Special Topics in the Theory of Piezoelectricity

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Editor

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ISBN 978-0-387-89497-3 e-ISBN 978-0-387-89498-0
DOI 10.1007/978-0-387-89498-0
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2009929638

Mathematics Subject Classification (2000): 74-xx

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Piezoelectric or, more generally, electroelastic materials, exhibit electromechanical coupling. They experience mechanical deformations when placed in an electric field and become electrically polarized under mechanical loads. These materials have been used to make various electromechanical devices. Examples include transducers for converting electric energy to mechanical energy or vice versa, resonators and filters for telecommunication and timekeeping, and sensors for information collection.

Piezoelectricity has been a steadily growing field for more than a century, progressed mainly by researchers from applied physics, acoustics, materials science and engineering, and electrical engineering. After World War II, piezoelectricity research has gradually concentrated in the IEEE Society of Ultrasonics, Ferroelectrics, and Frequency Control. The two major research focuses have always been the development of new piezoelectric materials and devices. All piezoelectric devices for applications in the electronics industry require two phases of design. One aspect is the device operation principle and optimal operation which can usually be established from linear analyses; the other is the device operation stability against environmental effects such as a temperature change or stress, which is usually involved with nonlinearity. Both facets of design usually present complicated electromechanical problems.

Due to the application of piezoelectric sensors and actuators in civil, mechanical, and aerospace engineering structures for control purposes, piezoelectricity has also become a topic for mechanics researchers. Mechanics can provide effective tools for piezoelectric device and material modeling. For example, the finite element and boundary element methods for numerical analysis and the one- and two-dimensional theories of piezoelectric beams, plates, and shells are effective tools for the design and optimization of piezoelectric devices. Mechanics theories of composites are useful for predicting material behaviors.

In spite of the wide and growing applications of piezoelectric devices, books published on the topic of piezoelectricity are relatively few. Following the editor's previous book, *An Introduction to the Theory of Piezoelectricity*, Springer ©2005, this book addresses more advanced topics that require a collective effort. Each self-contained chapter has been written by a group of international experts and includes quite a few advanced topics in the theory of piezoelectricity. Each chapter attempts to present a basic picture of the subject area addressed.

Piezoelectricity is a broad field and, practically speaking, this volume can only cover a fraction of the many relatively advanced topics. Following a brief summary of the three-dimensional theory of linear piezoelectricity, Chapters 2 through 5 discuss selected topics within the linear theory. The linear theory of piezoelectricity assumes a reference state free of deformations and fields. When initial deformations and/or fields are present, the theory for small incremental fields superimposed on a bias is needed, which is the subject of Chapter 6. The theory for incremental fields needs to be obtained from the fully nonlinear theory by linearization about an initial state, and, therefore, is a subject that is inherently nonlinear. Chapter 7 covers the fully dynamic effects due to electromagnetic coupling. Chapter 8 addresses nonlocal and gradient effects of electric field variables.

I would like to take this opportunity to thank all chapter contributors. My thanks also go to Patricia A. Worster and Ziguang Chen of the College of Engineering at the University of Nebraska-Lincoln for their editing assistance on Chapters 1, 7, and 8.

Jiashi Yang
January 2009

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Basic Equations

Jiashi Yang

1.1 Introduction

This chapter presents a brief summary of the basic theory of linear piezoelectricity based mainly on the *IEEE Standard on Piezoelectricity* [1] and the classical book on piezoelectricity [2] by H. F. Tiersten who also wrote the theoretical part of [1]. The organization of this chapter is essentially a shortened version of Chapter 2 of *An Introduction to the Theory of Piezoelectricity* [3]. This chapter uses Cartesian tensor notation, the summation convention for repeated tensor indices, and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index. A superimposed dot represents a time derivative.

1.2 Basic Equations

The equations of linear piezoelectricity can be obtained by linearizing the nonlinear electroelastic equations [4, 5] under the assumption of infinitesimal deformation and fields. The equations of motion and the charge equation are

$$T_{ji,j} + \rho_0 f_i = \rho_0 \ddot{u}_i, \qquad D_{i,i} = q, \tag{1.1}$$

where $\mathbf{T}$ is the stress tensor, ρ_0 is the reference mass density, $\mathbf{f}$ is the body force per unit mass, $\mathbf{u}$ is the displacement vector, $\mathbf{D}$ is the electric displacement vector, and q is the body free charge density which is usually zero. Within the linear theory, the conservation of mass that determines the present

