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Representations of Hecke Algebras at Roots of Unity

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Representations of Hecke Algebras at Roots of Unity

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Preface

One of the major open problems in the representation theory of finite groups is the determination of the irreducible representations of the symmetric group $\mathfrak{S}_n$ over a field of characteristic $p > 0$. Thanks to the work of James [179] in the 1970s, we do have a natural parametrisation of the irreducible representations in the framework of the theory of Specht modules, but explicit combinatorial formulae for their dimensions are not known in general! Note that the analogous problem in characteristic 0 has been solved for a long time, by the work of Frobenius around 1900.

In a wider context, this problem is a special case of the problem of determining the irreducible representations of Iwahori–Hecke algebras. These algebras arise naturally in the representation theory of finite groups of Lie type, but they can also be defined abstractly as certain deformations of group algebras of finite Coxeter groups, where the deformation depends on one or several parameters. For the purposes of this introduction, let us assume that all the parameters are integral powers of a fixed element in the base field. If this base parameter has infinite order and the base field has characteristic 0, then we are in the "generic case" where the algebras are semisimple; this case is quite well understood [132], [231]. Also note that, both for historical reasons and as far as applications are concerned, the case where all parameters are equal is particularly important.

The main focus in this text will be on the "modular case" where the algebras are non-semisimple. This situation typically occurs over fields of positive characteristic (a familiar phenomenon from the representation theory of finite groups), but it also occurs over fields of characteristic 0 when the base parameter is a root of unity. While leading to a highly interesting and rich theory in its own right, it turns out that the study of the characteristic 0 situation also provides a crucial step for understanding the positive characteristic case, which is most important for applications to finite groups of Lie type.

Over the last two decades, there has been considerable progress on the characteristic 0 situation. One of the most spectacular advances is the "LLT conjecture" [208] (where "LLT" stands for Lascoux, Leclerc, Thibon) and its proof by Ariki [7], [10]. This brings deep geometric methods and the combinatorics of crystal/canonical bases of quantum groups into the picture, opening the way for a variety of new theoretical connections and practical applications. Combined with sophisticated computational methods, the theory has now reached the following state:

- The classical theory of "Specht modules" has been generalised to Iwahori–Hecke algebras associated to arbitrary finite Coxeter groups, giving rise to natural parametrisations of the irreducible representations.
- Explicit descriptions of these parametrisations are now known in terms of so-called "canonical basic sets". Also, the dimensions of the irreducible representations are known, either by purely combinatorial algorithms (for the classical types) or in the form of explicit tables (for the exceptional types).

These results remain valid over fields of characteristic $p > 0$, as soon as p is larger than some bound depending on the type of the algebra. As far as the parametrisation of the irreducible representations is concerned, the bound is very mild. For example, in the equal-parameter case, it will turn out that it is sufficient to assume that the characteristic is "good" in the sense of the theory of algebraic groups. However, as far as the dimensions of the irreducible representations are concerned, no explicit bound on p is known at the present state of knowledge.

But there is a general conjecture – first formulated by James [181] in type A – specifying such a bound. This conjecture has been verified in a number of cases, including algebras of type A_n for $n \leqslant 9$ (see [181]) and all algebras of exceptional type (see Geck, Lux, and Müller [94], [126], [129]). If true, this conjecture would also yield explicit results about the dimensions of the irreducible representations of the symmetric group $\mathfrak{S}_n$ in characteristic $p > 0$ where p is such that $p^2 > n$.

The purpose of this book is to develop the general theory along the above lines and to show how it is transformed into explicit results. In a sense, this book tries to do for representations of Iwahori–Hecke algebras at roots of unity what the book by Geck and Pfeiffer [132] did for the "generic case". However, while [132] was essentially self-contained, the situation is more complex here. In fact, in order to obtain our main results, we rely on the following sources:

- Ariki's proof [7] of the LLT conjecture.
- Certain deep properties of Kazhdan–Lusztig cells [222], [231] which do not seem to be accessible by elementary methods.
- The existence and basic properties of "canonical bases" and "crystals" for the Fock space representations of certain quantised enveloping algebras.

The first two ultimately rely on deep geometric theories, an exposition of which would go far beyond the scope of this text. Fortunately, this material is now more readily accessible through a number of books; for example, Kirwan [201], Chriss and Ginzburg [50], Hotta et al. [159], Kiehl and Weissauer [197]. Also note that the geometry only plays a role in the proofs, but not in the formulation of the results! (It is not completely impossible that, some day, more direct and purely algebraic proofs will be found.) Much of what we need about crystal and canonical bases can be found in Ariki's book [10]; see also Jantzen [185], Kashiwara [191], Lusztig [230]. Our general policy regarding these topics is that we shall introduce the required notation to state the results that we need, but we will not endeavour to give the proofs. In this way, we can keep the size of this text within reasonable limits, and yet present some substantial results and applications.

The origin of the theory of Iwahori–Hecke algebras lies in the representation theory of finite groups of Lie type, where these algebras arise as endomorphism algebras of certain induced representations. Via some natural functors, a well-defined part of the representation theory of a finite group of Lie type is controlled by the representation theory of Iwahori–Hecke algebras. Thus, the theory and the results that we are going to present in this book form a contribution to the general project of determining the irreducible representations of all non-abelian finite simple groups. Note that such a group is either an alternating group of degree at least 5, or a simple group of Lie type, or one of 26 sporadic simple groups; see Gorenstein et al. [142].

A rough outline of the contents of this book now follows.

Chapters 1 and 2 provide a general introduction to the representation theory of Iwahori–Hecke algebras and, thus, may be of some independent interest. The discussion will be based on the Kazhdan–Lusztig theory of "cells" [195], [219]. In Lusztig's work [220] on characters of reductive groups over finite fields, a crucial role is played by the "**a**-function", which associates with every irreducible representation E of a finite Coxeter group a numerical invariant $\mathbf{a}_E$. One of the main themes of this book will be to show that these invariants play a similarly important role for "modular" representations. In Theorem 2.6.12, this culminates in the construction of a "cell datum" in the sense of Graham and Lehrer [144], giving rise to a general theory of "Specht modules" for Iwahori–Hecke algebras. (These results originally appeared in [111], [112].) Thus, we now see that the original Specht module theory in type A, due to Dipper and James [62] and Murphy [256], [257] (see also the exposition by Mathas [245]), is the prototype of a picture which applies to all Iwahori–Hecke algebras associated with finite Coxeter groups.

In our exposition, we pay a particular attention to treating Iwahori–Hecke algebras of type A as a model case. The required results on Kazhdan–Lusztig cells will be established in a complete and self-contained manner, where no use of geometry is required; see Section 2.8. This treatment of type A is new and entirely independent of the original approach by Dipper, James, and Murphy.

In Chapter 3, we study non-semisimple Iwahori–Hecke algebras in the spirit of Brauer's classical "modular representation theory" involving, in particular, blocks and decomposition numbers. We shall assume that the reader has some familiarity which the basic features of this theory (for a general finite-dimensional associative algebra); this is readily accessible in standard reference texts, like Curtis and Reiner [53] and Feit [83]. In this setting, we define the key concept of a "canonical basic set" in Section 3.2. This concept is independent of the existence of a Graham–Lehrer cell datum, but, in a sense, it captures precisely those features of a cell datum which can be seen by looking only at the decomposition matrix of the algebra. Again, we treat Iwahori–Hecke algebras of type A as a model case. In Section 3.5 we give a new proof of the classification of the modular irreducible representations of these algebras. For this purpose, we have found it convenient to introduce the formal concept of an "abstract Fock datum" in Section 3.4. In another direction, we present a factorisation result for decomposition matrices and formulate a general version of

James's conjecture. The exposition in Section 3.7 unifies the original formulation of James [181] with the further developments in [92], [98], [129], [133].

In Chapter 4 we explain the fundamental connection between Iwahori–Hecke algebras and representations of a finite group of Lie type $G(\mathbb{F}_q)$ (where q is a power of a prime number and $\mathbb{F}_q$ denotes a finite field with q elements). We begin with a self-contained discussion of the Schur functor and its variations, where we combine the original approach of Dipper [58] with later developments by Cline et al. [51] and Schubert [279]. Following [109], we then show in Theorem 4.4.1 how our results on "cell data" and "canonical basic sets" lead to a natural parametrisation of the modular irreducible representations of $G(\mathbb{F}_q)$ which admit non-zero vectors fixed under a Borel subgroup. This generalises classical results from the characteristic 0 situation (due to Bourbaki, Iwahori, Tits, ...) to positive characteristic. We also explain how this fits into a (conjectural) classification of all irreducible representations of $G(\mathbb{F}_q)$ in the "non-defining characteristic case".

The determination of canonical basic sets for the classical types B_n and D_n has turned out to be an extremely difficult problem. At the end of Chapter 4 we shall discuss some cases that can be dealt with by elementary methods, based on the work of Dipper, James, and Murphy [66], [68]. The solution in the general case requires completely new methods; this will be achieved as a consequence of the results presented in Chapters 5 and 6.

For this purpose, it will be convenient to work in the framework of the theory of *Ariki–Koike algebras*, which are generalisations of Iwahori–Hecke algebras of type B_n. The main idea of Chapter 5 is to try to generalise as much as possible the combinatorial constructions involved in the discussion of type A in Chapter 3. This leads us to consider in Section 5.7 certain special choices of the parameters which arise from the combinatorics of "FLOTW multipartitions" (where FLOTW stands for Foda, Leclerc, Okado, Thibon, Welsh [88]); these special choices cover, in particular, the equal parameter case for Iwahori–Hecke algebras of type B_n and D_n. As a consequence, in Theorem 5.8.2, we can state the main result concerning the determination of canonical basic sets for this choice of parameters. The methods in Chapter 5 do not allow us to complete the proof of this theorem. The missing piece is a result about the number of irreducible representations of Ariki–Koike algebras which is due to Ariki and Mathas [15] and which relies on the deep work of Ariki [7] on the proof of the LLT conjecture. This will be discussed in Chapter 6.

The idea that FLOTW multipartitions are relevant in the modular representation theory of Iwahori–Hecke algebras of classical type first appeared in the work of Jacon [172], [173], [174]. Originally, the base field for the algebras was assumed to be of characteristic 0. The new approach developed in Chapter 5 shows that these results also hold for fields of positive characteristic.

In Chapter 6 we introduce the quantised enveloping algebra $\mathscr{U}_q(\widehat{\mathfrak{sl}_e})$ and study the canonical bases of certain Fock space representations. The associated "crystals" carry some rich combinatorial structure which will be discussed in detail. We can state (without proof) Ariki's theorem [7] which links the canonical bases of the Fock space representations to the irreducible representations of Ariki–Koike algebras at roots of unity. This allows us to complete the proofs of the main results of the previous chapter; see Section 6.3. Since this only covers certain choices of the parameters for Ariki–Koike algebras, we then go further and present some deep results of Uglov [291] on the canonical bases of the Fock space representations. We show how this leads to an explicit description of the "canonical basic sets" for Ariki–Koike algebras at roots of unity – and, hence, of Iwahori–Hecke algebras of classical type – for any choice of the parameters, assuming that the base field is of characteristic 0; see Theorem 6.7.2. We also derive purely combinatorial algorithms for computing decomposition numbers and the dimensions of the irreducible representations (in characterictic 0).

