, y \in X$, $\lambda \in [0, 1]$,

$$f(x) \leq f(y) \Rightarrow f(\lambda x + (1 - \lambda)y) \leq f(y).$$

Proposition 1 [2, 6] *A differentiable function f is quasiconvex on an open convex set X of $\mathbb{R}^n$ if and only if, for every pair of points $x, y \in X$, we have*

$$f(x) \leq f(y) \Rightarrow \langle \nabla f(y), x - y \rangle \leq 0.$$

Remark 1 [3] Every pseudoconvex function is quasiconvex, but the converse is not necessarily true.