Gavin R. Thomson
Christian Constanda

Stationary Oscillations of Elastic Plates

A Boundary Integral Equation Analysis

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ISBN 978-0-8176-8240-8 e-ISBN 978-0-8176-8241-5
DOI 10.1007/978-0-8176-8241-5
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2011931291

Mathematics Subject Classification (2010): 31A10, 74K20, 74H45

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For my parents
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For Lia
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Preface

Many important problems in mathematical physics can be modeled by means of elliptic partial differential equations or systems. Such equations arise in the study of, for example, steady-state heat conduction (the Laplace equation), acoustics (the Helmholtz equation), elasticity (the Lamé system), and electromagnetism (the Maxwell system).

An important tool for investigating boundary value problems associated with equations of this type is the boundary integral equation technique, which relies on the derivation of Fredholm or quasi-Fredholm integral equations over the boundary of the region of interest and leads to a very convenient representation of the solution. The kernels of the ensuing integral equations are expressed in terms of a two-point (scalar or matrix) function that is, in fact, a fundamental solution of the governing linear differential operator.

Boundary integral equation methods are extremely useful for a variety of reasons. First, they reduce the problem from one involving an unbounded partial differential operator to one with an integral operator, making it much more appealing from an analytic perspective; second, the methods are very general in that they can be applied to any linear second-order elliptic boundary value problem with constant coefficients; and third, the methods are attractive from a numerical point of view because they yield closed-form solutions and, therefore, lend themselves readily to boundary element treatment.

Boundary integral equation methods come in many versions. Thus, the classical indirect approach seeks the solution in an appropriate form that is chosen *a priori*. This method is 'indirect' in the sense that the unknown function in the corresponding integral equation has no physical significance, being merely a convenient mathematical abstraction. By contrast, in the direct methods the unknown function in the integral formulation is an actual physical quantity. For example, in elasticity the solution of the integral equation may represent the displacement or the moment/stress on the boundary of the elastic body.

Another main class of boundary integral equation methods makes use of modified fundamental solutions. This approach was developed to address problems of existence of nonunique solutions to the integral equations derived by the classical techniques. In certain instances, integral equations formulated for boundary value problems known to have at most one solution may themselves admit multiple solutions. Intuitively, this should not be the case. For this reason, we consider ways of modifying the standard fundamental solution so that it leads to uniquely solvable integral equations.

This book investigates an elliptic system of equations arising in the theory of elasticity which characterizes the stationary oscillations of thin elastic plates. The system is obtained by assuming a Mindlin-type form (also known as Kirchhoff's kinematic hypothesis) for the displacements.

Approximate theories describing the bending of plates are important because they reduce the equations of classical three-dimensional elasticity to a system involving only two independent space variables, while highlighting the important bending characteristics of the elastic structure. Such theories have been used successfully in many practical engineering applications. The Mindlin-type model differs from the classical Kirchhoff theory in that it accounts for transverse shear deformation as well, thereby offering additional useful information to practitioners.

Boundary integral equation methods have been widely used in the study of various elliptic systems arising in the theory of elasticity and beyond, such as equilibrium and dynamic problems in the process of deformation of two- and three-dimensional elastic bodies, and the equilibrium and time-dependent bending of elastic and thermoelastic plates with transverse shear deformation.

Although the equations governing the stationary oscillations of Mindlin-type plates are related in a certain way to the equilibrium equations, the two systems display very different characteristics. The main difference is the presence in the oscillatory case of so-called eigenfrequencies. These are values of the oscillation frequency for which the main homogeneous boundary value problems in a bounded domain have nonzero solutions. The book will show how such difficulties can be resolved and how the problems in question can be reduced to uniquely solvable integral equations.

Here is a brief description of the contents.

Chapter 1 presents a derivation of the system of equations modeling the stationary oscillations of elastic plates with transverse shear deformation. A fundamental integral formula, analogous to Green's second identity from potential theory, is also deduced.

The aim in Chapter 2 is to define the generalized single-layer and double-layer plate potentials and to describe their essential properties. All subsequent discussion of boundary value problems relies on the boundary properties of these integral functions. In order to construct the potentials, a suitable matrix of fundamental solutions is made available and its behavior, together with the behavior of a so-called matrix of singular solutions, near the boundary of the plate is investigated.

Chapter 3 deals with the setting up and smoothness properties of a particular solution to the inhomogeneous system obtained in Chapter 1.

In Chapter 4 we introduce the Dirichlet and Neumann problems associated with the governing system of equations. It is natural to discuss these two problems together because they are intrinsically linked in the analysis of their solvability. Radiation conditions that ensure the uniqueness of the solutions of the exterior problems are given, which are then shown to be satisfied by the potential functions defined in Chapter 2. The bulk of the chapter is concerned with establishing integral representations for the regular solutions of the system, to be used later as a starting point in the direct boundary integral equation method.