Finally, Chapter 7 contains explicit results concerning Iwahori–Hecke algebras of exceptional type H_3, H_4, F_4, E_6, E_7, E_8. We also explain some basic algorithmic methods, including Parker's MeatAxe. The project of computing the decomposition matrices for these algebras (over fields of characteristic 0) was started almost 20 years ago in [126] and finally completed in [129]; the matrices for type E_8 appear here for the first time in print. From these matrices, one can simply read off the corresponding "canonical basic sets".

Acknowledgments. While the idea of writing this book has been around for some time, the actual work began in 2008. Much of the writing of the first chapters was done while the first author enjoyed the hospitality of the Newton Institute (Cambridge, UK) during the special semester on algebraic Lie theory (January to June 2009). We thank the anonymous referees for detailed comments on early versions of the manuscript which, we hope, led to considerable improvements. We are indebted to Gunter Malle, who carefully read all chapters and made numerous and detailed comments, and Cédric Lecouvey for his comments on Chapters 5 and 6.

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Aberdeen and Besançon, *Meinolf Geck*
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Contents

Chapter 1
Generic Iwahori–Hecke Algebras

In this chapter we introduce the main objects of our study: Finite Coxeter groups, generic Iwahori–Hecke algebras, and their representations. The groups and algebras are defined in a purely algebraic way, in terms of generators and defining relations. Thus, a generic Iwahori–Hecke algebra **H** is seen to be a deformation of the group algebra of a Coxeter group W, where the deformation depends on one or several parameters. We recall the relevant definitions and basic results in Sections 1.1 and 1.2, but we tacitly assume that the reader already has some familiarity with them.

Following Lusztig [219], [231], we specify the parameters of **H** by a "weight function" $L\colon W \to \Gamma$, where Γ is a totally ordered abelian group. The typical and most familiar example is the case where $\Gamma = \mathbb{Z}$, with its natural order. More general choices for Γ are useful for several reasons: first of all, this provides the greatest level of generality and flexibility; furthermore, it brings out more clearly the role that is played by the given total order on Γ.

In Section 1.3, the total order is used to define Lusztig's **a**-function, which associates with every irreducible representation $E \in \mathrm{Irr}_{\mathbb{C}}(W)$ an element $\mathbf{a}_E \in \Gamma$. The construction relies in an essential way on the generic algebra **H** and the known connection (via Tits's deformation theorem) between the irreducible representations of W and those of **H**. The explicit results summarized at the end of Section 1.3 show the remarkable dependence of the function $E \mapsto \mathbf{a}_E$ on the total order of Γ.

The study of the **a**-function, and its subtle relation with the Kazhdan–Lusztig basis of **H**, will be one of the main themes of this book. As a first step we will introduce in Section 1.5 an "asymptotic" version of **H**. Our construction, following [112], is logically independent of Lusztig's construction of the asymptotic ring **J** in [223], [231, Chap. 18], but it is, of course, motivated by it. The advantage of our approach is that it does not rely on certain deep properties of the Kazhdan–Lusztig basis of **H** which are not (yet) known to hold in the general multiparameter case. Instead, we rely on properties of the "balanced representations" in Section 1.4.

In Section 1.6, using the asymptotic version of **H**, we can then give a first definition of the partition of W into left, right, and two-sided "cells" and establish some basic properties of them. This is followed by the discussion of a number of examples and further results in Sections 1.7 and 1.8.

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1.1 Coxeter Groups and Weight Functions

We briefly recall the basic definitions concerning Coxeter groups and Iwahori–Hecke algebras. More details and references can be found in [29], [132], [231].

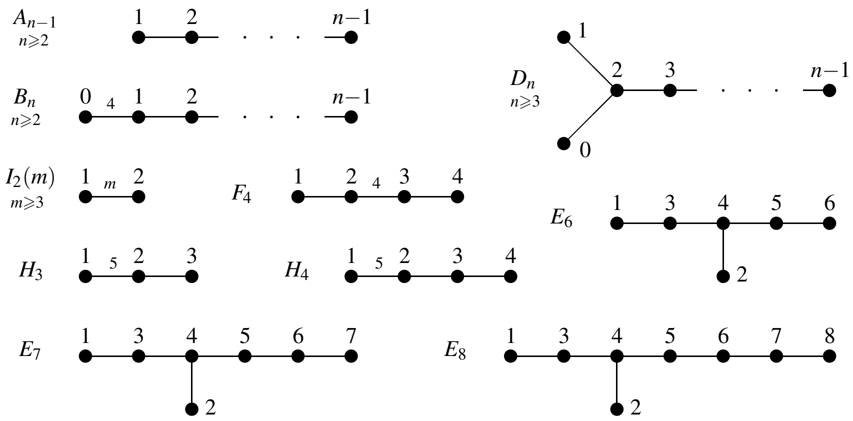
1.1.1. Let S be a finite set and $M = (m_{st})_{s,t \in S}$ be a matrix satisfying $m_{ss} = 1$ for all $s \in S$, and $m_{st} = m_{ts} \in \{2,3,4,5,\ldots\} \cup \{\infty\}$ for all $s \neq t$ in S. Such a matrix is called a *Coxeter matrix*. Then we define a group $W = W(M)$ by a presentation with generators S and defining relations as follows:

- $s^2 = 1$ for all $s \in S$;
- $(st)^{m_{st}} = 1$ for all $s \neq t$ in S with $m_{st} < \infty$.

The pair (W,S) is called a *Coxeter system* and W is called a *Coxeter group*.

We encode the above presentation in a graph, called the *Coxeter graph* of W. It has vertices labelled by the elements of S, and two vertices labelled by $s \neq t$ are joined by an edge if $m_{st} \geqslant 3$. Moreover, if $m_{st} \geqslant 4$, we label the edge by m_{st}. If the graph is connected, we say that W is an *irreducible Coxeter group*. If this is not the case, we have a direct product decomposition $W = W_1 \times \cdots \times W_d$, where $W_i = \langle S_i \rangle$ and each subset $S_i \subseteq S$ corresponds to the vertices in a connected component of the Coxeter graph; furthermore, each W_i is a Coxeter group with generating set S_i. The groups W_i will be called the *irreducible components* of W.

Table 1.1 Coxeter graphs of irreducible finite Coxeter groups



(The numbers on the vertices correspond to a chosen labelling of the elements of S.)

Type	A_{n-1}	B_n	D_n	$I_2(m)$
Order	$n!$	$2^n n!$	$2^{n-1} n!$	$2m$

Type	H_3	H_4	F_4	E_6	E_7	E_8
Order	120	14400	1152	51840	2903040	696729600

There is a complete classification of the finite Coxeter groups. The graphs corresponding to the irreducible finite Coxeter groups, and the group orders, are given in Table 1.1. We say that W is a finite *Weyl group* or is of *crystallographic type* if $m_{st} \in \{2,3,4,6\}$. These are precisely the finite Coxeter groups which arise, for example, in the theory of finite-dimensional semisimple complex Lie algebras, or in the theory of connnected reductive algebraic groups (see also Chapter 4).

The standard example is type A_{n-1}, where W can be identified with the symmetric group $\mathfrak{S}_n$, generated by the basic transpositions $s_i = (i, i+1)$ for $1 \leqslant i \leqslant n-1$. This is the Weyl group for the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of $n \times n$-matrices with trace 0, or for the simple algebraic group $\mathrm{SL}_n(k)$ of $n \times n$-matrices with determinant 1 over any algebraically closed field k.

The groups of type H_3, H_4 or $I_2(m)$ ($m = 5$ or $m > 7$) are non-crystallographic.

1.1.2. Let k be any commutative ring (with 1) and $\{\xi_s \mid s \in S\} \subseteq k^\times$ be a collection of elements such that $\xi_s = \xi_t$ whenever $s,t \in S$ are conjugate in W. Then, by Bourbaki [29, Chap. IV, §2, Exc. 23], we have a corresponding *Iwahori–Hecke algebra*

$$H_k = H_k(W,S,\{\xi_s\}).$$

This is an associative algebra over k which is free as a k-module, with basis $\{T_w \mid w \in W\}$; the multiplication is uniquely determined by the rule

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (\xi_s - \xi_s^{-1})T_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$. Here, $l\colon W \to \mathbb{Z}_{\geqslant 0}$ is the length function on W. Recall that, given $w \in W$, we can write $w = s_1 \cdots s_p$, where $s_i \in S$. If p is minimal with this property, we say that this is a *reduced expression* for w; then $l(w) = p$ is called the length of w. In this case, the above rules imply that $T_w = T_{s_1} \cdots T_{s_p}$.

We note that T_1 is the identity element of H_k. The elements $\{\xi_s\}$ are called the *parameters* of H_k. We also remark that if $W = W_1 \times \cdots \times W_d$ is the decomposition into irreducible components (where $W_i = \langle S_i \rangle$ as above), then we have

$$H_k \cong H_k(W_1, S_1, \{\xi_s\}_{s\in S_1}) \otimes_k \cdots \otimes_k H_k(W_d, S_d, \{\xi_s\}_{s\in S_d});$$

see [132, Exc. 8.4]. In this way, many questions about Iwahori–Hecke algebras in general can be reduced to the case where (W,S) is irreducible.

Example 1.1.3. (a) Assume that $\xi_s = 1$ for all $s \in S$. Then the map $w \mapsto T_w$ defines an isomorphism of k-algebras from kW (the group algebra of W over k) onto H_k.

(b) Assume that $\xi_s = \xi_t$ for all $s,t \in S$; this case will be referred to as the *equal-parameter case*. We are automatically in this case when W is of type A_{n-1}, D_n, $I_2(m)$ (m odd), H_3, H_4, E_6, E_7 or E_8 (since all generators in S are conjugate in W).

(c) Assume that W is finite and irreducible. Then unequal parameters can only occur in types B_n, F_4 or $I_2(m)$ (m even). In these cases, the set S falls into two classes under conjugation by W; see also Example 1.1.11(b) below.

1.1.4. The purpose of this book is to address the following problem.

Fundamental Problem. *Assume that W is finite and k is a field. Then determine the irreducible representations of $H_k(W,S,\{\xi_s\})$.*

By Example 1.1.3, this includes the problem of determining the irreducible representations of the symmetric group $\mathfrak{S}_n \cong W(A_{n-1})$ over fields of positive characteristic. Note that there are many open questions even in this special case; in particular, the dimensions of the irreducible representations are not known!

