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To our wives,
Paige, Sung-Ock, and Randi

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Preface

Grand visions in mathematics can begin with simple observations. It is hardly more than a homework exercise to prove that what we nowadays call the Poincaré metric on the unit disc is invariant under the biholomorphic maps of the unit disc to itself. But this easily established fact, when combined with the (profound) uniformization theorem of Poincaré and Koebe, yields the striking conclusion that, with a small number of exceptions, every Riemann surface has a canonical complete Hermitian metric of constant Gauss curvature -1. This result became a basic tool for the study of Riemann surfaces. From this result also grew the whole subject of canonical metrics, an area which has become central in transcendental algebraic geometry and in the topology of low-dimensional manifolds.

It is natural to ask what analogue there might be in higher complex dimensions of the Poincaré metric on the unit disc. Indeed, this was asked not long after the era in the early 1900s of the uniformization theorem (Theorem 2.5.1) and the canonical metric idea for Riemann surfaces. The higher dimensional situation is inevitably different from the situation in complex dimension 1 because the Riemann mapping theorem fails in higher dimensions. It was Poincaré again who showed that the unit ball in $\mathbb{C}^2$ was not biholomorphic to the product of the unit disc with itself. In a similar vein, it was understood around the same time that uniformization of algebraic surfaces was not possible in the same form as the Riemann surface result: there is no single simply connected cover for all the algebraic surfaces with only a few exceptions, no analogue to the unit disc being the universal cover of all but a few Riemann surfaces. But quite early on, in the 1920s, Stefan Bergman showed how to attach to each bounded domain in $\mathbb{C}^n$, $n \geq 1$, a canonical metric with the biholomorphic invariance properties of the Poincaré metric on the unit disc: each biholomorphic mapping of a bounded domain to itself was an isometry of the metric, and moreover, any biholomorphic mapping of one bounded domain to another was an isometry of their respective metrics. Uniformization was a failure, but invariant metrics were successful indeed.

The Bergman metric is only numerically computable in most instances, not given by formulas, and for some time it remained primarily an intriguing general idea rather than a specifically useful one. But the development of the detailed theory of the $\overline{\partial}$ operator by Hörmander, Andreotti-Vesentini, Kohn, and many others made accessible information about the behavior of the Bergman kernel and metric, especially on strongly pseudoconvex bounded domains with smooth boundary. The Bergman kernel is expressible directly in terms of the solutions of $\overline{\partial}$ that are orthogonal to holomorphic functions, and this expression means that the kernel and hence the metric can be analyzed in $\overline{\partial}$ terms. In particular, Fefferman's asymptotic expansion of the Bergman kernel (1974) near the boundary of a C^∞ bounded, strongly pseudoconvex domain opened up the possibility of realizing the grand vision of unifying complex function theory and geometry in this case.

This unification of function theory and geometry for domains in $\mathbb{C}^n$ is the subject of this book—hence its title. In particular, the use of geometric methods yields many results about biholomorphic mappings in general and especially about automorphisms, that is, biholomorphic maps of a domain to itself. The fact that a biholomorphic map is an isometry means that the curvature invariants of differential geometry are preserved by biholomorphic maps, and this provides a powerful method of studying the biholomorphic mappings themselves.

While the Bergman metric has become over the years a familiar item in several complex variables that occurs in many texts on the subject, the study via curvature of the geometry of the Bergman metric has been largely confined to research papers up to now. Thus it seemed to the authors that the body of information on this and related topics was both large enough and coherent enough to justify its treatment in a book. That it was large enough is clear from the length of this book. The question of being coherent we leave to the reader, with hope for the best.

This book is not self-contained: on occasion we use, without apology and sometimes without proof, standard results of several complex variables and in particular of the theory of the $\overline{\partial}$ operator. Even so, we have tried to make the book as accessible as possible to the nonspecialist. Most of the arguments can be followed convincingly by simply taking the unproved background results on faith, these being usually very specific and easily stated, if not easily proven. In this sense, the book will be accessible, we hope, to anyone with a basic background in complex analysis and differential geometry. We have also separated out the more technical aspects of the differential geometry so that the complex analyst can most appreciate the shape of the arguments involving curvature by simply knowing that somehow curvature attaches differential invariants to each point that must be preserved under isometries and hence preserved under biholomorphic maps. Really detailed information on differential geometry is rather seldom needed. Geodesics, for example, hardly occur in the book at all. We have tried, in short, to make almost everything accessible to as many readers as possible without short-changing the readers with more specific expertise. Brave words, but we did try.

This book is wide-ranging, though all the topics are related. And a description of the mathematical prerequisites of the book as a whole and of the various chapters specifically may be useful. All of the book presumes basic knowledge of complex analysis in several variables, with the exception of Chapter 2, which concerns one variable only. Especially important is some working knowledge of normal families. A quick summary of what is needed is given in Chapter 1. Chapter 1 also provides a summary of what is known and needed about automorphism groups being Lie groups. These results can be taken on faith if need be. Chapter 1 also begins to talk about Riemannian metrics. Not much depth is needed here nor will be needed later about Riemannian geometry, but the reader is presumed to have in mind what a Riemannian metric is, at least. Chapter 2 is about automorphisms of Riemann surfaces. The results there provide motivation for later developments, but as it happens, the contents of this chapter are not explicitly used anywhere else in the book. Again, metric concepts are used but at a quite elementary level—Gauss curvature and some ideas about geodesics suffice. In Chapter 3, the idea of the Bergman metric is introduced, and the geometry of the Bergman metric is systematically exploited. The Bergman metric is by nature a Kähler metric, but rather little is needed here about Kähler geometry in detail. Indeed, it is not really necessary to know what a Kähler metric is. What is needed is the realization that attached to a metric structure, a Riemannian metric in general, are some second-order differential invariants which are preserved by mappings that preserve the metric itself. Of course, the deeper meaning of these curvature invariants, if known, will enhance the reader's appreciation of the power and elegance of their application to complex analysis. But in the strictly logical sense, one could think of them as simply formulas, which happened to have certain important invariance properties. The same remarks apply to the continuation of these developments in Chapter 4.

Chapter 5 involves some considerable background in Lie group theory, especially in its second half, on the Bedford–Dadok argument. But Chapter 5 is not needed for the later parts of the book, and the reader who is so inclined can simply take as answered the question of which compact Lie groups occur as the automorphism group of a smoothly bounded strongly pseudoconvex domain in Euclidean space—first all of them do—and skip this chapter altogether.

Chapter 6 is similarly not needed for subsequent developments. It answers a natural and interesting (in the authors' view) question, and the argument in the noncompact case is not far outside the usual ways of thinking in several complex variables. The compact case involves some ideas from further afield, in algebraic geometry, and can be omitted without penalty if desired.

Chapter 7 reviews some metric ideas more general than the smooth Riemannian metrics that were used earlier. These more general metrics are of fundamental importance in several complex variables and are likely somewhat familiar to complex analysts in any case. References are given to further details about these metrics. This material is of central importance to the whole subject, though it is not needed in subsequent chapters as such. Automorphisms of Reinhardt domains, the subject of Chapter 8, require some information about Lie algebras if they are to be studied in detail, but the reader can gain a good impression without this.

Chapters 9 and 10 are in fact the natural continuation of Chapters 3 and 4 and can be read effectively immediately after Chapter 4, with the intervening chapters skipped. Chapters 9 and 10 introduce what is known as the scaling method at a rather more leisurely pace than is followed in the rest of the book, since this material is both very important and not so widely available in systematic form. Indeed some of the material here is new. Chapter 11 looks back on the whole book and discusses where the results could have been stated and proved more generally. For ease of reading, many of the results in the earlier parts of the book were stated in special cases—e.g., for domains in Euclidean spaces rather than complex manifolds—and Chapter 11 clarifies what additional generality holds without the introduction of fundamentally new arguments.