The presence of eigenfrequencies in the interior problems, which makes the system of stationary oscillations so different from the corresponding equilibrium system, is investigated in Chapter 5. The proof of the existence of eigenfrequencies relies, however, on the relationship between the two systems.

Chapter 6 is concerned with the solvability of the boundary value problems mentioned above. This issue is approached through a classical indirect formulation that results in quasi-Fredholm integral equations of the second kind. Unfortunately, owing to the existence of eigenfrequencies in the interior problems, the solvability of the latter is not always guaranteed. Furthermore, the connection between the solvability of the Dirichlet and Neumann problems leads to difficulties regarding the unique solvability of the integral equations for the exterior problems as well, an effect that, given the available uniqueness results, is not expected.

The application of the direct boundary integral equation method is the subject of Chapter 7, where a coupled pair of equations for each problem—one a quasi-Fredholm second-kind equation and one an equation of the first kind—is obtained. Their analysis is simplified through the use of composition formulas relating various boundary integral operators of interest. It is shown that, as physically expected, each pair of equations for the exterior problems admits exactly one solution. A composite equation consisting of a linear combination of the first-kind and second-kind equations is also studied.

In Chapter 8 a theory of modified integral equations is developed. This is motivated by the need for uniquely solvable equations from which the solutions of the exterior boundary value problems can then be constructed. An indirect method is employed, where the solutions are postulated in the form of modified potentials that lead to quasi-Fredholm second-kind equations. Two different types of modification are considered, with existence and uniqueness results proved for each. The chapter concludes with a look at how uniquely solvable first-kind equations can be derived (again, by an indirect method).

The Robin boundary value problems are introduced in Chapter 9. After the question of uniqueness of solution has been investigated in three separate cases, integral equation methods analogous to those used in Chapters 6–8 are also constructed for these problems.

Chapter 10 considers a fourth type of fundamental boundary value problem associated with the stationary oscillations of thin elastic plates, namely, the transmission problem. The existence of the solution is proved by means of an indirect method after some regularization of the operators involved. A more refined method of solution is then described, based on a direct method in conjunction with a modified fundamental solution.

Finally, in Chapter 11 the null field method is examined. Though, strictly speaking, this is not an integral equation method, it is closely connected to much of the work from the preceding chapters. Facts concerning the unique solvability of the null field equations and the completeness of certain sets of functions are presented.

Brief announcements of the some of the results discussed in the book can be found in [58]–[69].

The methodology and results presented in this monograph should prove useful to applied mathematicians, scientists, and engineers engaged in the research of oscillatory phenomena and other similar models, as well as to graduate students in those disciplines.

We would like to express our sincerest thanks to Tom Grasso and Ben Cronin at Birkhäuser Boston for their highly professional and efficient handling of this project.

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May 2011

Contents

Chapter 1
The Mathematical Model

In this chapter we derive the system characterizing the stationary oscillations of thin elastic plates with transverse shear deformation proposed in [56]. We assume that the body forces have a time-harmonic form, which we substitute into the full classical three-dimensional elasticity model to obtain a time-independent system. The so-called kinematic hypothesis is then applied and, by averaging over the thickness of the plate, we arrive at the desired system of equations. The boundary moment–stress operator is also defined.

The chapter concludes with the reciprocity relation, which connects the solutions of the system in a bounded domain with the values of the displacements and of the moments and stresses on the boundary.

Unless stated otherwise, throughout what follows Latin and Greek indices take the values 1, 2, 3 and 1, 2, respectively, and the convention of summation over repeated indices is understood. Also, a superscript T denotes matrix transposition.

Consider a three-dimensional homogeneous and isotropic elastic body, and let $t_{ij} = t_{ji}$ be the internal stresses, V_j the displacements, F_j the body forces, λ and μ the Lamé constants of the material, and ρ the (constant) mass density. If $x = (x_1, x_2, x_3)^{\mathrm{T}}$ is a generic point in $\mathbb{R}^3$, then the equations of motion are [56]

$$t_{ij,i}(x,t) + F_j(x,t) = \rho \frac{\partial^2 V_j}{\partial t^2}(x,t), \tag{1.1}$$

with the constitutive relations written as

$$t_{ij}(x,t) = \lambda \delta_{ij} V_{k,k}(x,t) + \mu (V_{i,j} + V_{j,i})(x,t), \tag{1.2}$$

where $(\dots)_{,i} = \partial(\dots)/\partial x_i$ and δ_{ij} is the Kronecker delta. The components of the resultant stress vector t in a direction $n = (n_1, n_2, n_3)^{\mathrm{T}}$ are