If $H_k(W,S,\{\xi_s\})$ is semisimple, then the above problem is essentially solved; see [132], [231]. So the main focus in this text will be on the non-semisimple situation. The first step consists of noting that any algebra H_k as above can be obtained from a suitable "generic" Iwahori–Hecke algebra by a process of specialisation. For this purpose, we introduce the following notion where, following a suggestion of Bonnafé [21], we combine the two settings in [219], [231].

Definition 1.1.5 (Lusztig). Let Γ be an abelian group (written additively). We say that a function $L\colon W \to \Gamma$ is a *weight function* if the following condition holds:

$$L(ww') = L(w) + L(w') \text{ for all } w, w' \in W \text{ such that } l(ww') = l(w) + l(w').$$

Note that L is uniquely determined by the values $\{L(s) \mid s \in S\}$. Furthermore, if $\{c_s \mid s \in S\}$ is a collection of elements in Γ such that $c_s = c_t$ whenever $s,t \in S$ are conjugate in W, then there is a (unique) weight function $L\colon W \to \Gamma$ such that $L(s) = c_s$ for all $s \in S$. (This follows from Matsumoto's lemma; see [132, §1.2].)

1.1.6. Let us assume that a weight function $L\colon W \to \Gamma$ has been fixed. Let $R \subseteq \mathbb{C}$ be a subring and $A = R[\Gamma]$ be the free R-module with basis $\{\varepsilon^g \mid g \in \Gamma\}$. There is a well-defined ring structure on A such that $\varepsilon^g \varepsilon^{g'} = \varepsilon^{g+g'}$ for all $g, g' \in \Gamma$. We write $1 = \varepsilon^0 \in A$. Given $a \in A$ we denote by a_g the coefficient of ε^g, so that $a = \sum_{g \in \Gamma} a_g \varepsilon^g$.

We apply the general construction in 1.1.2 to the ring A and the collection of elements $\{v_s \mid s \in S\}$ where $v_s := \varepsilon^{L(s)}$ for $s \in S$. The corresponding algebra will be denoted by $\mathbf{H} = \mathbf{H}_A(W,S,L)$ and called the *generic Iwahori–Hecke algebra* associated with W, L. Thus, $\mathbf{H}$ is an associative algebra which is free as an A-module, with basis $\{T_w \mid w \in W\}$; the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (v_s - v_s^{-1}) T_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$. The element T_1 is the identity element.

In the setting of 1.1.2, assume that there is a ring homomorphism $\theta\colon A \to k$ such that $\theta(v_s) = \xi_s$ for all $s \in S$. Then we can regard k as an A-module (via θ), and we find that H_k is obtained by extension of scalars from $\mathbf{H}$:

$$H_k(W,S,\{\xi_s\}) \cong k \otimes_A \mathbf{H}_A(W,S,L).$$

In this situation, we say that $\theta\colon A \to k$ is a *specialisation* and that $H_k(W,S,\{\xi_s\})$ is obtained from $\mathbf{H}$ by *specialisation* (via θ). For example, if $\theta_1\colon A \to k$ is a ring homomorphism such that $\theta_1(\varepsilon^g) = 1$ for all $g \in \Gamma$, then $k \otimes_A \mathbf{H}_A(W,S,L) \cong kW$.

Example 1.1.7. Let $\Gamma = \mathbb{Z}$. Then $A = R[v, v^{-1}]$ is the ring of Laurent polynomials over R in one indeterminate $v := \varepsilon$. Let $L \colon W \to \mathbb{Z}$ be a weight function and set $c_s = L(s)$ for $s \in S$. Then the relations in $\mathbf{H}$ read as follows, where $s \in S$ and $w \in W$:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (v^{c_s} - v^{-c_s}) T_w & \text{if } l(sw) < l(w). \end{cases}$$

This is the setting of Lusztig [231]; it is particularly relevant for applications to the representation theory of reductive groups over finite fields; see Section 4.2.

Remark 1.1.8. For various applications, it will be convenient to set $\dot{T}_w := \varepsilon^{L(w)} T_w$ for all $w \in W$ and $u_s := v_s^2$ for all $s \in S$. Then, clearly, $\{\dot{T}_w \mid w \in W\}$ also is an A-basis of $\mathbf{H}$; furthermore, we have the multiplication rules:

$$\dot{T}_s \dot{T}_w = \begin{cases} \dot{T}_{sw} & \text{if } l(sw) > l(w), \\ u_s \dot{T}_{sw} + (u_s - 1)\dot{T}_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$. Thus, the introduction of the basis $\{\dot{T}_w \mid w \in W\}$ shows that $\mathbf{H}$ is already defined over the subring $\mathbb{Z}[u_s \mid s \in S] \subseteq A$; that is, all structure constants with respect to this basis lie in $\mathbb{Z}[u_s \mid s \in S]$.

Example 1.1.9. A "universal" weight function is given as follows. For $s, t \in S$, we write $s \sim t$ if s, t are conjugate in W. Let $S' \subseteq S$ be a set of representatives for the equivalence classes of S under this relation. Let Γ_0 be the group of all tuples $(n_s)_{s \in S'}$ where $n_s \in \mathbb{Z}$ for all $s \in S'$. (The addition is defined componentwise.) Let $L_0 \colon W \to \Gamma_0$ be the weight function given by sending $s \in S$ to the tuple $(n_t)_{t \in S'}$, where $n_t = 1$ if t is conjugate to s and $n_t = 0$ otherwise. Let $A_0 = R[\Gamma_0]$ and $\mathbf{H}_0 = \mathbf{H}_{A_0}(W, S, L_0)$ be the associated Iwahori–Hecke algebra; we denote the parameters by $\{v_s^\circ \mid s \in S\}$ in this case. Note that A_0 is nothing but the ring of Laurent polynomials in $\{v_s^\circ \mid s \in S'\}$ (and these elements are algebraically independent).

Any algebra $H_k(W, S, \{\xi_s\})$ as above is obtained by specialisation from $\mathbf{H}_0$ (where we take $R = \mathbb{Z}$). Indeed, since $\{v_s^\circ \mid s \in S'\}$ are algebraically independent, we can certainly find a unital ring homomorphism $\theta_0 \colon A_0 \to k$ such that $\theta_0(v_s^\circ) = \xi_s$ for all $s \in S$. Thus, $H_k(W, S, \{\xi_s\}) \cong k \otimes_{A_0} \mathbf{H}_0$ (via θ_0).

1.1.10. As in [219], we shall assume that Γ admits a total ordering $\leqslant$ which is compatible with the group structure; that is, whenever $g, g' \in \Gamma$ are such that $g \leqslant g'$, we have $g + h \leqslant g' + h$ for all $h \in \Gamma$. Such an order will be called a *monomial order*. We usually assume that $L(s) \geqslant 0$ for all $s \in S$. (We will see in Lemma 1.1.12 below that this is no severe restriction.) The existence of a monomial order on Γ implies that Γ is torsion free. Furthermore, we can write any $0 \neq a \in A$ uniquely in the form

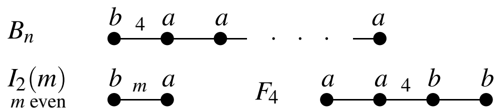
$$a = a_1 \varepsilon^{g_1} + \cdots + a_d \varepsilon^{g_d} \quad \text{where} \quad 0 \neq a_i \in R,\ g_i \in \Gamma \ \text{ and } \ g_1 < \ldots < g_d.$$

We denote $\mathrm{lt}(a) := a_1 \varepsilon^{g_1}$ and call this the *leading term* of a. We also set $\mathrm{lt}(0) := 0$. Then one easily checks that $\mathrm{lt}(aa') = \mathrm{lt}(a)\mathrm{lt}(a')$ for any $a, a' \in A$. In particular, if $a \neq 0$ and $a' \neq 0$, then $\mathrm{lt}(aa') \neq 0$ and so $aa' \neq 0$; hence, A is an integral domain.

Finally, since S is a finite set, it is usually sufficient to consider the case where Γ is finitely generated. Consequently, in this case, we have $\Gamma \cong \mathbb{Z}^r$ for some $r \geqslant 1$, which means that A is a ring of Laurent polynomials in r variables. Then specifying a monomial order $\leqslant$ on Γ amounts to specifying a total order on the monomials in A (compatible with the multiplication).

Example 1.1.11. (a) In the set-up of Example 1.1.7, there is a natural monomial order on $\Gamma = \mathbb{Z}$, and we will usually assume that $c_s \geqslant 0$ for all $s \in S$.

(b) Assume that W is of type B_n, F_4 or $I_2(m)$ (m even). Then, in general, L depends on two values $a, b \in \Gamma$, which are attached to the generators in S:



The possible choices of monomial orders that are available here can best be seen by taking $L = L_0$ to be the "universal" weight function in Example 1.1.9, where $\Gamma_0 = \mathbb{Z}^2$, $b = (1,0)$ and $a = (0,1)$. Then $A_0 = R[V^{\pm 1}, v^{\pm 1}]$ is the ring of Laurent polynomials in the indeterminates $V := \varepsilon^{(1,0)}$ and $v := \varepsilon^{(0,1)}$. A familiar monomial order is the *pure lexicographic order* given by

$$(i,j) \leqslant_{\text{lex}} (i',j') \quad \overset{\text{def}}{\Leftrightarrow} \quad i \leqslant i' \quad \text{or} \quad i = i' \text{ and } j \leqslant j' \qquad (i,i',j,j' \in \mathbb{Z}).$$

More generally, for any $\alpha \in \mathbb{R}$, we have a monomial ordering $\leqslant_\alpha$ given by

$$(i,j) \leqslant_\alpha (i',j') \qquad \overset{\text{def}}{\Leftrightarrow} \qquad \begin{array}{l} i + \alpha j < i' + \alpha j' \text{ or} \\ i + \alpha j = i' + \alpha j' \text{ and } j \leqslant j' \end{array}.$$

In particular, we see that there are infinitely many monomial orders on Γ_0. For a classification of all orderings on Γ_0, see Tutorial 10 in [206, §1.4] and also [271].

(c) Now assume that W is any finite Coxeter group and $L \colon W \to \Gamma$ is a weight function. Let $\leqslant$ be a monomial order on Γ. By analogy to Bonnafé and Iancu [26], we say that we are in the *asymptotic case* if $L(s) > 0$ for all $s \in S$ and if, on any irreducible component of type B_n, F_4 or $I_2(m)$ (m even), where L takes values $a, b \in \Gamma$ as above, we have $b > ra > 0$ for all $r \in \mathbb{Z}_{\geqslant 1}$.

As already mentioned above, we will usually assume that $L(s) \geqslant 0$ for all $s \in S$. This is justified by the following result, observed by Bonnafé [22, Cor. 5.8].