This book has been under construction for some considerable time. The authors have benefited during this effort from interactions with many colleagues. We thank them all. In particular, the third named author (Krantz) thanks Alexander Isaev for his collaboration and for many helpful ideas over the years. Several institutions have offered us mathematical hospitality during the writing. In addition to our home institutions, we thank MSRI, the Technical University of Denmark, the American Institute of Mathematics, and l'École Polytechnique de France (Palaiseau). We thank Ms. Ae-Ryoung Seo of POSTECH and Mr. Felipe Garcia Hernandez of UCLA, who each read the whole manuscript and made helpful suggestions. It goes without saying that any remaining errors are the authors' sole responsibility.

Some mathematical subjects begin slowly, by accumulation of many small contributions, like a river forming from many small streams. The general idea of the deep relationship between function theory and geometry does indeed have many historical sources in the nineteenth century, as indicated briefly in the opening paragraphs of this preface. But the specific subject of this book began definitely and quite suddenly with the work of Stefan Bergman. Without his work, this book would not have existed. We dedicate it to his memory.

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1

Preliminaries

1.1 Automorphism Groups

A subset $\Omega \subseteq \mathbb{C}^n$ will be called a *domain* if it is connected and open. The *automorphism group* $\mathrm{Aut}(\Omega)$ of Ω is by definition the set of all holomorphic mappings $f : \Omega \to \Omega$ with inverse map f^{-1} existing and also holomorphic. The group operation is the composition of mappings, and it is easy to check that this binary operation makes $\mathrm{Aut}(\Omega)$ into a group. When $n = 1$, it is well known and easy to prove that f^{-1} will be automatically holomorphic when it is defined. This follows from the argument principle because a locally injective holomorphic function has nowhere zero first derivative. This result is also true in several complex variables, but requires more effort to prove. One must show that a locally injective, equi-dimensional holomorphic mapping has nowhere vanishing holomorphic Jacobian determinant; from this it follows immediately that f^{-1} is holomorphic. This result is conceptually fundamental, but plays little explicit role in what follows and will not be discussed further. [See, e.g., [Narasimhan 1971] for a proof.]

The definition of automorphism group can obviously be extended to the case where Ω is replaced by a complex manifold M. The same observation applies to the redundancy of the hypothesis that f^{-1} be holomorphic since the proof of that result can be performed in local coordinates. Much of the theory of automorphism groups of domains in space can be transferred, without any extra work, directly to the complex manifold case; we shall often treat the two situations simultaneously. Other results are quite different for manifolds than for domains in $\mathbb{C}^n$, and we shall indicate some of these distinctions later.

Just as, in one complex variable, the study of Riemann surfaces can clarify basic function-theoretic questions, the study of manifolds in higher dimensions can clarify the situation for domains in space. However, little detailed knowledge of complex manifold theory will be needed for the reading of this book.

The subject of the geometry of open sets in $\mathbb{C}^n$ and of the geometry of open complex manifolds in general divides itself rather naturally into two

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parts. It is really two subjects. In one of these, the domains and manifolds are such that their automorphism groups are finite dimensional and indeed are Lie groups. In the other, the automorphism groups involve infinitely many parameters. The one-variable, Riemann surface situation (for example) is deceptively simple. The group $\mathrm{Aut}\,(M)$ when M is a Riemann surface is *always* a Lie group, as we shall prove in Chapter 2. By contrast, if one takes $\Omega = \mathbb{C}^2$, then the group $\mathrm{Aut}\,(\Omega)$ is *not* a Lie group but rather is infinite dimensional in a certain sense. For example, if $f : \mathbb{C} \to \mathbb{C}$ is any entire function, then $(z_1, z_2) \mapsto (z_1 + f(z_2), z_2)$ is an automorphism of $\mathbb{C}^2$.

The present book is primarily about the situations in which $\mathrm{Aut}\,(\Omega)$ is a (finite-dimensional) Lie group and satisfies an additional condition that the action is *proper* in the following sense: the action map $A : \mathrm{Aut}\,(\Omega) \times \Omega \to \Omega \times \Omega$ defined by $(\varphi, z) \mapsto (\varphi(z), z)$ is proper. That is, $A^{-1}(C)$ is compact for each compact subset C of $\Omega \times \Omega$. In particular, the isotropy group $I_p \times \{p\} := \{\varphi \in \mathrm{Aut}\,(\Omega) : \varphi(p) = p\}$ is compact for any $p \in \Omega$ since $I_p = A^{-1}(p, p)$. For a statement like this to make sense, we need to define a topology on $\mathrm{Aut}\,(\Omega)$. The appropriate topology, which will be used throughout, is the compact-open topology, equivalently the topology of uniform convergence on compact sets. [It should be noted that all the complex manifolds that we shall consider in the sequel will be paracompact; thus no topological pathologies will arise. In particular, the compact-open topology is metrizable in this case.]

If Ω is a bounded domain in $\mathbb{C}^n$, then $\mathrm{Aut}\,(\Omega)$ is necessarily a Lie group. This was proved specifically by H. Cartan ([Cartan 1935]). Our approach to this will be via normal families and the Bochner–Montgomery theorem (Theorem 1.3.11 below), which characterizes the subgroups of the diffeomorphism group which are Lie groups. Our approach will also yield the properness of the action of $\mathrm{Aut}\,(\Omega)$ on Ω (Theorem 1.3.12).

Any covering-space quotient of a manifold M with $\mathrm{Aut}\,(M)$ acting properly, and in particular any covering-space quotient of a bounded domain, also has its automorphism group acting properly. Also, any Riemann surface except the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and $\mathbb{C}$ itself has this proper-action property.[1]

In addition to bounded domains in $\mathbb{C}^n$ and their quotients, there are other classes of complex manifolds for which the automorphism group action is proper. Some aspects of this phenomenon will be considered in Chapter 7.

The role of proper action can be made explicit even at this early stage of our development. This condition is necessary for the existence of a (smooth) Riemannian metric for which all the elements of the automorphism group are isometries. Actually, the condition of proper action is also sufficient for the

[1] That the property holds for tori and for $\mathbb{C}$ with one point removed is, in a sense, accidental: for these Riemann surfaces are both covered by $\mathbb{C}$, which itself does *not* have the desired property that the action of the automorphism group is proper. But all other Riemann surfaces (except the sphere and the cylinder) are quotients of the unit disc $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, and for these the general principle applies.

existence of such an "invariant metric" [Palais 1961].[2] This will be discussed in more detail in Section 1.3.

Thus, for the domains and manifolds that we shall consider, the automorphism group, which is at first sight a function-theoretic object, will turn out to be also a geometric one via the existence of an invariant metric. These matters will usually be treated here by constructing explicitly an invariant metric rather than by appealing to the general results of Lie group theory.

In Riemann surface theory, this idea of relating function theory to geometry goes back at least to Poincaré and even Riemann. In higher dimensions, some aspects of the idea also have a long history, but many developments have occurred in recent times as well. It is this interaction between function theory and geometry that makes the whole subject so varied and interesting. And while we begin with the function theory, geometry soon takes center stage and plays a major role thereafter.

1.2 Some Fundamentals from Complex Analysis of Several Variables

We shall use systematically the standard notational conventions for coordinates in $\mathbb{C}^n$, first

$$z = (z_1, \ldots, z_n) \qquad \text{and} \qquad w = (w_1, \ldots, w_n).$$

We shall also write

$$|z| = \left(\sum_{j=1}^{n} |z_j|^2 \right)^{\frac{1}{2}} .$$

Thus a mapping from an open subset of $\mathbb{C}^n$ into $\mathbb{C}^m$ is given by an m-tuple of complex-valued functions of n complex variables:

$$w = (w_1, \ldots, w_n) = f(z) = (f_1(z_1, \ldots, z_n), \ldots, f_m(z_1, \ldots, z_n)).$$

Such a map is, by definition, holomorphic if each of the functions f_j, $j = 1, \ldots, m$, is holomorphic in one and hence any of the various equivalent senses of the word "holomorphic."