$$t_i = t_{ij} n_j.$$

If F is of the form

$$F(x,t) = \mathrm{Re}\left[f(x)e^{-i\omega t}\right], \tag{1.3}$$

G.R. Thomson and C. Constanda, *Stationary Oscillations of Elastic Plates*,
A Boundary Integral Equation Analysis, DOI 10.1007/978-0-8176-8241-5_1,
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where f is a complex-valued vector function and $\omega \in \mathbb{R}$, and if the boundary conditions are separable in the same way with respect to the space and time variables, then the body performs stationary oscillations of frequency ω, and its expected displacements are of the form

$$V(x,t) = \mathrm{Re}\left[v(x)e^{-i\omega t}\right]. \tag{1.4}$$

Substituting (1.3) and (1.4) in (1.1) and (1.2) yields

$$t_{ij,i}(x) + f_j(x) + \rho\omega^2 v_j(x) = 0, \tag{1.5}$$

$$t_{ij}(x) = \lambda\delta_{ij}v_{k,k}(x) + \mu(v_{i,j} + v_{j,i})(x). \tag{1.6}$$

These are the equations of stationary oscillations in classical elasticity [40].

Suppose that the body is an elastic plate occupying a region $\bar{S} \times [-h_0/2, h_0/2]$ in $\mathbb{R}^3$, where S is a domain in $\mathbb{R}^2$ bounded by a simple, closed C^2-curve ∂S and h_0, $0 < h_0 \ll \operatorname{diam} S$, is the constant thickness of the plate (see Figure 1). The bounded domain enclosed by ∂S is denoted by S^+, and we write $S^- = \mathbb{R}^2 \backslash (S^+ \cup \partial S)$.

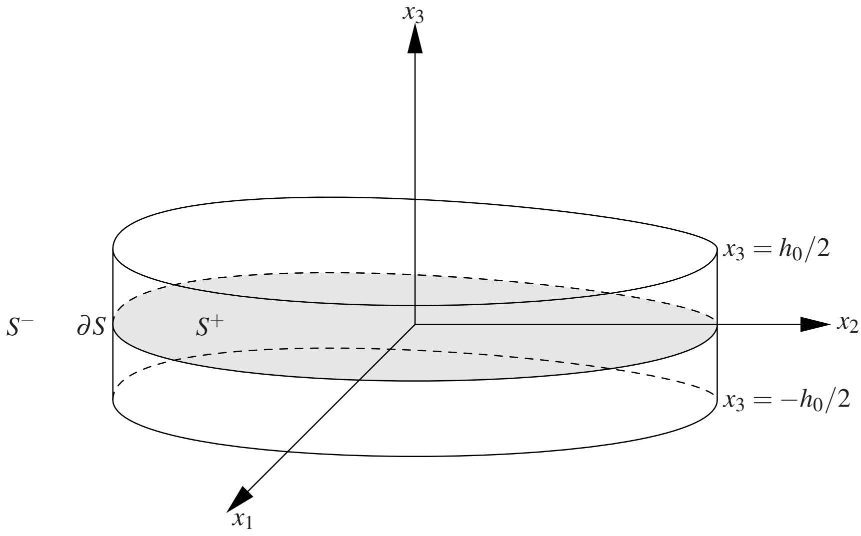

Fig. 1. Geometric configuration of a plate.

We assume a Mindlin-type displacement field; that is,

$$\begin{aligned} v_\alpha(x_1,x_2,x_3) &= x_3 u_\alpha(x_1,x_2), \\ v_3(x_1,x_2,x_3) &= u_3(x_1,x_2), \end{aligned} \tag{1.7}$$

which ensures that this model takes into account the effects of transverse shear forces [14].

Consider the averaging operators $\mathscr{I}_{\alpha-1}$ and $\mathscr{J}_{\alpha-1}$ defined by

$$(\mathscr{I}_{\alpha-1}g)(x_1,x_2) = \frac{1}{h_0}\left[x_3^{\alpha-1}g(x_1,x_2,x_3)\right]_{x_3=-h_0/2}^{x_3=h_0/2},$$

$$(\mathscr{J}_{\alpha-1}g)(x_1,x_2) = \frac{1}{h_0}\int\limits_{-h_0/2}^{h_0/2} x_3^{\alpha-1}g(x_1,x_2,x_3)\,dx_3.$$

Substituting the kinematic assumption (1.7) in (1.5) and writing

$$\begin{aligned} N_{\alpha\beta} &= \mathscr{J}_1 t_{\alpha\beta}, \\ N_{3\alpha} &= \mathscr{J}_0 t_{3\alpha}, \\ H_\alpha &= -(\mathscr{J}_1 f_\alpha + \mathscr{I}_1 t_{3\alpha}), \\ H_3 &= -(\mathscr{J}_0 f_3 + \mathscr{I}_0 t_{33}), \end{aligned}$$

we arrive at the system of equations

$$\begin{aligned} N_{\alpha\beta,\beta} - N_{3\alpha} + \rho\omega^2 h^2 u_\alpha &= H_\alpha, \\ N_{3\beta,\beta} + \rho\omega^2 u_3 &= H_3, \end{aligned} \tag{1.8}$$

where $h^2 = h_0^2/12$.