Lemma 1.1.12. *Let $\leqslant$ be a monomial order on Γ. For $s \in S$, set $\delta_s = 1$ if $L(s) \geqslant 0$, and $\delta_s = -1$ if $L(s) < 0$. Then there is a well-defined weight function $L' \colon W \to \Gamma$ such that $L'(s) = \delta_s L(s)$ ($s \in S$); note that $L'(s) \geqslant 0$ for all $s \in S$.*

Let $\mathbf{H}'$ be the generic Iwahori–Hecke algebra associated with W, L' and let $\{T'_w \mid w \in W\}$ be the standard basis of $\mathbf{H}'$. Then there is a unique A-algebra isomorphism $\mathbf{H} \to \mathbf{H}'$ such that $T_s \mapsto \delta_s T'_s$ for all $s \in S$.

Proof. To show that there is a weight function L' as above, we need to check that $\delta_s = \delta_t$ whenever $s, t \in S$ are conjugate in W. But, if $s, t \in S$ are conjugate, then

$L(s) = L(t)$ and so δ_s, δ_t will either both be $+1$ or both be -1. Now let us show that there is an algebra isomorphism $\mathbf{H} \to \mathbf{H}'$ as above. By [132, 4.4.5], the algebra $\mathbf{H}$ has a presentation with generators $\{T_s \mid s \in S\}$ and defining relations

$$\begin{aligned} T_s^2 &= T_1 + (\varepsilon^{L(s)} - \varepsilon^{-L(s)})T_s \qquad && (s \in S), \\ \underbrace{T_sT_tT_s\cdots}_{m_{st}\text{ terms}} &= \underbrace{T_tT_sT_t\cdots}_{m_{st}\text{ terms}} && (s,t \in S, s \neq t, m_{st} < \infty). \end{aligned}$$

Hence, all we need to check is whether the elements $\{\delta_s T_s' \mid s \in S\}$ satisfy the above relations. Now, we have

$$(\delta_s T_s')^2 = T_{s'}^2 = T_1' + (\varepsilon^{L'(s)} - \varepsilon^{-L'(s)})T_s' = T_1' + (\delta_s\varepsilon^{L'(s)} - \delta_s\varepsilon^{-L'(s)})(\delta_s T_s').$$

It remains to note that if $\delta_s = -1$, then $\delta_s\varepsilon^{L'(s)} - \delta_s\varepsilon^{-L'(s)} = -\varepsilon^{-L(s)} + \varepsilon^{L(s)}$, as required. Now let us check the second type of relations. Let $s \neq t$ in S be such that $m_{st} < \infty$. Then the verification reduces to proving that $\delta_s\delta_t\delta_s\cdots = \delta_t\delta_s\delta_t\cdots$ (with m_{st} factors on both sides). If $\delta_s = \delta_t$, this is clear. Now assume that $\delta_s \neq \delta_t$. In particular, $L(s) \neq L(t)$ in this case and so m_{st} must be even. But then, on both sides of the above identity, we have $m_{st}/2$ factors corresponding to s and $m_{st}/2$ factors corresponding to t. Hence, we get the same result on both sides. $\square$

Remark 1.1.13. Let $\theta\colon A \to k$ be a specialisation into a field k and consider the specialised algebra $\mathbf{H}_k$. Then, in the setting of Lemma 1.1.12, the algebras $\mathbf{H}_k$ and $\mathbf{H}_k'$ are isomorphic. Hence, if we have solved our "fundamental problem" in 1.1.4 for $\mathbf{H}_k'$, then this problem is automatically solved for $\mathbf{H}_k$ as well. Thus, indeed, it will be sufficient to consider weight functions such that $L(s) \geqslant 0$ for all $s \in S$.

1.2 Representations of H

We will assume from now on that W is finite. Let $L\colon W \to \Gamma$ be a weight function and $\mathbf{H}$ be the associated generic Iwahori–Hecke algebra. Let us now turn to the representation theory of W and of $\mathbf{H}$.

1.2.1. We set $\mathbb{Z}_W := \mathbb{Z}[2\cos(2\pi/m_{st}) \mid s,t \in S] \subseteq \mathbb{R}$. For example, $\mathbb{Z}_W = \mathbb{Z}$ if W is a finite Weyl group. We shall always assume that $\mathbb{Z}_W \subseteq R$ (where $R \subseteq \mathbb{C}$ is the subring used to define A). Then the field of fractions of R, which will be denoted by $\mathbb{K}$, is a splitting field for W; see [132, Theorem 6.3.8]. Throughout, we use the following notation for the irreducible representations of W (up to isomorphism):

$$\mathrm{Irr}_{\mathbb{K}}(W) = \{E^\lambda \mid \lambda \in \Lambda\},$$

where Λ is a finite indexing set and each E^λ is a $\mathbb{K}$-vector space with a given $\mathbb{K}W$-module structure. We also use the notation

$$d_\lambda = \dim E^\lambda, \qquad M(\lambda) = \text{ an indexing set for a basis of } E^\lambda$$

(where $M(\lambda)$ is ordered in some way so that it makes sense to write down matrices with rows and columns indexed by $M(\lambda)$).

Finally, we assume that Γ admits a monomial ordering as in 1.1.10. As we have seen, this implies that A is an integral domain. Let K be the field of fractions of A; by extension of scalars, we obtain a finite-dimensional K-algebra $\mathbf{H}_K = K \otimes_A \mathbf{H}$ which is known to be split semisimple; see [132, Theorem 9.3.5]. Let $\mathrm{Irr}(\mathbf{H}_K)$ denote the set of irreducible representations of $\mathbf{H}_K$ (up to isomorphism). By *Tits's deformation theorem*, there is a bijection between this set and $\mathrm{Irr}_{\mathbb{K}}(W)$; see [132, 8.1.7] and also Exercises 26 and 27 of Bourbaki [29, Chap. IV, §2]. Thus, we can write

$$\mathrm{Irr}(\mathbf{H}_K) = \{E_\varepsilon^\lambda \mid \lambda \in \Lambda\} \qquad (d_\lambda = \dim E_\varepsilon^\lambda),$$

where each E_ε^λ is a K-vector space with a given $\mathbf{H}_K$-module structure. The correspondence $E^\lambda \leftrightarrow E_\varepsilon^\lambda$ is uniquely determined by the condition

$$\mathrm{trace}(w, E^\lambda) = \theta_1\big(\mathrm{trace}(T_w, E_\varepsilon^\lambda)\big) \qquad \text{for all } w \in W,$$

where $\theta_1 \colon A \to R$ is the unique R-linear ring homomorphism such that $\theta_1(\varepsilon^g) = 1$ for all $g \in \Gamma$. Note that, by [132, Theorem 9.3.5], we have

$$\varepsilon^{L(w)}\,\mathrm{trace}(T_w, E_\varepsilon^\lambda) \in \mathbb{Z}_W[v_s \mid s \in S] \qquad \text{for all } w \in W;$$

in particular, these traces lie in A and so it makes sense to apply θ_1 to them. It also follows that, for *any* $\mathbf{H}_K$-module V, we have $\mathrm{trace}(T_w, V) \in A$ for all $w \in W$.

Remark 1.2.2. The proofs of the statements summarized in 1.2.1, especially the statements concerning splitting fields for W and $\mathbf{H}_K$, are by no means easy. In fact, various chapters of [132] are concerned with these questions, where case-by-case arguments (according to the classification of finite Coxeter groups) are required. A number of authors have contributed to the establishment of these results, over an extended period of time; see the bibliographic comments in [132, §5.7 and §9.5].

If one is mainly interested in finite Weyl groups and the equal-parameter case, then more conceptual arguments are available via the geometry of an associated algebraic group; see Springer [282] and Lusztig [216] (see also Example 2.5.7).

Example 1.2.3. Let $\mathrm{Cl}(W)$ be the set of conjugacy classes of W. For $C \in \mathrm{Cl}(W)$, let $w_C \in C$ be a representative which has minimal length in C. Then the matrix

(a) $$X(\mathbf{H}) := \big(\mathrm{trace}(\dot{T}_{w_C}, E_\varepsilon^\lambda)\big)_{\lambda \in \Lambda, C \in \mathrm{Cl}(W)}$$

is called the *character table* of $\mathbf{H}$, where we define $\dot{T}_w := \varepsilon^{L(w)} T_w$ for any $w \in W$, as in Remark 1.1.8. By a result due to Geck and Pfeiffer [132, 8.2.9], $X(\mathbf{H})$ does not depend on the choice of the representatives $\{w_C \mid C \in \mathrm{Cl}(W)\}$; furthermore, there is a unique set of polynomials $\{f_{w,C} \mid w \in W, C \in \mathrm{Cl}(W)\} \subseteq \mathbb{Z}[v_s^2 \mid s \in S]$ such that

(b) $$\mathrm{trace}(\dot{T}_w, E_\varepsilon^\lambda) = \sum_{C \in \mathrm{Cl}(W)} f_{w,C}\,\mathrm{trace}(\dot{T}_{w_C}, E_\varepsilon^\lambda) \qquad \text{for any } \lambda \in \Lambda \text{ and } w \in W.$$

The tables $X(\mathbf{H})$ are explicitly known for all W, L; see [132, Chap. 10 and 11] and the references there. When we apply the specialisation homomorphism $\theta_1 : A \to R$ to the entries of $X(\mathbf{H})$, we obtain the classical character table of the finite group W.

Example 1.2.4. We shall frequently apply the following "specialisation argument" for representations. Let V be an $\mathbf{H}_K$-module and V' be a $\mathbb{K}W$-module such that

$$(*) \qquad \operatorname{trace}(w, V') = \theta_1\big(\operatorname{trace}(T_w, V)\big) \qquad \text{for all } w \in W,$$

where θ_1 is defined as above; recall that $\mathbb{K}W \cong \mathbb{K} \otimes_A \mathbf{H}$, where $\mathbb{K}$ is regarded as an A-module via θ_1. For example, $(*)$ will certainly hold when $V \cong K \otimes_A M$ and $V' \cong \mathbb{K} \otimes_A M$, where M is an $\mathbf{H}$-module which is finitely generated and free over A.

For any $\lambda \in \Lambda$, denote by $m(V, \lambda)$ the multiplicity of E_ε^λ as an irreducible constituent of V, and denote by $m(V', \lambda)$ the multiplicity of E^λ as an irreducible constituent of V'. Thus, we have

$$\operatorname{trace}(T_w, V) = \sum_{\lambda \in \Lambda} m(V, \lambda) \operatorname{trace}(T_w, E_\varepsilon^\lambda) \qquad \text{for all } w \in W,$$
$$\operatorname{trace}(w, V') = \sum_{\lambda \in \Lambda} m(V', \lambda) \operatorname{trace}(w, E^\lambda) \qquad \text{for all } w \in W.$$

Applying θ_1 and using Tits's deformation theorem, we obtain that

$$\sum_{\lambda \in \Lambda} m(V, \lambda) \operatorname{trace}(w, E^\lambda) = \sum_{\lambda \in \Lambda} m(V', \lambda) \operatorname{trace}(w, E^\lambda) \qquad \text{for all } w \in W.$$

Since the trace functions associated with the irreducible representations of W are linearly independent, we deduce that $m(V, \lambda) = m(V', \lambda)$ for all $\lambda \in \Lambda$.