Here and elsewhere we take for granted basic elements of the theory of functions of several complex variables, for which see [Grauert/Fritzsche 1976], [Hörmander 1990], or [Krantz 2001] for instance. In particular, we assume that

[2] It is a familiar fact that the group of isometries of a (smooth) Riemannian manifold acts properly. But the partial converse, that a properly-acting subgroup of the group of diffeomorphisms acts as isometries for some smooth metric, is not obvious.

the reader is aware that, for $\mathbb{C}$-valued functions $f(z_1, \ldots, z_n)$ defined on an open subset of $\mathbb{C}^n$, the following ideas are equivalent:

- The function f is holomorphic in each variable separately; [3]
- The function f is real-continuously differentiable (C^1) and satisfies the Cauchy–Riemann equations in each variable separately;
- The function f has at each point $p = (p_1, p_2, \ldots, p_n)$ of its domain a power series expansion

$$f(z) = \sum_{i_1, i_2, \ldots, i_n \geq 0} a_{i_1 i_2 \cdots i_n} (z_1 - p_1)^{i_1} (z_2 - p_2)^{i_2} \cdots (z_n - p_n)^{i_n}$$

  which converges absolutely to f for all $(z_1, z_2, \ldots, z_n)$ in some open neighborhood of p.

As will be taken for granted here, many of the ideas of one complex variable have more or less automatic extensions to several variables. These include the Cauchy integral formula in several variables: recall that the polydisc $D^n(p, r)$ of polyradius $r = (r_1, \ldots, r_n)$ with $r_j > 0$ for every j is defined to be

$$D^n(p, r) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j - p_j| < r_j \text{ for every } j\}.$$

If the closure $\mathrm{cl}(D^n(p, r))$ of this polydisc is contained in the (open) domain of definition of a holomorphic function f then, for each $(z_1, \ldots, z_n)$ in the open polydisc,

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n} \oint_{|\zeta_1 - p_1| = r_1} \cdots \oint_{|\zeta_n - p_n| = r_n} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_n \cdots d\zeta_1,$$

where the integral is an iterated line integral. This reconstructs the power series expansion of f around $(p_1, \ldots, p_n)$, by expansion of the integrand and integration term-by-term. Differentiation of this formula under the integral sign together with obvious estimates also yields the following, which we shall apply repeatedly: if a sequence $\{f_j\}$ of $\mathbb{C}$-valued holomorphic functions on an open subset U of $\mathbb{C}^n$ converges uniformly on each compact subset of U, then every derivative (of any order) of the sequence also converges uniformly on each compact subset, and the derivative of the limit is equal to the limit of the derivative.

This last result, which is a direct analogue of a familiar fact about one-variable theory, will be especially important to us since, in effect, it says that the compact-open topology for holomorphic functions is the same as the C^∞ topology. Thus sets or groups of holomorphic mappings have a natural, unique topology. This means that the subtle questions associated to the phrase

[3] In the background here is the famous theorem of Hartogs that a function holomorphic in each variable separately is automatically continuous, indeed real analytic.

"Hilbert's fifth problem" play no role here; such matters are automatically straightforward.

Hurwitz's theorem in one variable on limits of zero-free functions has a direct generalization to several variables: first, if $f_j\colon \Omega \to \mathbb{C}$, $j = 1, 2, 3, \ldots$, are holomorphic functions from a domain (i.e., a connected open set) in $\mathbb{C}^n$ with $0 \notin f_j(\Omega)$, and if the sequence $\{f_j\}$ converges uniformly on compact subsets of Ω to a (necessarily holomorphic) limit $f_0\colon \Omega \to \mathbb{C}$, then either $f_0(\Omega) = \{0\}$, i.e., $f_0 \equiv 0$, or $0 \notin f_0(\Omega)$, i.e., f_0 is nowhere zero. The proof is obtained by observing that, if $f_0(z_0) = 0$ for some $z_0 \in \Omega$, then, by the one-variable Hurwitz theorem, the function $\zeta \mapsto f_0(z_0 + a\zeta)$, for $\zeta \in \mathbb{C}$ with $|\zeta|$ small and for $a \in \mathbb{C}^n$ with $\|a\| = 1$, is defined and identically zero. Then that $f_0 \equiv 0$ follows by analytic continuation.

Since one of the main subjects of this book is self-mappings of domains in $\mathbb{C}^n$ or, on occasion, complex manifolds, we have some special interest in holomorphic mappings where domain and range have equal dimension; first, n-tuples $(f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$ of holomorphic functions of n variables. Attached to this situation is the holomorphic Jacobian determinant $\mathcal{J}$, first, the ordinary determinant of the $n \times n$ complex matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n} \end{pmatrix}.$$

A linear algebra calculation shows that the Jacobian determinant of the mapping considered as a real mapping from an open subset of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ is $|\mathcal{J}|^2$. This is a generalization of the familiar fact from one variable that the real differential of a holomorphic function is a rotation followed by dilation by a factor of $|f'|$, so that its action on the area element is multiplication by $|f'|^2$.

Returning to the $\mathbb{C}^n$ situation in general, we see that the holomorphic mapping from an open subset into $\mathbb{C}^n$ again is nonsingular as a real mapping at a given point if and only if its holomorphic Jacobian determinant $\mathcal{J}$ is nonzero at that point. Combining this observation with Hurwitz's theorem, we see that the limit (uniformly on compact sets) of everywhere nonsingular mappings of a connected open set in $\mathbb{C}^n$ to $\mathbb{C}^n$ is either everywhere nonsingular or everywhere singular. In the latter case, the limit mapping has image with empty interior (by Sard's theorem (Theorem 5.3.2)). This line of thought is associated to the idea that the limit of biholomorphic mappings is either biholomorphic or in some sense "degenerate." This point will be explored in detail in later sections.

It is of interest to characterize holomorphic mappings in terms of their real differentials. This is done in effect by way of the Cauchy–Riemann equations. Let $(f_1(z_1, \ldots, z_n), \ldots, f_m(z_1, \ldots, z_n))$ be a holomorphic mapping into $\mathbb{C}^m$ defined on an open subset of $\mathbb{C}^n$. Then we write $f_j = u_j + \sqrt{-1}v_j$, where u_j, v_j are real-valued. The Cauchy–Riemann equations are as usual

$$\frac{\partial u_j}{\partial x_\ell} = \frac{\partial v_j}{\partial y_\ell} \qquad \text{and} \qquad \frac{\partial u_j}{\partial y_\ell} = -\frac{\partial v_j}{\partial x_\ell}, \quad j = 1, \ldots, m, \quad \ell = 1, \ldots, n.$$

We write here, by convention, $z_\ell = x_\ell + \sqrt{-1}y_\ell$. This can be thought of in a less coordinate-dependent fashion as follows. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by sending $(z_1, \ldots, z_n)$ to $(x_1, y_1, \ldots, x_n, y_n)$. Define an $\mathbb{R}$-linear map J_{2n} of $\mathbb{R}^{2n}$ to itself by sending $(x_1, y_1, \ldots, x_n, y_n)$ to $(-y_1, x_1, \ldots, -y_n, x_n)$. Then the Cauchy–Riemann equations for a map $F : U \to \mathbb{C}^m$, with U open in $\mathbb{C}^n$, are equivalent to

$$J_{2m} \circ dF = dF \circ J_{2n},$$

where dF is the real differential of F considered as a C^∞ function from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$.

This characterization of holomorphicity has an immediate consequence that is important for the theory of complex manifolds. first, if two complex local coordinate systems $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ are holomorphically related, then the J operator determined from the z-coordinates is the same operator as the J operator determined from the w-coordinates. The meaning of this assertion is familiar in Riemann surface theory: J is rotation by $90°$ counterclockwise in the orientation determined by the Riemann surface structure. The meaning of this is the same in any holomorphic coordinate system because the real differential of the coordinate change is orientation-preserving and conformal. In higher dimensions, there is again a coordinate-invariant operator J on the real tangent space at each point of a complex manifold. This operator corresponds to the J operator in any coordinate system, and the observation in the previous paragraph shows that it is independent of coordinate choice.