It can be shown that $N_{3\alpha}$, $N_{\alpha\alpha}$ (α not summed) and $N_{12} = N_{21}$ are, respectively, the averaged transverse shear forces and averaged bending and twisting moments with respect to the middle plane, acting on the face of a vertical cross-sectional element of the plate perpendicular to the x_α-axis [14]. Similarly, H_α and H_3 are related to the averaged body forces and moments and resultant averaged forces and moments acting on the faces $x_3 = \pm h_0/2$.

Using the same averaging procedure and taking (1.7) into account, we bring the constitutive relations (1.6) to the form

$$\begin{aligned} N_{\alpha\beta} &= h^2\left[\lambda u_{\gamma,\gamma}\delta_{\alpha\beta} + \mu(u_{\alpha,\beta} + u_{\beta,\alpha})\right], \\ N_{3\alpha} &= \mu(u_\alpha + u_{3,\alpha}). \end{aligned} \tag{1.9}$$

In view of (1.9), system (1.8) can now be rewritten as

$$\begin{aligned} A^\omega(\partial_x)u(x) &= A(\partial/\partial x_1, \partial/\partial x_2)u(x) \\ &\quad + \begin{pmatrix} \rho\omega^2 h^2 & 0 & 0 \\ 0 & \rho\omega^2 h^2 & 0 \\ 0 & 0 & \rho\omega^2 \end{pmatrix} u(x) = H(x), \end{aligned} \tag{1.10}$$

where $A(\xi_1,\xi_2)$ is the matrix

$$\begin{pmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 - \mu & h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu & -\mu\xi_2 \\ \mu\xi_1 & \mu\xi_2 & \mu\Delta \end{pmatrix}, \tag{1.11}$$

$u = (u_1, u_2, u_3)^{\mathrm{T}}$, $H = (H_1, H_2, H_3)^{\mathrm{T}}$, and $\Delta = \xi_1^2 + \xi_2^2$. System (1.10) was derived in [56]. The matrix operator $A(\partial_x)$ defined by (1.11) arises in the equilibrium bending of plates (see [14]).

The components of the moments and stress on the boundary of the plate are

$$\begin{aligned} N_\alpha &= N_{\alpha\beta}\nu_\beta, \\ N_3 &= N_{3\beta}\nu_\beta, \end{aligned} \tag{1.12}$$

where $\nu = (\nu_1, \nu_2)^{\mathrm{T}}$ is the unit outward normal to ∂S in the middle plane of the plate. By (1.9), equations (1.12) may be written in terms of u as

$$N = T(\partial_x)u(x).$$

Here, $N = (N_1, N_2, N_3)^{\mathrm{T}}$ and $T(\partial_x) = T(\partial/\partial x_1, \partial/\partial x_2)$, where $T(\xi_1, \xi_2)$ is the matrix

$$\begin{pmatrix} h^2[(\lambda+2\mu)\nu_1\xi_1 + \mu\nu_2\xi_2] & h^2(\mu\nu_2\xi_1 + \lambda\nu_1\xi_2) & 0 \\ h^2(\lambda\nu_2\xi_1 + \mu\nu_1\xi_2) & h^2[\mu\nu_1\xi_1 + (\lambda+2\mu)\nu_2\xi_2] & 0 \\ \mu\nu_1 & \mu\nu_2 & \mu\nu_\alpha\xi_\alpha \end{pmatrix}. \tag{1.13}$$

A solution u of (1.10) is called regular in $S^\pm$ if $u \in C^2(S^\pm) \cap C^1(\bar{S}^\pm)$.

In what follows we assume that

$$\lambda + \mu > 0, \quad \mu > 0, \tag{1.14}$$

$$\omega^2 > \frac{1}{h^2}\frac{\mu}{\rho}. \tag{1.15}$$

Clearly, (1.15) indicates that we are concentrating on high-frequency oscillations.