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J. Yang (ed.), *Special Topics in the Theory of Piezoelectricity*, DOI: 10.1007/978-0-387-89498-0_1,

mass density ρ takes the following form,

$$\rho_0 \cong \rho(1+u_{k,k}), \tag{1.2}$$

which can be treated separately once the displacement has been obtained. Constitutive relations are given by an electric enthalpy function H,

$$H(S_{kl}, E_k) = \frac{1}{2}\, c^E_{ijkl} S_{ij} S_{kl} - e_{ijk} E_i S_{jk} - \frac{1}{2}\, \varepsilon^S_{ij} E_i E_j \tag{1.3}$$

through

$$\begin{aligned} T_{ij} &= \frac{\partial H}{\partial S_{ij}} = c^E_{ijkl} S_{kl} - e_{kij} E_k, \\ D_i &= -\frac{\partial H}{\partial E_i} = e_{ikl} S_{kl} + \varepsilon^S_{ik} E_k, \end{aligned} \tag{1.4}$$

where the strain tensor $\mathbf{S}$ and the electric field vector $\mathbf{E}$ are related to the displacement $\mathbf{u}$ and the electric potential, ϕ, by

$$S_{ij} = (u_{i,j} + u_{j,i})/2, \qquad E_i = -\phi_{,i}. \tag{1.5}$$

c^E_{ijkl}, e_{ijk}, and ε^S_{ij} are the elastic, piezoelectric, and dielectric constants. The superscript E in c^E_{ijkl} indicates that the independent electric constitutive variable is the electric field $\mathbf{E}$. The superscript S in ε^S_{ij} indicates that the mechanical constitutive variable is the strain tensor $\mathbf{S}$. The material constants have the following symmetries.

$$c^E_{ijkl} = c^E_{jikl} = c^E_{klij}, \qquad e_{kij} = e_{kji}, \qquad \varepsilon^S_{ij} = \varepsilon^S_{ji}. \tag{1.6}$$

We also assume that the elastic and dielectric tensors are positive definite in the following sense.

$$\begin{aligned} &c^E_{ijkl} S_{ij} S_{kl} \geq 0 \quad \text{for any } S_{ij} = S_{ji}, \quad \text{and} \quad c^E_{ijkl} S_{ij} S_{kl} = 0 \Rightarrow S_{ij} = 0, \\ &\varepsilon^S_{ij} E_i E_j \geq 0 \quad \text{for any } E_i, \quad \text{and} \quad \varepsilon^S_{ij} E_i E_j = 0 \Rightarrow E_i = 0. \end{aligned} \tag{1.7}$$

The internal energy density per unit volume can be obtained from H through a Legendre transform by

$$U(\mathbf{S}, \mathbf{D}) = H(\mathbf{S}, \mathbf{E}(\mathbf{S}, \mathbf{D})) + \mathbf{E}(\mathbf{S}, \mathbf{D}) \cdot \mathbf{D}. \tag{1.8}$$

Constitutive relations in the following form then follow.

$$\mathbf{T} = \frac{\partial U}{\partial \mathbf{S}}, \qquad \mathbf{E} = \frac{\partial U}{\partial \mathbf{D}}, \tag{1.9}$$

or

$$T_{ij} = c^D_{ijkl} S_{kl} - h_{kij} D_k, \qquad E_i = -h_{ikl} S_{kl} + \beta^S_{ik} D_k. \tag{1.10}$$

It can be shown that U is positive definite:

$$\begin{aligned} U &= H + E_i D_i \\ &= \frac{1}{2}\, c^E_{ijkl} S_{ij} S_{kl} - e_{ijk} E_i S_{jk} - \frac{1}{2}\, \varepsilon^S_{ij} E_i E_j + E_i (e_{ikl} S_{kl} + \varepsilon^S_{ik} E_k) \\ &= \frac{1}{2}\, c^E_{ijkl} S_{ij} S_{kl} + \frac{1}{2}\, \varepsilon^S_{ij} E_i E_j \\ &\geq 0. \end{aligned} \tag{1.11}$$

Similar to Equations (1.4) and (1.10), linear constitutive relations can also be written as

$$S_{ij} = s^E_{ijkl} T_{kl} + d_{kij} E_k, \qquad D_i = d_{ikl} T_{kl} + \varepsilon^T_{ik} E_k, \tag{1.12}$$

and

$$S_{ij} = s^D_{ijkl} T_{kl} + g_{kij} D_k, \qquad E_i = -g_{ikl} T_{kl} + \beta^T_{ik} D_k. \tag{1.13}$$

With successive substitutions from Equations (1.4) and (1.5), Equation (1.1) can be written as four equations for **u** and ϕ

$$\begin{aligned} c_{ijkl} u_{k,lj} + e_{kij} \phi_{,kj} + \rho f_i &= \rho \ddot{u}_i, \\ e_{ikl} u_{k,li} - \varepsilon_{ij} \phi_{,ij} &= q, \end{aligned} \tag{1.14}$$

where we have neglected the superscripts of the material constants and the subscript of the reference mass density.