The J operator thus obtained provides a way to connect real Riemannian geometry with complex behavior, since J is a real $(1, 1)$ tensor but it completely determines which (locally defined) functions are holomorphic. This approach to the geometry of complex manifolds is presented systematically in, e.g., [Greene 1987], [Wells 1979]; see also [Kobayashi/Nomizu 1963].

1.3 Normal Families and Automorphisms

Let $D \subset \mathbb{C}$ denote the open unit disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Also $D(p, r) \subset \mathbb{C}$ denotes the open disc with radius r centered at p. For $r > 0$ we let

$$D^n(0, r) \equiv \underbrace{D(0, r) \times \cdots \times D(0, r)}_{n \text{ times}}.$$

Further, if $p = (p_1, \ldots, p_n) \in \mathbb{C}^n$ and $r > 0$, then

$$D^n(p, r) \equiv D(p_1, r) \times \cdots \times D(p_n, r).$$

If $f : D \to D \subset \mathbb{C}$ is a holomorphic function with $f(0) = 0$ and $|f'(0)| = 1$, then f has the form $f(z) = f'(0)z$. In particular, if $f \in \mathrm{Aut}\,(D)$ and if such an f has $f'(0) = 1$, then $f(z) = z$. This is part of the classical Schwarz lemma. The following result is a direct generalization to several variables, and to arbitrary bounded domains. There are many possible generalizations of the Schwarz lemma, some of which will be discussed later on in this book, but this one is the one that will play the most direct role in our investigations. For example, it will enable us to see that, if Ω is a bounded domain, then $\mathrm{Aut}\,(\Omega)$ has compact isotropy group at each point.

Theorem 1.3.1 (H. Cartan). *Suppose that Ω is a bounded domain in $\mathbb{C}^n$. Let $\phi : \Omega \to \Omega$ be holomorphic and suppose that, for some $p \in \Omega$, $\phi(p) = p$ and $d\phi(p) = \mathrm{id}$. [Here $d\phi$ is the n-dimensional complex differential.] Then ϕ is the identity mapping from Ω to itself.*

Boundedness of Ω is an essential hypothesis: consider the automorphism of $\mathbb{C}^2$ given by $(z_1, z_2) \mapsto (z_1 + z_2^2, z_2)$.

Proof of Theorem 1.3.1. We may assume that $p = \mathbf{0}$ (the origin). For proof by contradiction, assume that ϕ does not coincide with the identity mapping. Expanding ϕ in a power series about $p = \mathbf{0}$ (and remembering that ϕ is vector-valued, hence so is the expansion) yields

$$\phi(z) = z + P_k(z) + O(|z|^{k+1}),$$

where P_k is the first nonvanishing homogeneous polynomial (of degree k) of order exceeding 1 in the Taylor expansion. Defining $\phi^j(z) = \phi \circ \cdots \circ \phi$ (j times); direct computation then gives that

$$\begin{aligned}
\phi^2(z) &= z + 2P_k(z) + O(|z|^{k+1}) \\
\phi^3(z) &= z + 3P_k(z) + O(|z|^{k+1}) \\
&\vdots \\
\phi^j(z) &= z + jP_k(z) + O(|z|^{k+1}).
\end{aligned}$$

Choose polydiscs $D^n(0, a) \subseteq \Omega \subseteq D^n(0, b)$. The Cauchy estimates imply then that, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| := \alpha_1 + \cdots + \alpha_n = k$,

$$j \cdot \left| \left(\frac{\partial}{\partial z} \right)^{\alpha} \phi \bigg|_{\mathbf{0}} \right| = \left| \left(\frac{\partial}{\partial z} \right)^{\alpha} \phi^j \bigg|_{\mathbf{0}} \right| \leq n \cdot \frac{b \cdot \alpha!}{a^k},$$

where

$$\left(\frac{\partial}{\partial z} \right)^{\alpha} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}.$$

Note that the rightmost item in this estimate is independent of j. Hence, for each such multi-index α with $|\alpha| = k$, $(\partial/\partial z)^{\alpha} \phi \big|_{\mathbf{0}} = \mathbf{0}$. Thus $P_k = 0$, a contradiction. $\square$

This argument in particular applies when the dimension $n = 1$ and the domain Ω is the unit disc. There it gives a conceptually direct proof of the corresponding part of the classical Schwarz lemma.

Cartan's result has some further immediate but surprising consequences.

Corollary 1.3.2. *Suppose that Ω is a bounded, circular domain in $\mathbb{C}^n$, that is $(e^{i\theta}z_1, e^{i\theta}z_2, \ldots, e^{i\theta}z_n) \in \Omega$ whenever $(z_1, z_2, \ldots, z_n) \in \Omega$ for every $\theta \in \mathbb{R}$. If $\mathbf{0} \in \Omega$ and $f \in \mathrm{Aut}\,(\Omega)$ with $f(\mathbf{0}) = \mathbf{0}$, then f is a linear mapping.*

Proof. For $\theta \in \mathbb{R}$ and $z \in \Omega$, let $F(z) = e^{-i\theta} f(e^{i\theta} z)$. Then $F \in \mathrm{Aut}\,(\Omega)$, since Ω is circular. By the chain rule it follows that

$$d(f^{-1} \circ F)\big|_0 = \mathrm{id}.$$

Hence

$$f^{-1} \circ F = \mathrm{id}$$

on Ω, or equivalently $f = F$. If we write $f = (f_1, \ldots, f_n)$, $F = (F_1, \ldots, F_n)$, and

$$f_j(z) = \sum_{|N|=1}^{+\infty} a_N z^N$$

is the Taylor expansion of f_j, then the Taylor expansion of F_j is, by definition of F and by substitution,

$$F_j = \sum_{|N|=1}^{+\infty} e^{-i\theta} a_N e^{i|N|\theta} z^N.$$

But $F_j = f_j$. Therefore $e^{i(|N|-1)\theta} a_N = a_N$ for all multi-indices N and all $\theta \in \mathbb{R}$. This implies that $a_N = 0$ for $|N| \geq 2$.[4] Thus each f_j is linear. □

It is easy to modify this argument to show that, if Ω_1, Ω_2 are two bounded, circular domains containing the origin $\mathbf{0}$ and if $F : \Omega_1 \to \Omega_2$ is biholomorphic with $F(\mathbf{0}) = \mathbf{0}$, then F is linear. This immediately implies that, when $n \geq 2$, the unit ball $\{(z_1, \ldots, z_n) \colon |z_1|^2 + \cdots + |z_n|^2 < 1\}$ and the unit polydisc $\{(z_1, \ldots, z_n) \colon |z_j| < 1, j = 1, \ldots, n\}$ are not biholomorphic: If there were a biholomorphic map between them, then applying suitable biholomorphic maps to each variable in the unit polydisc separately would produce a biholomorphic map that took $\mathbf{0}$ to $\mathbf{0}$. This would then have to be linear, which is not possible, since, e.g., the ball has smooth boundary and the polydisc does not (when $n \geq 2$). Thus the direct analogue of the Riemann mapping theorem fails in $\mathbb{C}^n$, $n \geq 2$: (bounded) domains can be homeomorphic to the ball without being biholomorphic to it. This failure, even for small perturbations of the ball, will be explained in much more detail in later chapters.

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[4]Here $N = (n_1, \ldots, n_n)$ and $|N| = n_1 + \cdots + n_n$.

The second corollary will play an important role in what follows.

Corollary 1.3.3. *If Ω is a bounded domain in $\mathbb{C}^n$ and $p \in \Omega$, then the mapping*

$$f \longmapsto df\big|_p$$

is an injective homomorphism of the group

$$I_p \equiv \{f \in \mathrm{Aut}\,(\Omega) : f(p) = p\}$$

into $GL(n, \mathbb{C})$.