Theorem 1.1. *The system of equations (1.10) is elliptic.*

Proof. Consider the matrix

$$A_0^\omega(\xi) = \begin{pmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 & h^2(\lambda+\mu)\xi_1\xi_2 & 0 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 & 0 \\ 0 & 0 & \mu\Delta \end{pmatrix},$$

which corresponds to the second-order derivatives in (1.10). Then

$$\begin{aligned} \det A_0^\omega(\xi) &= h^4\mu\Delta\left[\left(\mu\Delta + (\lambda+\mu)\xi_1^2\right)\left(\mu\Delta + (\lambda+\mu)\xi_2^2\right) - (\lambda+\mu)^2\xi_1^2\xi_2^2\right] \\ &= h^4\mu\Delta\left[\mu^2\Delta^2 + \mu(\lambda+\mu)\Delta^2\right] \\ &= h^4\mu^2(\lambda+2\mu)\Delta^3 \\ &= h^4\mu^2(\lambda+2\mu)(\xi_1^2+\xi_2^2)^3. \end{aligned}$$

By (1.14), $\det A_0^\omega(\xi) > 0$ for $\xi \neq 0$, so $A_0^\omega(\xi)$ is invertible; therefore, (1.10) is an elliptic system [47]. □

The next assertion is known as the reciprocity relation.

Theorem 1.2. *If* $u, v \in C^2(S^+) \cap C^1(\bar{S}^+)$, *then*

$$\int_{S^+} \left(v^{\mathrm{T}} A^{\omega} u - u^{\mathrm{T}} A^{\omega} v\right) da = \int_{\partial S} \left(v^{\mathrm{T}} T u - u^{\mathrm{T}} T v\right) ds.$$

Proof. The corresponding reciprocity relation for the equilibrium bending of plates [14] states that if $u, v \in C^2(S^+) \cap C^1(\bar{S}^+)$, then

$$\int_{S^+} \left(v^{\mathrm{T}} A u - u^{\mathrm{T}} A v\right) da = \int_{\partial S} \left(v^{\mathrm{T}} T u - u^{\mathrm{T}} T v\right) ds.$$

Consequently,

$$\begin{aligned}\int_{S^+} \big[v^{\mathrm{T}} A u + \rho \omega^2 (h^2 u_1 v_1 + h^2 u_2 v_2 + u_3 v_3) & \\ - u^{\mathrm{T}} A v - \rho \omega^2 (h^2 u_1 v_1 + h^2 u_2 v_2 + u_3 v_3)\big] \, da & \\ = \int_{\partial S} (v^{\mathrm{T}} T u - u^{\mathrm{T}} T v) \, ds, &\end{aligned}$$

and the required formula follows from (1.10). □

Remark 1.1. (i) Theorem 1.1 enables us to replace the discussion of the solvability of system (1.10) in the (two-dimensional) domain S by the analysis of the solvability of some related integral equations on the (one-dimensional) boundary curve ∂S. This is a distinct advantage since integral operators have 'better' properties than their differential counterparts.

(ii) The reciprocity relation (Theorem 1.2) is used in the construction of representation formulas (Chapter 4) and in the proof of existence of solutions by means of direct methods (Chapters 7 and 8).

Chapter 2
Layer Potentials

We solve boundary value problems associated with system (1.10) (or, rather, its homogeneous version) by means of potential-type functions with a suitably chosen kernel. In this chapter we construct a matrix of fundamental solutions for the operator $A^{\omega}(\partial_x)$, which we can then use to define generalized single-layer and double-layer plate potentials. The method used is analogous to the one employed in [14] to construct the corresponding matrix for the operator $A(\partial_x)$ defined by (1.11), which occurs in the study of the equilibrium bending of plates.

Later, we write the matrix of fundamental solutions in a form that allows us to decompose it into an infinite series of so-called wavefunctions. This form, constructed in Theorem 2.1, is similar to the corresponding matrix in the theory of plane elastodynamics [48] with the added complication of certain computational constants.

In Section 2.2 we investigate the singularities of the matrix of fundamental solutions and of its associated matrix of singular solutions. Thus, in Theorems 2.3 and 2.4 we find that these singularities coincide with those of the corresponding matrices from equilibrium plate theory. Therefore, the single-layer and double-layer potentials introduced in Section 2.3 behave in the same way as the potentials considered in [14]. The important properties of these functions, used extensively in the subsequent analysis, are contained in Theorems 2.5–2.7.