Example 1.2.5. It is known that every $w \in W$ is conjugate to its inverse; see [132, 3.2.14]. Hence, we have $\operatorname{trace}(w, E^\lambda) = \operatorname{trace}(w^{-1}, E^\lambda)$ for all $\lambda \in \Lambda$. A similar property holds on the level of $\mathbf{H}_K$; that is, we have

$$\text{(a)} \qquad \operatorname{trace}(T_w, E_\varepsilon^\lambda) = \operatorname{trace}(T_{w^{-1}}, E_\varepsilon^\lambda) \qquad \text{for all } w \in W.$$

This is seen as follows. It is easily checked that the A-linear map $h \mapsto h^\flat$ defined by $T_w^\flat = T_{w^{-1}}$ $(w \in W)$ is an anti-involution of $\mathbf{H}$. So we can define the *contragredient module* $\hat{E}_\varepsilon^\lambda := \operatorname{Hom}_K(E_\varepsilon^\lambda, K)$ where T_w acts via $T_w \colon \varphi \mapsto \varphi \circ T_{w^{-1}}$ for $\varphi \in \hat{E}_\varepsilon^\lambda$.

For any $w \in W$, we have $\operatorname{trace}(T_w, \hat{E}_\varepsilon^\lambda) = \operatorname{trace}(T_{w^{-1}}, E_\varepsilon^\lambda)$ and, hence,

$$\theta_1\big(\operatorname{trace}(T_w, \hat{E}_\varepsilon^\lambda)\big) = \operatorname{trace}(w^{-1}, E^\lambda) = \operatorname{trace}(w, E^\lambda) = \theta_1\big(\operatorname{trace}(T_w, E_\varepsilon^\lambda)\big).$$

By Tits's deformation theorem, this implies that $\hat{E}_\varepsilon^\lambda \cong E_\varepsilon^\lambda$ and so (a) holds.

Example 1.2.6. Let sgn denote the sign representation of W, which is given by the group homomorphism sending each $w \in W$ to $(-1)^{l(w)}$. Via tensoring with sgn, we obtain a bijection $\lambda \mapsto \lambda^\dagger$ of Λ such that

$$\text{(a)} \qquad E^{\lambda^\dagger} \cong E^\lambda \otimes \operatorname{sgn} \qquad \text{for all } \lambda \in \Lambda.$$

This operation can be lifted to representations of $\mathbf{H}_K$. Namely, there is a unique A-algebra automorphism $\dagger\colon \mathbf{H} \to \mathbf{H}$ such that $T_s^\dagger = -T_s^{-1}$ for all $s \in S$; see [132, Exc. 8.2]. By extension of scalars, this induces a K-algebra automorphism of $\mathbf{H}_K$, which we denote by the same symbol. Given any finite-dimensional $\mathbf{H}_K$-module V, denote by $V^\dagger$ the $\mathbf{H}_K$-module with the same underlying vector space V, but where $h \in \mathbf{H}_K$ acts via $h^\dagger$. Then, by [132, Prop. 9.4.1], we have

(b) $$(E^{\lambda^\dagger})_\varepsilon \cong (E_\varepsilon^\lambda)^\dagger \qquad \text{for all } \lambda \in \Lambda.$$

The trace of T_w on $E_\varepsilon^{\lambda^\dagger}$ is determined as follows. There is a unique R-linear ring homomorphism $A \to A$, $a \mapsto \bar{a}$, such that $\overline{\varepsilon^g} = \varepsilon^{-g}$ for all $g \in \Gamma$. Then we have

(c) $$\operatorname{trace}(T_w, E_\varepsilon^{\lambda^\dagger}) = (-1)^{l(w)}\overline{\operatorname{trace}(T_w, E_\varepsilon^\lambda)} \qquad \text{for all } w \in W;$$

see [132, Prop. 9.4.1].

Example 1.2.7. Let $w_0 \in W$ be the longest element. Then $T_{w_0}^2$ lies in the centre of $\mathbf{H}_K$ and, hence, acts by a scalar in every irreducible representation of $\mathbf{H}_K$. This scalar can be explicitly described, as follows. Let $T := \{wsw^{-1} \mid s \in S, w \in W\}$ be the set of all reflections in W. Let $S' \subseteq S$ be a set of representatives of the conjugacy classes of W which are contained in T. For $s \in S'$, let N_s be the cardinality of the conjugacy class of s; thus, $|T| = \sum_{s\in S'} N_s$. Let $\rho^\lambda : \mathbf{H}_K \to M_{d_\lambda}(K)$ be a representation afforded by E_ε^λ. Then, by an argument due to Springer (see [132, Theorem 9.2.2]), we have

(a) $$\rho^\lambda(T_{w_0}^2) = \varepsilon^{2N_\lambda} I_{d_\lambda} \quad \text{where} \quad N_\lambda := \sum_{s\in S'} \left(\frac{N_s \operatorname{trace}(s, E^\lambda)}{\dim E^\lambda}\right) L(s) \in \Gamma$$

and I_{d_λ} denotes the identity matrix of size d_λ. Note that, since every $s \in S'$ has order 2, we have $\operatorname{trace}(s, E^\lambda) \in \mathbb{Z}$ and so, by a well-known result in the character theory of finite groups, the quantity $N_s \operatorname{trace}(s, E^\lambda)/\dim E^\lambda$ also is an integer. Thus, the expression defining N_λ is a well-defined element of Γ.

Now let us set $P := \varepsilon^{-N_\lambda}\rho^\lambda(T_{w_0})$. Then $P^2 = I_{d_\lambda}$ and so P is a diagonalisable matrix with eigenvalues ± 1; in particular, $m := \operatorname{trace}(P) \in \mathbb{Z}$. Thus, we obtain that $\operatorname{trace}(T_{w_0}, E_\varepsilon^\lambda) = m\varepsilon^{N_\lambda}$. Applying the specialisation homomorphism $\theta_1 : A \to R$, we conclude that $m = \operatorname{trace}(w_0, E^\lambda)$ and, hence,

(b) $$\operatorname{trace}(T_{w_0}, E_\varepsilon^\lambda) = \operatorname{trace}(w_0, E^\lambda)\, \varepsilon^{N_\lambda}.$$

Thus, $\operatorname{trace}(T_{w_0}, E_\varepsilon^\lambda)$ is explicitly described in terms of character values of W.

Remark 1.2.8. We have already remarked in 1.2.1 that $\operatorname{trace}(T_w, E_\varepsilon^\lambda) \in A$ for all $w \in W$. So it is natural to ask if it is even possible to find a representation ρ^λ afforded by E_ε^λ such that $\rho^\lambda(T_w) \in M_{d_\lambda}(A)$ for all $w \in W$. This is indeed the case, but the proof requires some deep results on the Kazhdan–Lusztig basis of $\mathbf{H}$ and a case-by-case analysis; see [132, 9.3.8] and the references there. (We will recover this result in the context of cellular algebras in Corollary 2.7.14.) However, we can establish a weak version of this statement by a general argument, and this will be useful in our discussion of cells in Section 1.6. This relies on the following ring-theoretic result.

Lemma 1.2.9 (Rouquier [272]). *Assume that $\Gamma = \mathbb{Z}$, so that A is the ring of Laurent polynomials in one indeterminate $v = \varepsilon$. Then the subring*

$$\mathscr{R} := \left\{ \frac{f}{g} \;\middle|\; f \in \mathbb{Z}[v, v^{-1}], g \in \mathbb{Z}[v] \text{ and } g \text{ has constant term } 1 \right\} \subseteq K.$$

is a principal ideal domain.

Proof. We need some standard results from commutative ring theory; a suitable reference is Matsumura [248]. Following Broué and Kim [33, §2.B], the first and crucial step is to show that $\mathscr{R}$ is a Dedekind domain. For this purpose, by [248, Theorem 11.6], it is enough to show that $\mathscr{R}$ is a one-dimensional noetherian domain which is integrally closed in $\mathbb{Q}(v)$ (its field of fractions). Now note that $\mathscr{R} \subseteq \mathbb{Q}(v)$ is the localisation of $\mathbb{Z}[v, v^{-1}]$ with respect to the multiplicatively closed set

$$M = \{g \in \mathbb{Z}[v] \mid g \text{ has constant term } 1\}.$$

But the ring $\mathbb{Z}[v, v^{-1}]$ is known to be noetherian and integrally closed in $\mathbb{Q}(v)$. These properties pass on to localisations and, hence, $\mathscr{R}$ is noetherian and integrally closed in $\mathbb{Q}(v)$. In order to show that $\mathscr{R}$ is one-dimensional, we must show that every non-zero prime ideal of $\mathscr{R}$ is maximal. So let $\mathfrak{p}$ be a non-zero prime ideal of $\mathscr{R}$. By [248, Theorem 4.1], $\mathfrak{p}$ is generated by a prime ideal $I \subseteq \mathbb{Z}[v, v^{-1}]$ such that $I \cap M = \varnothing$. The prime ideals in $\mathbb{Z}[v, v^{-1}]$ are explicitly known (see, for example, [132, Exc. 7.9]). Thus, I is either principal (generated by a prime number in $\mathbb{Z}$ or by an irreducible polynomial in $\mathbb{Z}[v]$) or generated by two elements ℓ and f where $\ell > 0$ is a prime number and $f \in \mathbb{Z}[v]$ is a monic polynomial whose reduction modulo ℓ is an irreducible polynomial in $\mathbb{F}_l[v]$. But ideals of the latter type have non-empty intersection with M. Indeed, every irreducible polynomial in $\mathbb{F}_l[v]$ divides $v^{l^m} - v$ for some $m \geqslant 1$. Hence, we have $v^{l^m} - v \in I$ for some $m \geqslant 1$. Since v is a unit in $\mathbb{Z}[v, v^{-1}]$, this implies that $1 - v^{l^m - 1} \in I \cap M$ and so $I \cap M$ is non-empty, as claimed.