Proof. If $df\big|_p = dg\big|_p$ for $f, g \in I_p$, then the chain rule gives that $d(f^{-1} \circ g)\big|_p =$ id, where the identity map id is given by the $n \times n$ identity matrix $I_n \in GL(n, \mathbb{C})$. By Theorem 1.3.1, $f^{-1} \circ g : \Omega \to \Omega$ is the identity mapping. Hence $f \equiv g$. We conclude that $f \mapsto df\big|_p$ is injective on I_p. The homomorphism property is a special case of the chain rule. □

If a group G acts on a space X through an action $G \times X \to X$, and if $x \in X$, then the *orbit* $\mathcal{O}_x$ of the point x is the set $\{gx : g \in G\}$. In a natural sense the orbit is the image of the group G. Indeed, $\mathcal{O}_x$ is naturally identified with the quotient G/I_x, where $I_x = \{g \in G \colon gx = x\}$. We shall be particularly interested in boundary points that are accumulation points of some orbit for the action of the automorphism group $\mathrm{Aut}\,(\Omega)$ on Ω. If the orbit $\mathcal{O}_x \subseteq \Omega$, considered as a point set, has a boundary point $p \in \partial\Omega$ as an accumulation point then we call p a *boundary orbit accumulation point*. These will be discussed in detail in Section 1.5.

Corollary 1.3.3 immediately yields the following observation. Fix $p_0 \in \Omega$. Then each $f \in \mathrm{Aut}\,(\Omega)$ is uniquely determined by $f(p_0)$ and $df|_{p_0}$. Now the possibilities for $f(p_0)$ range at most over Ω and for $df|_{p_0}$ over $\mathbb{C}^{n^2}$ (identifying $df|_{p_0}$ with its complex $n \times n$ matrix). So in a general sense $\mathrm{Aut}\,(\Omega)$ is parameterized by a subset of $\mathbb{C}^n \times \mathbb{C}^{n^2}$. Thus one might expect $\mathrm{Aut}\,(\Omega)$ to be a finite-dimensional group, and hence a Lie group. This expectation turns out to be justified. But of course this depends on adding the topology into the picture of $\mathrm{Aut}\,(\Omega)$: as it stands, this "parameterization" is only set-theoretic. We have already discussed the appropriate topology for $\mathrm{Aut}\,(\Omega)$, first the compact-open topology. Clearly the association $f \mapsto (f(p_0), df|_{p_0}) \in \mathbb{C}^n \times \mathbb{C}^{n^2}$ is continuous (for the second factor, by Cauchy estimates). To pursue this matter further, we shall need some results from normal families, to which we shall turn next.

Among results also associated to normal families and the closure properties of the group $\mathrm{Aut}\,(\Omega)$, when Ω is a bounded domain in $\mathbb{C}^n$, the following principle will in particular play an important role in our later considerations. While in a sense this is just an application of standard normal families ideas, the details are surprisingly subtle in this general, multi-variable situation.

Theorem 1.3.4 (Normal Families of Automorphisms). *Let Ω be a bounded domain in $\mathbb{C}^n$. If $\{f_j\}$ is a sequence in $\mathrm{Aut}\,(\Omega)$ which converges* *uniformly on compact subsets of* Ω *and if, for some* $p_0 \in \Omega$*, the limit* $\lim_{j\to\infty} f_j(p_0)$ *is a point of* Ω*, then the limit holomorphic mapping* $f_0 \equiv \lim f_j : \Omega \to cl(\Omega)$ *has image equal precisely to* Ω *and* $f_0 \in \mathrm{Aut}\,(\Omega)$*.*

Without the hypothesis about the point p_0, the conclusion can fail. For example, if $\Omega = D = \{z \in \mathbb{C} : |z| < 1\}$ and

$$f_j(z) = \frac{z - (1 - 1/j)}{1 - (1 - 1/j)z},$$

then $f_j \in \mathrm{Aut}\,(\Omega)$, but

$$\lim f_j = \text{the constant function } -1.$$

In one complex variable, such "degenerate limits," where $\lim f_j(p_0) \in \mathrm{cl}(\Omega)\setminus\Omega$ for some p_0 and hence (by the theorem) all $p_0 \in \Omega$, are necessarily constant functions. This is an easy consequence of Hurwitz's theorem on the limits of sequences of zero-free holomorphic functions. For, suppose to the contrary that $\lim f_j(p_0) = q \in \mathrm{cl}(\Omega) \setminus \Omega$. Then the limit of the zero-free functions $f_j(z) - q$ for $z \in \Omega$ has a zero at p_0 and is hence $\equiv 0$ on Ω.

This argument indeed shows that, under the hypotheses of the theorem, $\lim f_j$ is "interior," i.e., $(\lim f_j)(\Omega) \subset \Omega$, in the one-variable case. But the argument needed in general (i.e., higher dimensions) is much more intricate—even though Hurwitz's theorem on limits of sequences of zero-free holomorphic functions continues to play a role.

Proof of Theorem 1.3.4. Let $\mathcal{J}_{f_j}$ be the holomorphic Jacobian determinant of f_j as discussed earlier. Then $\mathcal{J}_{f_j}$ is zero-free on Ω. Write f_0 for the limit of the f_j. By Hurwitz's theorem, $\mathcal{J}_{f_0}$ is either identically 0 or is zero-free. To rule out the first possibility, we show that $\mathcal{J}_{f_0}(p_0) \neq 0$. For this, note that

$$\mathcal{J}_{f_0}(p_0) = \lim_{j\to\infty} \mathcal{J}_{f_j}(p_0) = \lim_{j\to\infty} \frac{1}{\mathcal{J}_{g_j}(f_j(p_0))},$$

where $g_j \equiv f_j^{-1}$.

Since $\lim f_j(p_0)$ exists by hypothesis and belongs to Ω, it follows that the set $\{f_j(p_0)\}$ belongs to a compact subset of Ω. Indeed it belongs to $\{\lim_j f_j(p_0)\} \cup \{f_j(p_0)\}$, which is surely compact. By Cauchy estimates, $\mathcal{J}_{g_j}$ is bounded on this compact set. Thus $\lim_j 1/\mathcal{J}_{g_j}(f_j(p_0)) \neq 0$, and that is what we wanted.

It would be pleasant if the fact that we just established, first that $\mathcal{J}_{f_0}$ is zero-free on Ω, implied immediately that $f_0(\Omega) \subset \Omega$. In the special case that Ω has a "nice boundary" (e.g., a regularly embedded C^2 hypersurface in $\mathbb{C}^n$), the result would actually follow. For in that case $\mathcal{J}_{f_0}$ being nowhere zero implies that $f_0(\Omega)$ is open in $\mathbb{C}^n$ and for a domain Ω with smooth boundary, every subset of the closure $\mathrm{cl}(\Omega)$ of Ω that is open in $\mathbb{C}^n$ is contained in Ω. But of course in a more general setting, wherein the boundary of Ω is not smooth,

$\mathrm{cl}(\Omega)$ can in fact contain points of $\mathrm{cl}(\Omega) \setminus \Omega$ in its interior (e.g., consider the case of Ω a punctured open ball). Thus a more refined argument is needed.

Fix a point $p \in \Omega$. Then $\mathcal{J}_{f_0}(p) \neq 0$ and of course the entire holomorphic Jacobian matrix of first derivatives of f_j at p converges to the matrix for f_0, which is nonsingular. Moreover, the second derivatives of the f_j on any fixed, closed ball $\mathrm{cl}(B^n(p, \epsilon)) \subset \Omega$, $\epsilon > 0$, are bounded uniformly in j by Cauchy estimates. Now it follows from the inverse function theorem (see, e.g., [Krantz/Parks 2002]) that there is a $\delta > 0$ such that $f_j(\Omega)$ contains an open ball of radius δ around $f_j(p)$. Here δ can be taken to be independent of j. In particular, since $f_j(\Omega) = \Omega$, the distance of $f_j(p)$ to $\mathbb{C}^n \setminus \Omega$ is at least δ for all j. It follows that $\lim_j f_j(p) = f_0(p)$ is in Ω, *not* in $\mathrm{cl}(\Omega) \setminus \Omega$. Thus, $f_0(\Omega) \subset \Omega$.