2.1 Fundamental and Singular Solutions

We construct a matrix of fundamental solutions for the operator $A^{\omega}(\partial_x)$ using the method described in [13]. If $A^{\omega *}(\xi)$ is the adjoint of the matrix $A^{\omega}(\xi)$, then

$$u(x) = A^{\omega *}(\partial_x)B(x), \tag{2.1}$$

where B satisfies

$$(\det A^{\omega})\,(\partial_x)B(x) = H(x). \tag{2.2}$$

From (1.10) and (1.11) it follows that

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A Boundary Integral Equation Analysis, DOI 10.1007/978-0-8176-8241-5_2,
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$$\begin{aligned}
&\det A^{\omega}(\xi)\\
&= \mu\xi_1\left[\mu^2\xi_1(h^2\Delta - 1) + \rho\omega^2 h^2\mu\xi_1\right] + \mu\xi_2\left[\mu^2\xi_2(h^2\Delta - 1) + \rho\omega^2 h^2\mu\xi_2\right]\\
&\quad + (\mu\Delta + \rho\omega^2)\left[h^4\mu(\lambda+2\mu)\Delta^2 + h^2(\lambda+3\mu)(\rho\omega^2h^2 - \mu)\Delta + (\rho\omega^2h^2 - \mu)^2\right]\\
&= \mu^3(\xi_1^2 + \xi_2^2)(h^2\Delta - 1) + \rho\omega^2 h^2\mu^2(\xi_1^2 + \xi_2^2)\\
&\quad + h^4\mu^2(\lambda+2\mu)\Delta^3 + \left[h^2\mu(\lambda+3\mu)(\rho\omega^2h^2 - \mu) + \rho\omega^2h^4\mu(\lambda+2\mu)\right]\Delta^2\\
&\quad + \left[\mu(\rho\omega^2h^2 - \mu)^2 + \rho\omega^2h^2(\lambda+3\mu)(\rho\omega^2h^2 - \mu)\right]\Delta + \rho\omega^2(\rho\omega^2h^2 - \mu)^2\\
&= h^4\mu^2(\lambda+2\mu)\Delta^3 + \left[\rho\omega^2h^4\mu(2\lambda+5\mu) - h^2\mu^2(\lambda+2\mu)\right]\Delta^2\\
&\quad + \rho\omega^2h^2(\rho\omega^2h^2 - \mu)(\lambda+4\mu)\Delta + \rho\omega^2(\rho\omega^2h^2 - \mu)^2.
\end{aligned}$$

Factoring this expression, we obtain

$$\det A^{\omega}(\xi) = h^4\mu^2(\lambda+2\mu)\left(\Delta^2 + \frac{\lambda+3\mu}{\lambda+2\mu}k^2\Delta + \frac{\mu}{\lambda+2\mu}k^2k_3^2\right)(\Delta + k_3^2),$$

where

$$k^2 = \frac{\rho\omega^2}{\mu} \tag{2.3}$$

and

$$k_3^2 = k^2 - \frac{1}{h^2}; \tag{2.4}$$

hence,

$$\det A^{\omega}(\xi) = h^4\mu^2(\lambda+2\mu)(\Delta + k_1^2)(\Delta + k_2^2)(\Delta + k_3^2), \tag{2.5}$$

where

$$\begin{aligned}
k_1^2 + k_2^2 &= \frac{\lambda+3\mu}{\lambda+2\mu}k^2,\\
k_1^2k_2^2 &= \frac{\mu}{\lambda+2\mu}k^2k_3^2.
\end{aligned} \tag{2.6}$$

Without loss of generality, we assume that $k_1^2 \geq k_2^2$.

Using (2.4) and (2.6), we find that k_1^2, k_2^2, and k_3^2 are connected by the equality

$$\begin{aligned}
h^2(k_1^2 - k_3^2)(k_2^2 - k_3^2) &= h^2\left[k_1^2k_2^2 - k_3^2(k_1^2 + k_2^2) + k_3^4\right]\\
&= h^2\left(\frac{\mu}{\lambda+2\mu}k^2k_3^2 - \frac{\lambda+3\mu}{\lambda+2\mu}k^2k_3^2 + k_3^4\right)\\
&= h^2(-k^2k_3^2 + k_3^4)\\
&= h^2k_3^2\left(-k^2 + k^2 - \frac{1}{h^2}\right),
\end{aligned}$$

from which

$$h^2(k_1^2 - k_3^2)(k_2^2 - k_3^2) = -k_3^2. \tag{2.7}$$

We claim that, under assumptions (1.14) and (1.15), k_1^2, k_2^2, and k_3^2 are real, strictly positive, and distinct. First, k_1^2 and k_2^2 are the two roots of the equation

$$x^2 + \frac{\lambda+3\mu}{\lambda+2\mu} k^2 x + \frac{\mu}{\lambda+2\mu} k^2 k_3^2 = 0.$$