Consequently, all the non-zero prime ideals in $\mathscr{R}$ are principal. In particular, this shows that every non-zero prime ideal in $\mathscr{R}$ is maximal and, hence, $\mathscr{R}$ is a Dedekind domain. Finally, in a Dedekind domain, every non-zero ideal is a product of a finite number of prime ideals. We have just seen that every prime ideal in $\mathscr{R}$ is principal. Hence, every ideal in $\mathscr{R}$ is principal. □

Corollary 1.2.10. *Assume that W is a Weyl group and $\Gamma = \mathbb{Z}$. Then, for each $\lambda \in \Lambda$, there exists a representation $\pi^\lambda : \mathbf{H}_K \to M_{d_\lambda}(K)$ afforded by E_ε^λ such that $\pi^\lambda(T_w) \in M_{d_\lambda}(\mathscr{R})$ for all $w \in W$.*

Proof. Since W is a Weyl group, we have $\mathbb{Z}_W = \mathbb{Z}$ and so we can take $R = \mathbb{Z}$. Hence, $K = \mathbb{Q}(v)$ is the field of fractions of the ring $\mathscr{R}$, which is a principal ideal domain by Lemma 1.2.9. But then a standard argument (see, for example, [132, 7.3.7]) shows that every irreducible representation of $\mathbf{H}_K$ can be realised over $\mathscr{R}$. □

1.2.11. We define an A-linear map $\tau\colon \mathbf{H} \to A$ by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$. Then we have $\tau(T_y T_x) = \tau(T_x T_y) = \delta_{x^{-1}y}$ (Kronecker delta); see [132, 8.1.1]. So τ defines a non-degenerate symmetric bilinear form on $\mathbf{H}$, where $\{T_w \mid w \in W\}$ and $\{T_{w^{-1}} \mid w \in W\}$ form a pair of dual bases. Thus, $\mathbf{H}$ is a *symmetric algebra*, with trace form τ. Let τ_K be the canonical extension to a trace form on $\mathbf{H}_K$.

For $\lambda \in \Lambda$, let χ^λ denote the *character* of E_ε^λ; we have $\chi^\lambda(h) = \operatorname{trace}(h, E_\varepsilon^\lambda) \in A$ for all $h \in \mathbf{H}$. Since $\mathbf{H}_K$ is split semisimple, the characters $\{\chi^\lambda \mid \lambda \in \Lambda\}$ form a basis of the vector space of all trace functions on $\mathbf{H}_K$. Hence, by the general theory of symmetric algebras, there is a unique expression (see [132, 7.2.6]):

(a) $$\tau_K = \sum_{\lambda \in \Lambda} \mathbf{c}_\lambda^{-1} \chi^\lambda \qquad \text{where} \qquad 0 \neq \mathbf{c}_\lambda \in K.$$

The elements $\mathbf{c}_\lambda$ were called *Schur elements* in [96]. By [132, Theorem 9.3.5], we have the following important integrality property:

(b) $$\varepsilon^{L(w_0)} \mathbf{c}_\lambda \in \mathbb{Z}_W[v_s \mid s \in S] \qquad \text{for all } \lambda \in \Lambda;$$

In particular, $\mathbf{c}_\lambda$ lies in A. The *generic degree* corresponding to E^λ is defined by

(c) $$\delta_\lambda := \mathbf{c}_\lambda^{-1} P_{W,L} \in K \qquad \text{where} \qquad P_{W,L} := \sum_{w \in W} \varepsilon^{2L(w)},$$

but note that this may no longer be an element of A.

Choosing a basis of E_ε^λ, indexed by $M(\lambda)$ as above, we obtain a matrix representation $\rho^\lambda\colon \mathbf{H}_K \to M_{d_\lambda}(K)$. Given $h \in \mathbf{H}_K$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, we denote by $\rho_{\mathfrak{st}}^\lambda(h)$ the $(\mathfrak{s}, \mathfrak{t})$-entry of $\rho^\lambda(h)$. We now have the following *Schur relations*. In particular, these yield an alternative characterisation of the elements $\mathbf{c}_\lambda$.

Proposition 1.2.12 (Schur relations; cf. [132, 7.2.2]). *Let $\{B_w \mid w \in W\}$ be any basis of $\mathbf{H}$ and $\{B_w^\vee \mid w \in W\}$ the corresponding dual basis, such that $\tau(B_x B_y^\vee) = \delta_{xy}$ for all $x, y \in W$. Given $\lambda, \mu \in \Lambda$, let $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$ and $\mathfrak{u}, \mathfrak{v} \in M(\mu)$. Then*

$$\sum_{w \in W} \rho_{\mathfrak{st}}^\lambda(B_w)\, \rho_{\mathfrak{uv}}^\mu(B_w^\vee) = \begin{cases} \mathbf{c}_\lambda & \text{if } \lambda = \mu,\ \mathfrak{s} = \mathfrak{v},\ \mathfrak{t} = \mathfrak{u}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, this implies the orthogonality relations

$$\sum_{w \in W} \chi^\lambda(B_w)\, \chi^\mu(B_w^\vee) = \begin{cases} d_\lambda \mathbf{c}_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

The origin of the definition of the elements $\mathbf{c}_\lambda$ and of the generic degrees δ_λ lies in the representation theory of finite groups of Lie type. Without going into much detail at this stage, let us briefly describe this connection.

1.2.13. Let us assume that W is of crystallographic type and arises as the Weyl group of a family of finite groups of Lie type

$$\mathscr{S} = \{G(q) \mid q \text{ any prime power}\}.$$

Assume that all groups $G(q)$ are of "split type". Some examples are given by

$$\begin{array}{lll} W = W(A_{n-1}) & \text{where} & G(q) = \mathrm{GL}_n(\mathbb{F}_q) \text{ for all } q, \\ W = W(B_n) & \text{where} & G(q) = \mathrm{SO}_{2n+1}(\mathbb{F}_q) \text{ for all } q, \\ W = W(E_8) & \text{where} & G(q) = E_8(\mathbb{F}_q) \text{ for all } q. \end{array}$$

Let $\Gamma = \mathbb{Z}$ and the weight function L be such that $L(s) = 1$ for all $s \in S$. Then $A = R[v, v^{-1}]$ is the ring of Laurent polynomials in one indeterminate $v = \varepsilon$. In this situation, it is known that $\mathbf{c}_\lambda$ divides $P_{W,L}$ in $\mathbb{K}[v, v^{-1}]$ and that there is a polynomial $D_\lambda \in \mathbb{Q}[u]$ (where u is an indeterminate) such that $\delta_\lambda = D_\lambda(v^2) \in \mathbb{Q}[v]$; see [132, 9.3.6]. This polynomial D_λ has the following interpretation.

Given a prime power q, let us consider the complex irreducible representations of $G(q)$. Let $B(q) \subseteq G(q)$ be a Borel subgroup and define

$$\mathrm{Irr}_{\mathbb{C}}(G(q), B(q)) := \left\{ \rho \in \mathrm{Irr}_{\mathbb{C}}(G(q)) \;\middle|\; \begin{array}{l} \rho \text{ occurs in the permutation} \\ \text{representation } \mathbb{C}[G(q)/B(q)] \end{array} \right\},$$

the set of (unipotent) *principal series representations*. Then, by classical results due to Iwahori, Tits, Benson and Curtis, there exists a bijection

$$\mathrm{Irr}_{\mathbb{C}}(W) \xrightarrow{\sim} \mathrm{Irr}_{\mathbb{C}}(G(q), B(q)), \qquad E^\lambda \mapsto \rho_q^\lambda,$$

such that $\dim \rho_q^\lambda = D_\lambda(q)$ for all $\lambda \in \Lambda$. (See [53, §68], [132, §8.4] or Section 4.3 in this book.) Thus, $D_\lambda(q)$ is the dimension of an irreducible representation of $G(q)$.

1.3 Lusztig's a-Invariants

In Lusztig's work [220] on characters of reductive groups over finite fields, a crucial role is played by a certain function which attaches to each irreducible representation E^λ of W an invariant $\mathbf{a}_\lambda \in \mathbb{Z}_{\geqslant 0}$. In the situation of 1.2.13, this is defined by

$$\mathbf{a}_\lambda := \max\{i \geqslant 0 \mid u^i \text{ divides } D_\lambda\} \qquad \text{for any } \lambda \in \Lambda.$$

(See also 2.2.12 and 4.3.12 for further interpretations of $\mathbf{a}_\lambda$.) One of the main themes of this book will be to show that these **a**-*invariants* play a similarly important role for "modular" representations.

Throughout this section, let W be a finite Coxeter group and $L\colon W \to \Gamma$ be a weight function. Let $\leqslant$ be a monomial order on Γ such that $L(s) \geqslant 0$ for all $s \in S$. Let $\Gamma_{\geqslant 0} := \{g \in \Gamma \mid g \geqslant 0\}$ and denote by $R[\Gamma_{\geqslant 0}]$ the set of all R-linear combinations of terms ε^g where $g \geqslant 0$. The notations $R[\Gamma_{>0}]$, $R[\Gamma_{\leqslant 0}]$, $R[\Gamma_{<0}]$ have a similar meaning. We are now ready to introduce the invariants $\mathbf{a}_\lambda$ in general.

Proposition 1.3.1 (Lusztig). *Let $\lambda \in \Lambda$. Then there exist some $\mathbf{a}_\lambda \in \Gamma_{\geqslant 0}$ and a strictly positive real number $f_\lambda \in \mathbb{Z}_W$ such that*

$$\varepsilon^{2\mathbf{a}_\lambda}\,\mathbf{c}_\lambda \in R[\Gamma_{\geqslant 0}] \quad \textit{and} \quad \varepsilon^{2\mathbf{a}_\lambda}\,\mathbf{c}_\lambda \equiv f_\lambda \mod R[\Gamma_{>0}].$$

Note that $\mathbf{a}_\lambda$ and f_λ are uniquely determined by these conditions. We have

$$\mathbf{a}_\lambda = \min\{g \in \Gamma_{\geqslant 0} \mid \varepsilon^g \chi^\lambda(T_w) \in \mathbb{K}[\Gamma_{\geqslant 0}] \textit{ for all } w \in W\}.$$

Proof. (Cf. [217, 1.9].) For the following discussion, it will be useful to assume that $\mathbb{K} \subseteq \mathbb{R}$. By Example 1.2.5(a), we have $\chi^\lambda(T_w) = \chi^\lambda(T_{w^{-1}})$ for all $w \in W$. Hence, using the orthogonality relations in Proposition 1.2.12 we obtain

$$\sum_{w \in W} \chi^\lambda(T_w)^2 = \sum_{w \in W} \chi^\lambda(T_w)\,\chi^\lambda(T_{w^{-1}}) = d_\lambda\,\mathbf{c}_\lambda.$$

Now we set $\mathbf{a}_\lambda := \min\{g \in \Gamma_{\geqslant 0} \mid \varepsilon^g \chi^\lambda(T_w) \in \mathbb{K}[\Gamma_{\geqslant 0}]$ for all $w \in W\}$. For each $w \in W$, denote by $c_{w,\lambda} \in \mathbb{K}$ the constant term of $\varepsilon^{\mathbf{a}_\lambda}\chi^\lambda(T_w)$; note that $c_{w,\lambda} \neq 0$ for at least one $w \in W$. Let $f'_\lambda := \sum_{w \in W} c_{w,\lambda}^2$. Since all $c_{w,\lambda}$ are real numbers, not all of which are zero, we conclude that f'_λ is a strictly positive real number. Now we obtain

$$\varepsilon^{2\mathbf{a}_\lambda} \sum_{w \in W} \chi^\lambda(T_w)^2 \equiv \sum_{w \in W} \left(\varepsilon^{\mathbf{a}_\lambda}\chi^\lambda(T_w)\right)^2 \equiv \sum_{w \in W} c_{w,\lambda}^2 \equiv f'_\lambda \mod \mathbb{K}[\Gamma_{>0}].$$

Hence, setting $f_\lambda = f'_\lambda/d_\lambda$, we see that $\varepsilon^{2\mathbf{a}_\lambda}\,\mathbf{c}_\lambda \in f_\lambda + \mathbb{K}[\Gamma_{>0}]$. Finally, since $\mathbf{c}_\lambda \in \mathbb{Z}_W[\Gamma]$, we must have $f_\lambda \in \mathbb{Z}_W$. □

Remark 1.3.2. Let $\lambda \in \Lambda$ and $w \in W$. By Proposition 1.3.1, we can write

$$\varepsilon^{\mathbf{a}_\lambda}\chi^\lambda(T_w) = c_{w,\lambda} + \mathbb{K}\text{-linear combination of terms } \varepsilon^g \text{ where } g > 0,$$

where $c_{w,\lambda} \in \mathbb{K}$. These are the "leading coefficients of character values" considered by Lusztig [220, Chap. 5], [225]. From the orthogonality relations in Proposition 1.2.12 (see also [132, Exc. 9.8]), we immediately deduce that

$$\sum_{w \in W} c_{w,\lambda}\,c_{w^{-1},\mu} = \begin{cases} f_\lambda d_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the coefficients $c_{w,\lambda}$ behave as if they were the character values of an algebra with a basis indexed by the elements of W; see Section 1.5 for a further discussion.