Now that we know that f_0 is "interior," i.e., *it maps the interior points to the interior points and hence no interior points are mapped to a boundary point*, we want to show that $f_0 \in \mathrm{Aut}\,(\Omega)$, i.e., that $f_0 : \Omega \to \Omega$ is one-to-one and onto. Passing to a subsequence if necessary, we can suppose that $\{g_j\} = \{f_j^{-1}\}$ converges uniformly on compact subsets to a limit $g_0 : \Omega \to \mathrm{cl}(\Omega)$. Our next goal is to show that g_0 is interior. By the argument used to show that f_0 was interior, it suffices to show that $g_0(f_0(p_0))$ belongs to Ω, not to $\mathrm{cl}(\Omega) \setminus \Omega$.

For this, choose $\lambda > 0$ such that the closed ball $\mathrm{cl}\big(B^n(f_0(p_0), 2\lambda)\big) \subset \Omega$. Notice that $f_j(p_0) \in \mathrm{cl}\big(B^n(f_0(p_0), \lambda)\big)$ whenever j is sufficiently large. Hence, by Cauchy estimates, there is a constant $M > 0$, independent of j, such that

$$\|g_j(f_j(p_0)) - g_j(f_0(p_0))\| \leq M \|f_j(p_0) - f_0(p_0)\|$$

for all j sufficiently large. But $g_j(f_j(p_0)) = p_0$. Hence

$$\|p_0 - g_j(f_0(p_0))\| \leq M \|f_j(p_0) - f_0(p_0)\|.$$

Since the righthand side goes to 0 as $j \to +\infty$, so does the lefthand side and hence

$$g_0(f_0(p_0)) = \lim_{j \to \infty} g_j(f_0(p_0)) = p_0.$$

We conclude that $g_0(f_0(p_0)) \in \Omega$ and therefore g_0 is interior.

We now must show that $f_0 \circ g_0 : \Omega \to \Omega$ and $g_0 \circ f_0 : \Omega \to \Omega$ are both identity maps of Ω to Ω. This of course will establish that $f_0 \in \mathrm{Aut}\,(\Omega)$. This final result is a consequence of the next lemma.

Lemma 1.3.5. *If $\{f_j : \Omega \to \Omega\}$ and $\{g_j : \Omega \to \Omega\}$ are sequences of holomorphic mappings which converge uniformly on compact subsets of Ω to interior limits $f_0 : \Omega \to \Omega$ and $g_0 : \Omega \to \Omega$, then the sequence $\{g_j \circ f_j : \Omega \to \Omega\}$ converges uniformly on compact subsets of Ω to $g_0 \circ f_0 : \Omega \to \Omega$.*

Assuming this lemma for the moment, we may apply it to f_j and g_j as before. Since $g_j \circ f_j$ is the identity map of Ω to Ω, for all j, it follows that $g_0 \circ f_0$ is also the identity map. Applying the lemma again with the roles of f and g interchanged gives that $f_0 \circ g_0$ is the identity. This completes the proof of the theorem. Thus, it remains to prove the lemma.

Proof of Lemma 1.3.5. Suppose that $K \subset \Omega$ is a compact subset. Then choose $\epsilon > 0$ such that

$$L_\epsilon \equiv \{z \in \Omega : \|z - w\| \leq \epsilon \text{ for some } w \in f_0(K)\}$$

is a compact subset of Ω. This choice is possible since $f_0(K)$ is a compact subset of Ω. For all j sufficiently large, $f_j(K) \subset L_\epsilon$. Furthermore, the members of $\{g_j\}$ are uniformly Lipschitz continuous on L_ϵ by Cauchy estimates. Thus, for $z \in K$ and j large, there is a j-independent constant M such that

$$\begin{aligned} \|g_j(f_j(z)) - g_0(f_0(z))\| &\leq \|g_j(f_j(z)) - g_j(f_0(z))\| + \|g_j(f_0(z)) - g_0(f_0(z))\| \\ &\leq M\|f_j(z) - f_0(z)\| + \|g_j(f_0(z)) - g_0(f_0(z))\|. \end{aligned}$$

Now $\|f_j(z) - f_0(z)\| \to 0$ uniformly for $z \in K$. Also, since $\{f_0(z) : z \in K\}$ is compact, $\|g_j(f_0(z)) - g_0(f_0(z))\| \to 0$ uniformly for $z \in K$. Thus $\lim_j g_j(f_j(z)) = g_0(f_0(z))$ uniformly for $z \in K$ as required. □

The proof of Theorem 1.3.4 is now complete. □

Corollary 1.3.6. *For each $p \in \Omega$, the orbit $\mathcal{O}_p := \{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$ is closed in Ω.*

Proof. We need to show that, if $\{f_j(p)\}$ converges to $q \in \Omega$, then $q \in \mathcal{O}_p$, i.e., that $q = f(p)$ for some $f \in \mathrm{Aut}\,(\Omega)$. Choose a subsequence of $\{f_j\}$ which converges uniformly on compact subsets of Ω to $f : \Omega \to \mathrm{cl}(\Omega)$.[5] By Theorem 1.3.4, $f \in \mathrm{Aut}\,(\Omega)$ and clearly $f(p) = \lim_j f_j(p) = q$. □

Corollary 1.3.7. *The injective homomorphism $f \mapsto df|_p$ of I_p (the isotropy group $\{f \in \mathrm{Aut}\,(\Omega) \colon f(p) = p\}$) onto dI_p is a homeomorphism of I_p (in the compact-open topology) onto a compact subgroup of $GL(n, \mathbb{C})$.*

Proof. That $f \mapsto df|_p$ is an injective homomorphism of I_p onto dI_p has already been established (Corollary 1.3.3). The continuity is an immediate consequence of the Cauchy estimates for first derivatives. For the compactness, note that a sequence $\{df_j|_p : f_j \in I_p\}$ has a subsequence $\{df_{j_k}|_p : f_{j_k} \in I_p\}$ for which $\{f_{j_k}\}$ converges uniformly on compact subsets of Ω and, by Theorem 1.3.4, to an element $f_0 \in \mathrm{Aut}\,(\Omega)$ that fixes p. Again by the Cauchy estimates, $df_{j_k}|_p$ converges in $GL(n, \mathbb{C})$ to $df_0|_p \in dI_p$. □

The compactness part of Corollary 1.3.7 is a special case of a more general result which has essentially the same proof.

Corollary 1.3.8. *If K is a compact subset of Ω and $p \in \Omega$, then* $\{f \in \mathrm{Aut}\,(\Omega) : f(p) \in K\}$ *is a compact subset of* $\mathrm{Aut}\,(\Omega)$.

[5] We shall use the notation $\mathrm{cl}(\Omega)$ for the closure of Ω, instead of the more familiar $\overline{\Omega}$, to avoid confusion with the complex conjugate.

Proof. Let $\{f_j\}$ be a sequence in $\mathrm{Aut}\,(\Omega)$ with $f_j(p) \in K$ for all j. Since K is compact, we see by passing to a subsequence (still called f_j) that $\lim_j f_j(p)$ exists and lies in K. By normal families considerations, a further passage to a subsequence yields a sequence that converges uniformly on compact sets. By Theorem 1.3.4, this sequential limit is itself an automorphism. Obviously this limit takes p to some point in K. □

Corollary 1.3.9. *If, for some* $p \in \Omega$, $\{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$ *is compact, then* $\mathrm{Aut}\,(\Omega)$ *is compact.*

Proof. In the corollary before this one, we simply take $K = \{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$. □

For all $p \in \Omega$, $\{f(p) : f \in \mathrm{Aut}\,(\Omega)\}$ is compact if $\mathrm{Aut}\,(\Omega)$ is compact, just because for a given p the mapping

$$\begin{aligned} F : \mathrm{Aut}\,(\Omega) &\to \Omega \\ f &\mapsto f(p) \end{aligned}$$

is continuous. Thus we have proved the following result.