The discriminant of this quadratic is

$$\begin{aligned}
&\frac{(\lambda+3\mu)^2}{(\lambda+2\mu)^2} k^4 - \frac{4\mu}{\lambda+2\mu} k^2 k_3^2 \\
&\qquad = \frac{k^2}{(\lambda+2\mu)^2}\left[k^2(\lambda^2+6\lambda\mu+9\mu^2) - 4\mu(\lambda+2\mu)\left(k^2 - \frac{1}{h^2}\right)\right] \\
&\qquad = \frac{k^2}{(\lambda+2\mu)^2}\left[k^2(\lambda^2+2\lambda\mu+\mu^2) + \frac{4\mu(\lambda+2\mu)}{h^2}\right] \\
&\qquad = \frac{k^2}{(\lambda+2\mu)^2}\left[k^2(\lambda+\mu)^2 + \frac{4\mu(\lambda+2\mu)}{h^2}\right] > 0.
\end{aligned}$$

Consequently, k_1^2 and k_2^2 are real and distinct. Also, by (2.4), assumption (1.15) implies that $k_3^2 > 0$. By (2.6) and (1.14), this means that

$$k_1^2 + k_2^2 > 0, \quad k_1^2 k_2^2 > 0.$$

Hence, k_1^2 and k_2^2 are strictly positive. Finally, from (2.7) and the fact that $k_3^2 \neq 0$ it follows that $k_1^2 \neq k_3^2$ and $k_2^2 \neq k_3^2$.

Replacing, in turn, each component of H by $-\delta(|x-y|)$, where δ is the Dirac delta distribution, and setting the other two equal to zero, from (2.1) and (2.2) we obtain the matrix of fundamental solutions

$$D^\omega(x,y) = A^{\omega*}(\partial_x)\left[t(x,y)E_3\right], \tag{2.8}$$

where, by (2.2) and (2.5), $t(x,y)$ is a solution of

$$h^4\mu^2(\lambda+2\mu)(\Delta+k_1^2)(\Delta+k_2^2)(\Delta+k_3^2)t(x,y) = -\delta(|x-y|). \tag{2.9}$$

We seek $t(x,y)$ in the form

$$t(x,y) = \sum_{j=1}^{3} b_j H_0^{(1)}(k_j|x-y|), \tag{2.10}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero and the b_j are constants to be determined from (2.9). This Hankel function is a fundamental solution of the Helmholtz operator and satisfies [75]

$$(\Delta+k_j^2)H_0^{(1)}(k_j|x-y|) = 4i\delta(|x-y|). \tag{2.11}$$

From (2.10) and (2.11) we find that

$$(\Delta + k_3^2)t(x,y) = b_1\left[4i\delta(|x-y|) + (k_3^2 - k_1^2)H_0^{(1)}(k_1|x-y|)\right]$$
$$+ b_2\left[4i\delta(|x-y|) + (k_3^2 - k_2^2)H_0^{(1)}(k_2|x-y|)\right] + 4ib_3\delta(|x-y|).$$

To eliminate the Dirac distribution in this equation, we require that

$$b_1 + b_2 + b_3 = 0. \tag{2.12}$$

Now, if (2.12) is satisfied, we see that

$$(\Delta + k_2^2)(\Delta + k_3^2)t(x,y)$$
$$= b_1(k_3^2 - k_1^2)\left[4i\delta(|x-y|) + (k_2^2 - k_1^2)H_0^{(1)}(k_1|x-y|)\right]$$
$$+ 4ib_2(k_3^2 - k_2^2)\delta(|x-y|),$$

so we must have

$$(k_3^2 - k_1^2)b_1 + (k_3^2 - k_2^2)b_2 = 0. \tag{2.13}$$

Since $t(x,y)$ satisfies (2.9), we obtain

$$-\frac{1}{h^4\mu^2(\lambda+2\mu)}\delta(|x-y|) = (\Delta + k_1^2)(\Delta + k_2^2)(\Delta + k_3^2)t(x,y)$$
$$= 4ib_1(k_1^2 - k_2^2)(k_1^2 - k_3^2)\delta(|x-y|),$$

from which we deduce that

$$b_1 = \frac{i}{4h^4\mu^2(\lambda+2\mu)(k_1^2-k_2^2)(k_1^2-k_3^2)}. \tag{2.14}$$

Substituting this into (2.12) and (2.13) yields

$$\begin{aligned} b_2 &= \frac{i}{4h^4\mu^2(\lambda+2\mu)(k_2^2-k_1^2)(k_2^2-k_3^2)}, \\ b_3 &= \frac{i}{4h^4\mu^2(\lambda+2\mu)(k_3^2-k_1^2)(k_3^2-k_2^2)}. \end{aligned} \tag{2.15}$$

The formula for b_3 can be simplified by means of (2.7):

$$b_3 = -\frac{i}{4h^2\mu^2(\lambda+2\mu)k_3^2}. \tag{2.16}$$

These constants are well defined since the k_j^2 are distinct.