Example 1.3.3. Assume that $L(s) > 0$ for all $s \in S$. Let $\lambda \in \Lambda$.

(a) If E^λ is the unit representation, then $\mathbf{a}_\lambda = 0$ and $f_\lambda = 1$.
(b) If E^λ is the sign representation, then $\mathbf{a}_\lambda = L(w_0)$ and $f_\lambda = 1$.
(c) If E^λ is neither the unit nor the sign representation, then $0 < \mathbf{a}_\lambda < L(w_0)$.

Here, $w_0 \in W$ denotes the longest element. (Note that these statements fail if $L(s) = 0$ for some $s \in S$; see Example 1.3.7 below.) First note that the unit and the sign representation of W correspond to the following representations of $\mathbf{H}_K$, respectively:

$$\begin{aligned} \mathrm{ind}_\varepsilon \colon \mathbf{H}_K \to K, \qquad & T_w \mapsto \varepsilon^{L(w)}, \\ \mathrm{sgn}_\varepsilon \colon \mathbf{H}_K \to K, \qquad & T_w \mapsto (-1)^{l(w)} \varepsilon^{-L(w)}. \end{aligned}$$

The condition that $L(s) > 0$ for all $s \in S$ implies that $0 < L(w) < L(w_0)$ for all $w \in W \setminus \{1, w_0\}$. This immediately yields (a) and (b). To prove (c), first note that $\mathrm{ind}_\varepsilon(T_w) \in R[\Gamma_{>0}]$ for $w \neq 1$. This yields that

$$\varepsilon^{\mathbf{a}_\lambda} \sum_{w \in W} \chi^\lambda(T_w)\, \mathrm{ind}_\varepsilon(T_{w^{-1}}) \equiv \varepsilon^{\mathbf{a}_\lambda} \chi^\lambda(T_1) \bmod R[\Gamma_{>0}].$$

Since $E_\varepsilon^\lambda \not\cong \mathrm{ind}_\varepsilon$, the sum on the left-hand side must be zero by Proposition 1.2.12. Hence, $\mathbf{a}_\lambda > 0$, as required. On the other hand, by 1.2.1, we have $\varepsilon^{L(w)}\chi^\lambda(T_w) \in R[\Gamma_{\geqslant 0}]$ for all $w \in W$. This already shows that $\mathbf{a}_\lambda \leqslant L(w_0)$. Now assume, if possible, that $\mathbf{a}_\lambda = L(w_0)$. Then $\varepsilon^{L(w_0)}\chi^\lambda(T_w) \in R[\Gamma_{>0}]$ for all $w \neq w_0$ and $\varepsilon^{L(w_0)}\chi^\lambda(T_{w_0})$ has a non-zero constant term. Using (b), it follows that

$$\varepsilon^{2L(w_0)} \sum_{w \in W} \chi^\lambda(T_w)\, \mathrm{sgn}_\varepsilon(T_{w^{-1}}) \equiv \pm \varepsilon^{L(w_0)} \chi^\lambda(T_{w_0}) \not\equiv 0 \bmod R[\Gamma_{>0}].$$

But, since $E_\varepsilon^\lambda \not\cong \mathrm{sgn}_\varepsilon$, we also deduce from Proposition 1.2.12 that the sum on the left-hand side is zero, which is a contradiction. So we have $\mathbf{a}_\lambda < L(w_0)$, as required.

Example 1.3.4. Recall from Example 1.2.6 that we have a "duality" operation $\lambda \mapsto \lambda^\dagger$ on Λ such that $E^{\lambda^\dagger} \cong E^\lambda \otimes \mathrm{sgn}$. By [132, Prop. 9.4.3], we have

$$\mathbf{c}_{\lambda^\dagger} = \bar{\mathbf{c}}_\lambda = \varepsilon^{-2N_\lambda} \mathbf{c}_\lambda \quad \text{and} \quad \mathbf{a}_{\lambda^\dagger} - \mathbf{a}_\lambda = N_\lambda, \quad \text{with } N_\lambda \text{ as in Example 1.2.7(a).}$$

Here, the map $A \to A$, $a \mapsto \bar{a}$, is defined as in Example 1.2.6.

Remark 1.3.5. Let $W = W_1 \times \cdots \times W_d$ be the decomposition into irreducible components. Correspondingly, we have

$$\mathrm{Irr}_{\mathbb{K}}(W) = \{E^{\lambda_1} \boxtimes \cdots \boxtimes E^{\lambda_d} \mid \lambda_i \in \Lambda_i\}, \quad \text{where} \quad \mathrm{Irr}_{\mathbb{K}}(W_i) = \{E^{\lambda_i} \mid \lambda_i \in \Lambda_i\}.$$

Thus, we can identify $\Lambda = \Lambda_1 \times \cdots \times \Lambda_d$. As already noted in 1.1.2, we have a tensor product decomposition $\mathbf{H} \cong \mathbf{H}_1 \otimes_A \cdots \otimes_A \mathbf{H}_d$, where $\mathbf{H}_i$ is the generic algebra associated with W_i and the restriction of L to W_i. Hence, we also have

$$\mathrm{Irr}(\mathbf{H}_K) = \{E_\varepsilon^{\lambda_1} \boxtimes \cdots \boxtimes E_\varepsilon^{\lambda_d} \mid \lambda_i \in \Lambda_i\}.$$

By [132, Exc. 8.5], this yields that $\mathbf{c}_\lambda = \mathbf{c}_{\lambda_1} \cdots \mathbf{c}_{\lambda_d}$, where $\lambda = (\lambda_1, \ldots, \lambda_d)$. Consequently, we have $\mathbf{a}_\lambda = \mathbf{a}_{\lambda_1} + \cdots + \mathbf{a}_{\lambda_d}$ and $f_\lambda = f_{\lambda_1} \cdots f_{\lambda_d}$. Thus, the determination of $\mathbf{a}_\lambda$ and f_λ can be reduced to the case where (W, S) is irreducible.

Remark 1.3.6. The elements $\mathbf{c}_\lambda$ are explicitly known for all types of W; see [132, Chap. 10 and 11] and the references there. It turns out that they have a quite special

Table 1.2 The invariants f_λ and $\mathbf{a}_\lambda$ for type F_4

	$b>2a>0$		$b=2a>0$		$2a>b>a>0$		$b=a>0$		$b>a=0$	
E^λ	f_λ	$\mathbf{a}_\lambda$	f_λ	$\mathbf{a}_\lambda$	f_λ	$\mathbf{a}_\lambda$	f_λ	$\mathbf{a}_\lambda$	f_λ	$\mathbf{a}_\lambda$
1_1	1	0	1	0	1	0	1	0	6	0
1_2	1	$12b-9a$	2	$15a$	1	$11b-7a$	8	$4a$	6	$12b$
1_3	1	$3a$	2	$3a$	1	$-b+5a$	8	$4a$	6	0
1_4	1	$12b+12a$	1	$36a$	1	$12b+12a$	1	$24a$	6	$12b$
2_1	1	$3b-3a$	2	$3a$	1	$2b-a$	2	a	12	$3b$
2_2	1	$3b+9a$	2	$15a$	1	$2b+11a$	2	$13a$	12	$3b$
2_3	1	a	1	a	1	a	2	a	3	0
2_4	1	$12b+a$	1	$25a$	1	$12b+a$	2	$13a$	3	$12b$
4_1	2	$3b+a$	2	$7a$	2	$3b+a$	8	$4a$	6	$3b$
9_1	1	$2b-a$	2	$3a$	1	$b+a$	1	$2a$	2	$2b$
9_2	1	$6b-2a$	1	$10a$	1	$6b-2a$	8	$4a$	2	$6b$
9_3	1	$2b+2a$	1	$6a$	1	$2b+2a$	8	$4a$	2	$2b$
9_4	1	$6b+3a$	2	$15a$	1	$5b+5a$	1	$10a$	2	$6b$
6_1	3	$3b+a$	3	$7a$	3	$3b+a$	3	$4a$	12	$3b$
6_2	3	$3b+a$	3	$7a$	3	$3b+a$	12	$4a$	12	$3b$
12_1	6	$3b+a$	6	$7a$	6	$3b+a$	24	$4a$	6	$3b$
4_2	1	b	1	$2a$	1	b	2	a	6	b
4_3	1	$7b-3a$	1	$11a$	1	$7b-3a$	4	$4a$	6	$7b$
4_4	1	$b+3a$	1	$5a$	1	$b+3a$	4	$4a$	6	b
4_5	1	$7b+6a$	1	$20a$	1	$7b+6a$	2	$13a$	6	$7b$
8_1	1	$3b$	1	$6a$	1	$3b$	1	$3a$	12	$3b$
8_2	1	$3b+6a$	1	$12a$	1	$3b+6a$	1	$9a$	12	$3b$
8_3	1	$b+a$	2	$3a$	1	$3a$	1	$3a$	3	b
8_4	1	$7b+a$	2	$15a$	1	$6b+3a$	1	$9a$	3	$7b$
16_1	2	$3b+a$	2	$7a$	2	$3b+a$	4	$4a$	6	$3b$

The notation for $\mathrm{Irr}_{\mathbb{K}}(W)$ is defined in [132, Appendix C.3].
Here, $a := L(s_1) = L(s_2)$ and $b := L(s_3) = L(s_4)$, cf. Table 1.1.

form. Indeed, one checks that there is a family $\{\Phi_d \mid d \in I\}$ of monic polynomials in one variable over $\mathbb{Z}_W$ such that

(a) $\mathbf{c}_\lambda = f_\lambda \varepsilon^{\gamma_\lambda} \prod_{d\in I} \Phi_d\big(\varepsilon^{\gamma_{\lambda,d}}\big)^{n_{\lambda,d}}$, where $\gamma_\lambda \in \Gamma$, $n_{\lambda,d} \geqslant 0$ and $0 \neq \gamma_{\lambda,d} \in \Gamma$;

(b) all the (complex) roots of Φ_d are roots of unity;

(c) the product of all $\mathbf{c}_\lambda$ (for $\lambda \in \Lambda$) lies in $\mathbb{Z}[u_s^{\pm 1} \mid s \in S]$, where $u_s := v_s^2$ ($s \in S$).