Proposition 1.3.10. *If one orbit of* $\mathrm{Aut}\,(\Omega)$ *is compact, then* $\mathrm{Aut}\,(\Omega)$ *is compact and all of its orbits are compact.*

We know from Corollary 1.3.6 that any orbit of $\mathrm{Aut}\,(\Omega)$ is closed in Ω. Thus the only way that an orbit of $\mathrm{Aut}\,(\Omega)$ can be noncompact is to "run out to the boundary" of Ω, i.e., the closure must contain an element of $\mathrm{cl}(\Omega) \setminus \Omega$. One of the main points of the present book is to study what happens when $\mathrm{Aut}\,(\Omega)$ is noncompact. And one of the main approaches will be to study $\mathrm{cl}(\Omega) \setminus \Omega$ in a neighborhood of such a "boundary orbit accumulation point," that is, an element of $\mathrm{cl}(\Omega) \setminus \Omega$ that lies in the closure of some orbit of the automorphism group action.

We now see that the automorphism group of a bounded domain is a (finite-dimensional) Lie group. For this we shall use the following general theorem.

Theorem 1.3.11 ([Bochner/Montgomery 1946]). *Let G be a subgroup of the diffeomorphism group of a smooth manifold. If it is locally compact, then G is a Lie group.*

When the action of the automorphism group is proper, the group is necessarily locally compact. first, as before, we define the action map $A\colon \mathrm{Aut}\,(\Omega) \times \Omega \to \Omega \times \Omega$ by $A(\varphi, z) = (\varphi(z), z)$. Then A^{-1} of a compact-closure neighborhood of (z, z) for any $z \in \Omega$ has compact closure in $\mathrm{Aut}\,(\Omega) \times \Omega$, when A is a proper map. This gives a compact-closure neighborhood of the identity in $\mathrm{Aut}\,(\Omega)$, by projection to the first factor of $\mathrm{Aut}\,(\Omega) \times \Omega$. Thus to show that $\mathrm{Aut}\,(\Omega)$ is a Lie group when Ω is a bounded domain in $\mathbb{C}^n$, it suffices, in the presence of the Bochner–Montgomery theorem (Theorem 1.3.11), to show:

Theorem 1.3.12. *If Ω is a bounded domain in $\mathbb{C}^n$, then the action of* $\mathrm{Aut}(\Omega)$ *on Ω is proper, i.e., the map $(\varphi, z) \mapsto (\varphi(z), z) : \mathrm{Aut}(\Omega) \times \Omega \to \Omega \times \Omega$ is proper.*

Proof. Properness means explicitly that, if $C \subset \Omega \times \Omega$ is a compact set, then $\{(\varphi, z) : (\varphi(z), z) \in C\}$ is a compact set in $\mathrm{Aut}(\Omega) \times \Omega$. To check this property for $\mathrm{Aut}(\Omega)$, suppose that $\{(\varphi_j, z_j) : j = 1, 2, \ldots\}$ is a sequence with $(\varphi_j(z_j), z_j) \in C$ for all j. Passing to a subsequence if necessary, one can assume that $\{z_j\}$ converges to a point $z_0 \in \Omega$ and that the sequence $\{\varphi_j(z_j)\}$ converges to $w_0 \in \Omega$.

Since Ω is bounded, Cauchy estimates imply that $\varphi_j(z_0)$ converges to w_0: in more detail, this follows by noting from the Cauchy estimates that, for some $\epsilon > 0$, $B(z_0, 2\epsilon) \subset \Omega$, so that there is a constant $M > 0$ independent of j such that the norm of the (real) differential of φ_j is less than M at each point of $B(z_0, \epsilon)$. Thus the distance from $\varphi_j(z_j)$ to $\varphi_j(z_0)$ is bounded by $M\|z_j - z_0\|$, and hence goes to 0.

Since $\varphi_j(z_0)$ converges now to $w_0 \in \Omega$, it follows from Corollary 1.3.8 that $\{\varphi_j\}$ has a subsequence that converges to some $\varphi_0 \in \mathrm{Aut}(\Omega)$. The compactness of $\{(\varphi, z) : (\varphi(z), z) \in C\}$ has thus been established. □

Corollary 1.3.13. *If Ω is a bounded domain in $\mathbb{C}^n$, then* $\mathrm{Aut}(\Omega)$ *is a Lie group.*

Proof. Combine Theorem 1.3.12 with the Bochner–Montgomery theorem (Theorem 1.3.11). □

As already noted at the end of Section 1.1, this result implies, from the result of Palais [Palais 1961], the existence of a smooth Riemannian metric on Ω invariant under $\mathrm{Aut}(\Omega)$. Averaging this with respect to the almost complex structure produces a Hermitian metric on Ω invariant under $\mathrm{Aut}(\Omega)$. In Chapter 3, an explicit construction of such a metric will be presented, but it is worth noting that the existence of such an invariant metric is guaranteed by the general principles we have discussed.

The general situation just described gives at least a philosophical idea of why $\mathrm{Aut}(\Omega)$ is a Lie group when Ω is a bounded domain. The precise version of this idea is Theorem 1.3.11 by Bochner and Montgomery. The main point is to describe the elements of $G := \mathrm{Aut}(\Omega)$ locally, in a neighborhood of the identity element, by a finite number of parameters so as to make the group itself a manifold (of finite dimension). A way to think of this is to look for a point of minimal isotropy dimension. This idea makes sense because all the isotropy groups are closed subgroups of $GL(n, \mathbb{C})$ (actually $U(n)$), so the idea of dimension is just submanifold dimension. If p is such a point, and its orbit $\mathcal{O}_p := \{\gamma(p) : \gamma \in G\}$, then elements γ near the identity can be determined by specifying $\gamma(p)$, which is near p, and $d\gamma\big|_p$, which is near the "identity map," where the “identity map” is just the map from the tangent space at p to the tangent space at $\gamma(p)$ arising from the coordinates in $\mathbb{C}^n$. The set of such $d\gamma$ in Euclidean coordinates is a submanifold of $GL(n,\mathbb{C})$, although it is not in general a subgroup (if $\gamma(p) \neq p$). Using submanifold coordinates from that observation and submanifold-of-$\mathbb{C}^n$ coordinates of $\mathcal{O}_p$ near p gives a local parameterization of $G = \mathrm{Aut}\,(\Omega)$ near the identity.

This picture will be clearer if one thinks of the case of Ω the unit disc and $p = 0$. Let γ be an element of $\mathrm{Aut}\,(\Omega)$. Near the identity, we can parameterize $\mathrm{Aut}\,(\Omega)$ by the image $\gamma(0)$ together with $d\gamma\big|_0$. The set of such $d\gamma\big|_0$ (when $\gamma(0)$ is near 0) is a submanifold of $GL(1,\mathbb{C}) = \mathbb{C} \setminus \{0\}$. It generally is not a subgroup:

$$\{d\gamma\big|_0 : \gamma(0) = a\} = \{\omega T_{-a}\big|_0 : |\omega| = 1\},$$

where $T_{-a} \in \mathrm{Aut}\,(\Omega)$ is defined by $T_{-a}(z) = (z+a)/(1+\bar{a}z)$. But we still get a legitimate smooth parameterization of $\mathrm{Aut}\,(\Omega)$ near the identity.

The reader is invited to consider the corresponding local parameterization of $\mathrm{Aut}\,(\Omega)$ when Ω is the unit ball in $\mathbb{C}^2$—after this group is discussed in some detail in the next section.

Note that one obtains here a view of the general fact that, for $G = \mathrm{Aut}\,(\Omega)$,

$$\dim \mathcal{O}_p + \dim\,(I_p) = \dim\, G,$$

when

$$\mathcal{O}_p = \text{orbit of } p = \{\gamma(p) : \gamma \in G\}.$$

[This holds in general: the restriction to minimal isotropy, maximal orbit dimensions we made was just for convenience of visualization purposes.]