To calculate the matrix of fundamental solutions $D^{\omega}(x,y)$ using (2.8), we first need to compute $A^{\omega*}(\partial_x)$. From (1.10) and (1.11) it follows that

$$
\begin{aligned}
A_{11}^{\omega*}(\xi) &= \begin{vmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu + \rho\omega^2 h^2 & -\mu\xi_2 \\ \mu\xi_2 & \mu\Delta + \rho\omega^2 \end{vmatrix} \\
&= h^2\mu^2\Delta\Delta + h^2\mu(\lambda+\mu)\Delta\xi_2^2 + (\rho\omega^2 h^2 - \mu)\mu\Delta + \rho\omega^2 h^2\mu\Delta \\
&\qquad + \rho\omega^2 h^2(\lambda+\mu)\xi_2^2 + \rho\omega^2(\rho\omega^2 h^2 - \mu) + \mu^2\xi_2^2 \\
&= h^2\mu^2\Delta\Delta + h^2\mu(\lambda+\mu)\Delta\Delta - h^2\mu(\lambda+\mu)\Delta\xi_1^2 - \mu^2\xi_1^2 + 2\rho\omega^2 h^2\mu\Delta \\
&\qquad + \rho\omega^2 h^2(\lambda+\mu)\Delta - \rho\omega^2 h^2(\lambda+\mu)\xi_1^2 + \rho\omega^2(\rho\omega^2 h^2 - \mu) \\
&= h^2\mu(\lambda+2\mu)\Delta\Delta - h^2\mu(\lambda+\mu)\Delta\xi_1^2 + \rho\omega^2 h^2(\lambda+3\mu)\Delta \\
&\qquad - \big(\mu^2 + \rho\omega^2 h^2(\lambda+\mu)\big)\,\xi_1^2 + \rho\omega^2(\rho\omega^2 h^2 - \mu).
\end{aligned}
$$

In the same way,

$$
\begin{aligned}
A_{22}^{\omega*}(\xi) = h^2\mu(\lambda+2\mu)\Delta\Delta - h^2\mu(\lambda+\mu)\Delta\xi_2^2 + \rho\omega^2 h^2(\lambda+3\mu)\Delta \\
- \big(\mu^2 + \rho\omega^2 h^2(\lambda+\mu)\big)\,\xi_2^2 + \rho\omega^2(\rho\omega^2 h^2 - \mu).
\end{aligned}
$$

Next,

$$
\begin{aligned}
A_{12}^{\omega*}(\xi) &= -\begin{vmatrix} h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ \mu\xi_2 & \mu\Delta + \rho\omega^2 \end{vmatrix} \\
&= -\begin{vmatrix} h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_2 \\ \mu\xi_1 & \mu\Delta + \rho\omega^2 \end{vmatrix} = A_{21}^{\omega*}(\xi) \\
&= -h^2\mu(\lambda+\mu)\Delta\xi_1\xi_2 - \big(\mu^2 + \rho\omega^2 h^2(\lambda+\mu)\big)\,\xi_1\xi_2, \\
A_{13}^{\omega*}(\xi) &= \begin{vmatrix} h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_1 \\ h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu + \rho\omega^2 h^2 & -\mu\xi_2 \end{vmatrix} \\
&= -\begin{vmatrix} h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu + \rho\omega^2 h^2 \\ \mu\xi_1 & \mu\xi_2 \end{vmatrix} = -A_{31}^{\omega*}(\xi) \\
&= -h^2\mu(\lambda+\mu)\xi_1\xi_2^2 \\
&\qquad + \big[h^2\mu^2\Delta\xi_1 + h^2\mu(\lambda+\mu)\xi_1\xi_2^2 + (\rho\omega^2 h^2 - \mu)\mu\xi_1\big] \\
&= h^2\mu^2\Delta\xi_1 + \mu(\rho\omega^2 h^2 - \mu)\xi_1, \\
A_{23}^{\omega*}(\xi) &= -\begin{vmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 - \mu + \rho\omega^2 h^2 & -\mu\xi_1 \\ h^2(\lambda+\mu)\xi_1\xi_2 & -\mu\xi_2 \end{vmatrix} = -A_{32}^{\omega*}(\xi) \\
&= h^2\mu^2\Delta\xi_2 + \mu(\rho\omega^2 h^2 - \mu)\xi_2.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&A_{33}^{\omega*}(\xi) \\
&\quad = \begin{vmatrix} h^2\mu\Delta + h^2(\lambda+\mu)\xi_1^2 - \mu + \rho\omega^2 h^2 & h^2(\lambda+\mu)\xi_1\xi_2 \\ h^2(\lambda+\mu)\xi_1\xi_2 & h^2\mu\Delta + h^2(\lambda+\mu)\xi_2^2 - \mu + \rho\omega^2 h^2 \end{vmatrix}
\end{aligned}
$$