Note that the monomials γ_λ and $\gamma_{\lambda,i}$ are not uniquely determined. In fact, depending on the monomial order $\leqslant$ on Γ, the terms involving those monomials are rearranged so as to produce the relation $\varepsilon^{2\mathbf{a}_\lambda}\mathbf{c}_\lambda \equiv f_\lambda \mod R[\Gamma_{>0}]$ in Proposition 1.3.1.

From the explicit knowledge of $\mathbf{c}_\lambda$ one can deduce explicit formulae for the invariants $\mathbf{a}_\lambda$ and f_λ. If $L(s) = 0$ for all $s \in S$, then $\mathbf{c}_\lambda = |W|/d_\lambda$. Hence, $\mathbf{a}_\lambda = 0$ and $f_\lambda = |W|/d_\lambda$ for all $\lambda \in \Lambda$ in this case. Now assume that $L(s) > 0$ for at least some $s \in S$. For W of exceptional type H_3, H_4, E_6, E_7, E_8 (where we are automatically in the equal-parameter case), see the tables in [220, Chap. 4] and in the Appendices C and E in [132]. For type F_4, see Table 1.2 (p. 16); note that, by the symmetry of the diagram, we can assume without loss generality that $L(s_1) = L(s_2) \leqslant L(s_3) = L(s_4)$. For the types $I_2(m)$, A_{n-1}, B_n and D_n, see the examples below.

Table 1.3 The irreducible representations of $\mathbf{H}_K$ in type $I_2(m)$

$$\begin{array}{llll}
1_W^{\varepsilon}: & T_{s_1} \mapsto v_{s_1}, & T_{s_2} \mapsto v_{s_2}, \\
\mathrm{sgn}^{\varepsilon}: & T_{s_1} \mapsto -v_{s_1}^{-1}, & T_{s_2} \mapsto -v_{s_2}^{-1}, \\
\mathrm{sgn}_1^{\varepsilon}: & T_{s_1} \mapsto v_{s_1}, & T_{s_2} \mapsto -v_{s_2}^{-1}, \\
\mathrm{sgn}_2^{\varepsilon}: & T_{s_1} \mapsto -v_{s_1}^{-1}, & T_{s_2} \mapsto v_{s_2}, \\
\sigma_j^{\varepsilon}: & T_{s_1} \mapsto \begin{pmatrix} -v_{s_1}^{-1} & 0 \\ \mu_j & v_{s_1} \end{pmatrix}, & T_{s_2} \mapsto \begin{pmatrix} v_{s_2} & 1 \\ 0 & -v_{s_2}^{-1} \end{pmatrix}
\end{array}$$

λ	1_W	(sgn_1)	σ_j	(sgn_2)	sgn	
$\mathbf{a}_\lambda$	0	a	a	a	ma	$(b=a>0)$
f_λ	1	$\frac{m}{2}$	$\frac{m}{2-\zeta^j-\zeta^{-j}}$	$\frac{m}{2}$	1	$(b=a>0)$
$\mathbf{a}_\lambda$	0	a	b	$\frac{m}{2}(b-a)+a$	$\frac{m}{2}(a+b)$	$(b>a\geqslant 0)$
f_λ	1	1	$\frac{m}{2-\zeta^{2j}-\zeta^{-2j}}$	1	1	$(b>a>0)$
f_λ	2	2	$\frac{m}{2-\zeta^{2j}-\zeta^{-2j}}$	2	2	$(b>a=0)$

(where $b := L(s_1)$, $a := L(s_2)$ and $\mu_j := v_{s_1}v_{s_2}^{-1} + \zeta^j + \zeta^{-j} + v_{s_1}^{-1}v_{s_2}$)

Example 1.3.7. Let W be of type $I_2(m)$ $(m \geqslant 3)$; that is, $W = \langle s_1, s_2 \rangle$, where $s_1^2 = s_2^2 = (s_1s_2)^m = 1$. In this case, $\mathbb{Z}_W = \mathbb{Z}[\zeta + \zeta^{-1}]$, where $\zeta \in \mathbb{C}$ is a root of unity of order m, chosen such that $\zeta + \zeta^{-1} = 2\cos(2\pi/m)$. By [132, §5.4], we have

$$\mathrm{Irr}_{\mathbb{K}}(W) = \begin{cases} \{1_W, \mathrm{sgn}, \sigma_1, \sigma_2, \ldots, \sigma_{(m-1)/2}\} & \text{if } m \text{ is odd,} \\ \{1_W, \mathrm{sgn}, \sigma_1, \sigma_2, \ldots, \sigma_{(m-2)/2}, \mathrm{sgn}_1, \mathrm{sgn}_2\} & \text{if } m \text{ is even,} \end{cases}$$

where 1_W is the unit and sgn is the sign representation, all σ_j are two-dimensional, and $\mathrm{sgn}_1, \mathrm{sgn}_2$ are two further one-dimensional representations when m is even, in which case we fix the notation such that s_1 acts as $+1$ in sgn_1 and as -1 in sgn_2. By [132, §8.3], explicit realisations of the corresponding representations of $\mathbf{H}_K$ are known; see Table 1.3. Using the formulae for $\mathbf{c}_\lambda$ in [132, Theorem 8.3.4], one obtains the invariants $\mathbf{a}_\lambda$ and f_λ. In Table 1.3, the columns are ordered such that the invariants $\mathbf{a}_\lambda$ are increasing from left to right; the columns corresponding to sgn_1, sgn_2 must be deleted if m is odd.

Let us give a concrete example to see how $\mathbf{a}_\lambda$ and f_λ are computed as a function of the monomial order $\leqslant$. Let $m = 4$ so that W is the dihedral group of order 8. Let $\zeta \in \mathbb{C}$ be a fourth root of unity; then, by the general formula in [132, 8.3.4], we have

$$\begin{aligned}\mathbf{c}_{\sigma_1} &= 4\frac{(v_{s_1}^2 v_{s_2}^2 - (\zeta+\zeta^{-1})v_{s_1}v_{s_2}+1)(v_{s_1}^2+(\zeta+\zeta^{-1})v_{s_1}v_{s_2}+v_{s_2}^2)}{v_{s_1}^2 v_{s_2}^2(2-(\zeta^2+\zeta^{-2}))}\\ &= v_{s_1}^{-2}v_{s_2}^{-2}(v_{s_1}^2 v_{s_2}^2+1)(v_{s_1}^2+v_{s_2}^2)\\ &= \varepsilon^{-2a}+\varepsilon^{-2b}+\varepsilon^{2a}+\varepsilon^{2b},\end{aligned}$$

where $b := L(s_1)$ and $a := L(s_2)$. Hence, if $b > a > 0$, then all four terms are different, $f_{\sigma_1} = 1$ and the "lowest" term is ε^{-2b}; so $\mathbf{a}_{\sigma_1} = b$. Similarly, if $a > b > 0$, then $f_{\sigma_1} = 1$ and $\mathbf{a}_{\sigma_1} = a$. If $b > a = 0$, then $\mathbf{c}_{\sigma_1} = \varepsilon^{-2b}+2+\varepsilon^{2b}$ and so $f_{\sigma_1} = 1$, $\mathbf{a}_{\sigma_1} = b$. Finally, if $a = b > 0$, then $\mathbf{c}_{\sigma_1} = 2(\varepsilon^{-2a}+\varepsilon^{2a})$ and so $f_{\sigma_1} = 2$, $\mathbf{a}_{\sigma_1} = a$.

In the next three examples, we describe the invariants $\mathbf{a}_\lambda$ for W of type A_{n-1}, B_n, D_n. In these cases, $\mathrm{Irr}(W)$ is parametrised by suitable partitions or pairs of partitions, where we follow the notational conventions of [220, Chap. 4], [132, Chap. 5].

Recall that a *partition* is a finite, weakly decreasing sequence of non-negative integers; we often write this in the form $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant 0)$. We say that λ is a partition of n, and write $\lambda \vdash n$, if $|\lambda| := \lambda_1 + \cdots + \lambda_N$ equals n; the numbers λ_i which are non-zero are called the *parts* of λ. As is usual, we will not distinguish between two partitions which have identical parts.

Example 1.3.8. Let W be of type A_{n-1}, where $W \cong \mathfrak{S}_n$. We are automatically in the equal-parameter case; write $a := L(s) > 0$ for $s \in S$. There is a standard labelling

$$\mathrm{Irr}_{\mathbb{K}}(W) = \{E^\lambda \mid \lambda \in \Lambda\}, \quad \text{where} \quad \Lambda = \{\text{set of all partitions of } n\}.$$

For example, the unit and the sign representation are labelled by (n) and (1^n), respectively; see [132, §5.4]. (Here, (1^n) denotes the partition which has n parts equal to 1.) By [132, Prop. 9.4.5], given a partition λ of n, we have

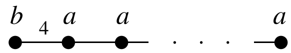
$$f_\lambda = 1 \qquad \text{and} \qquad \mathbf{a}_\lambda = n(\lambda)a, \quad \text{where} \quad n(\lambda) := \sum_{1\leqslant i\leqslant N}(i-1)\lambda_i;$$

here, we write $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N \geqslant 0)$ for some $N \geqslant 1$. We have

$$\lambda^\dagger = \lambda^* \qquad \text{and} \qquad \mathbf{a}_{\lambda^*} = a\sum_{1\leqslant i\leqslant N}\frac{1}{2}\lambda_i(\lambda_i-1),$$

where λ^* denotes the conjugate (or transpose) partition; see [132, 5.4.3, 5.4.9].

Example 1.3.9. Let W be of type B_n with weight function $L\colon W \to \Gamma$ given by

B_n 

where $a, b \geqslant 0$

There is a standard labelling $\mathrm{Irr}_{\mathbb{K}}(W) = \{E^\lambda \mid \lambda \in \Lambda\}$, where

$$\Lambda = \{\text{set of all pairs of partitions } (\lambda,\mu) \text{ such that } |\lambda|+|\mu| = n\}.$$

For example, the unit and the sign representation are labelled by $((n),\varnothing)$ and $(\varnothing,(1^n))$ respectively; see [132, §5.5]. We have $(\lambda,\mu)^\dagger = (\mu^*,\lambda^*)$; see [132, 5.5.6].