A closed subgroup of $GL(n,\mathbb{C})$ which acts on $\mathbb{C}^n$ isometrically is necessarily a closed subgroup of $U(n)$ and is hence compact. Conversely, if a subgroup of $GL(n,\mathbb{C})$ is compact, then there is a Hermitian metric on $GL(n,\mathbb{C})$ for which the subgroup acts isometrically and hence belongs to the $U(n)$ associated to the Hermitian metric. This follows from a standard argument using averaging of the standard metric with respect to the group action of the given subgroup of $GL(n,\mathbb{C})$.

The fact that every compact subgroup of $GL(n,\mathbb{C})$ acts isometrically relative to some Hermitian metric combined with Corollary 1.3.7 implies that, at each point $p \in \Omega$, there is a Hermitian metric for which I_p acts isometrically on the tangent space at p. This strongly suggests that one ought to seek a Hermitian metric on Ω which is $\mathrm{Aut}\,(\Omega)$-invariant. In other words, one ought to look for a C^∞ family h_p, $p \in \Omega$, of Hermitian metrics such that, for all $\gamma \in \mathrm{Aut}\,(\Omega)$ and $p \in \Omega$, the map $d\gamma\big|_p$ from the tangent space at p with metric h_p is an isometry onto the tangent space at $\gamma(p)$ with metric $h_{\gamma(p)}$. Indeed, it even suggests a way to do this: for some selection of distinguished points p, one in each orbit, choose h_p more or less arbitrarily except that in some sense it varies nicely with the choices of orbit. Then, for q in the orbit of such a point p, determine h_q by the requirement that $d\gamma\big|_p$ must be isometric for a γ_q with $\gamma_q(p) = q$. This is well defined by Corollary 1.3.7, independently of which γ_q is chosen. Thus the only question is whether this can be done so that the resulting metric on all of Ω is C^∞. This involves finding smooth “slices” for orbits. This is the point addressed in [Palais 1961]. But since we shall construct such $\mathrm{Aut}\,(\Omega)$-invariant metrics directly later on, we leave Palais's general construction as a philosophical observation.

1.4 The Basic Examples

In this section we shall collect a number of examples for which the automorphism groups are obtained explicitly. Some of these are well known and elementary, and the derivations of their automorphism groups need be outlined only briefly. But it will be convenient to have them all in one place; and looking at them all at once will suggest various paths of exploration that we follow later.

(1) $\mathrm{Aut}\,(\mathbb{C}) = \{z \mapsto az + b : a, b \in \mathbb{C}, a \neq 0\}$.

If $f : \mathbb{C} \to \mathbb{C}$ is injective, then the only possible singularity of f at ∞ is a simple pole. If instead ∞ were a removable singularity, then f would be constant by Liouville's theorem. If ∞ were an essential singularity, then f would not be injective in any neighborhood of ∞. Similarly, a pole at ∞ of higher order than 1 would preclude injectivity in a neighborhood of ∞. Thus the nonconstant injective function f is a polynomial of degree one. That any polynomial of degree one is an automorphism is clear. □

(2) $\mathrm{Aut}\,(D) = \{z \mapsto \omega \cdot (z - a)/(1 - \bar{a}z) : a, \omega \in \mathbb{C}, |\omega| = 1, |a| < 1\}$.

That

$$T_a : z \longmapsto \frac{z - a}{1 - \bar{a}z}$$

is defined and injective from D to D is easy algebra. Also $T_a(T_{-a}(z)) = z$; hence T_a is surjective.

Conversely, suppose that $f \in \mathrm{Aut}\,(D)$. Let $a = f^{-1}(0)$. Then $g := f/T_a$ is holomorphic and zero-free on D and

$$\lim_{|\zeta| \to 1} |g(\zeta)| = \lim_{|\zeta| \to 1} \left| \frac{f(\zeta)}{T_a(\zeta)} \right| = 1.$$

By the maximum principle applied to both g and $1/g$, we see that $|T_a/f| \equiv 1$ on D, hence $f = \omega T_a$ for some constant ω with $|\omega| = 1$.[6] □

[6] An alternative argument is to note that $T_a \circ f$ maps the disc to the disc and fixes 0. Then Schwarz's lemma implies that $|(T_a \circ f)(z)| \leq |z|$. Applying the same reasoning to the inverse of this mapping gives $|(T_a \circ f)(z)| \geq |z|$. Hence $|T_a \circ f(z)| \equiv |z|$ on D, and $T_a \circ f$ equals $w \cdot \mathrm{id}$ on D for some ω with $|\omega| = 1$.

(3) $\text{Aut}\,(\mathbb{C} \setminus \{0\}) = \{z \mapsto az^\epsilon : \epsilon = \pm 1, a \in \mathbb{C}, a \neq 0\}$.

If $f \in \text{Aut}\,(\mathbb{C}\setminus\{0\})$, then a connectivity argument shows that $\lim_{z\to 0} f(z) = 0$ or $\lim_{z\to 0} |f(z)| = +\infty$. Composing with an inversion, we may assume that the first alternative holds. But then f, considered as a holomorphic function, has a removable singularity at the origin. Thus the extension $f(0) = 0$ makes f an entire function that is an automorphism of the entire plane. From part (1), $f(z) = az$, for some $a \neq 0$. In case $\lim_{z\to 0} f(z) = \infty$, the same reasoning applied to $1/f$ gives $1/f(z) = az$. $\square$

(4) $\text{Aut}\,(\{z \in \mathbb{C} : 0 < r_1 < |z| < r_2\}) = \{z \mapsto \omega z : \omega \in \mathbb{C}, |\omega| = 1\} \cup \{z \mapsto \omega r_1 r_2 / z : \omega \in \mathbb{C}, |\omega| = 1\}$.

Denote the annulus by A. By a connectivity argument, for each $f \in \text{Aut}\,(A)$, either

(a) $\lim_{|z|\to r_2} |f(z)| = r_2$ and $\lim_{|z|\to r_1} |f(z)| = r_1$;

or

(b) $\lim_{|z|\to r_2} |f(z)| = r_1$ and $\lim_{|z|\to r_1} |f(z)| = r_2$.

In either case, repeated application of Schwarz reflection to the boundary circles extends f to an automorphism $\widehat{f} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ of $\mathbb{C} \setminus \{0\}$. Thus, by Example (3), $f(z) = az$ or $f(z) = a/z$ for some nonzero $a \in \mathbb{C}$. The condition $f(A) = A$ tells us then that $a = \omega$ in the first instance and that $a = \omega r_1 r_2$ in the second instance. $\square$

(5) $\text{Aut}\,(\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\})$.

The set

$$B^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$$

is of course the unit ball in $\mathbb{C}^2$. First notice that $I_{(0,0)} = U(2) \subset GL(2, \mathbb{C})$. Obviously $U(2) \subset I_{(0,0)}$. If $f \in I_{(0,0)}$, then f is $\mathbb{C}$-linear according to Corollary 1.3.2. Since f has to preserve the unit sphere (the boundary of B^2), it is immediate that $f \in U(2)$.

Now a direct calculation, analogous to that for the disc, shows that the mapping

$$T_{(a,0)}(z_1, z_2) \equiv \left(\frac{z_1 - a}{1 - \bar{a} z_1}, \frac{\sqrt{1 - |a|^2}\, z_2}{1 - \bar{a} z_1} \right)$$

sends the ball B^2 into itself. Furthermore, the inverse mapping to $T_{(a,0)}$ is $T_{(-a,0)}$. Thus $T_{(a,0)}$ is an automorphism.

If (z_1, z_2) is any point of B^2, then there is an element $\lambda \in U(2)$ that takes (z_1, z_2) to a point of the form $(a, 0)$. Also $T_{(a,0)}(a, 0) = (0, 0)$. These two pieces of information combined tell us that $\text{Aut}\,(B^2)$ acts transitively