Let the region occupied by a piezoelectric body be V and its boundary surface be S, as shown in Figure 1.1. Let the unit outward normal of S be **n**.

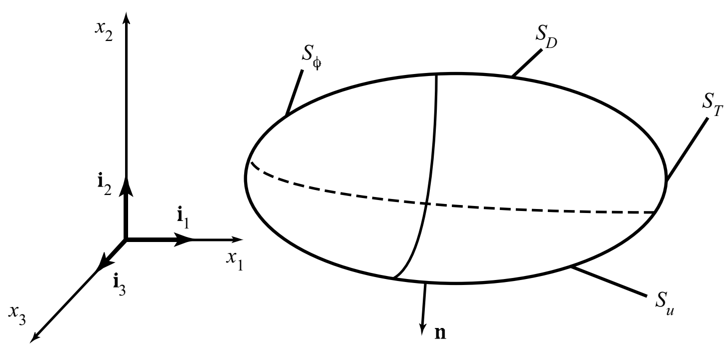

Fig. 1.1 A piezoelectric body and partitions of its boundary surface.

For boundary conditions, we consider the following partitions of S,

$$S_u \cup S_T = S_\phi \cup S_D = S, \qquad S_u \cap S_T = S_\phi \cap S_D = 0, \tag{1.15}$$

where S_u is the part of S on which the mechanical displacement is prescribed, and S_T is the part of S where the traction vector is prescribed. S_ϕ represents the part of S which is electroded where the electric potential is no more than a function of time, and S_D is the unelectroded part. For mechanical boundary conditions, we have prescribed displacement $\bar{u}_i$

$$u_i = \bar{u}_i \quad \text{on} \quad S_u, \tag{1.16}$$

and prescribed traction $\bar{t}_j$

$$T_{ij} n_i = \bar{t}_j \quad \text{on} \quad S_T. \tag{1.17}$$

Electrically, on the electroded portion of S,

$$\phi = \bar{\phi} \quad \text{on} \quad S_\phi, \tag{1.18}$$

where $\bar{\phi}$ does not vary spatially. On the unelectroded part of S, the charge condition can be written as

$$D_j n_j = -\bar{\sigma} \quad \text{on} \quad S_D, \tag{1.19}$$

where $\bar{\sigma}$ is the free charge density per unit surface area. In the above formulation, we assume very thin electrodes, and the mechanical effects, such as inertia and stiffness, of the electrodes are neglected. On an electrode S_ϕ, the total free electric charge Q can be represented by

$$Q = \int_{S_\phi} -n_i D_i dS. \tag{1.20}$$

The electric current flowing out of the electrode is given by

$$i = -\dot{Q}. \tag{1.21}$$

Sometimes there are two (or more) electrodes on a body, and the electrodes are connected to an electric circuit. In this case, circuit equation(s) need to be considered.

1.3 Principle of Superposition

The linearity of Equation (1.14) allows the superposition of solutions. Suppose the solutions under two different sets of loads $\{\mathbf{f}^{(1)}, q^{(1)}\}$ and $\{\mathbf{f}^{(2)}, q^{(2)}\}$

are $\{\mathbf{u}^{(1)}, \phi^{(1)}\}$ and $\{\mathbf{u}^{(2)}, \phi^{(2)}\}$, respectively. Then, under the combined load of $\{\mathbf{f}^{(1)}+\mathbf{f}^{(2)}, q^{(1)}+q^{(2)}\}$, the solution to Equation (1.14) is $\{\mathbf{u}^{(1)}+\mathbf{u}^{(2)}, \phi^{(1)}+\phi^{(2)}\}$. This is called the principle of superposition and can be shown as

$$
\begin{aligned}
& c_{ijkl}(u_k^{(1)}+u_k^{(2)})_{,lj}+e_{kij}(\phi^{(1)}+\phi^{(2)})_{,kj}+\rho(f_i^{(1)}+f_i^{(2)})-\rho\frac{\partial^2}{\partial t^2}(u_i^{(1)}+u_i^{(2)}) \\
&= c_{ijkl}u_{k,lj}^{(1)}+c_{ijkl}u_{k,lj}^{(2)}+e_{kij}\phi_{,kj}^{(1)}+e_{kij}\phi_{,kj}^{(2)}+\rho f_i^{(1)}+\rho f_i^{(2)}-\rho\ddot{u}_i^{(1)}-\rho\ddot{u}_i^{(2)} \\
&= (c_{ijkl}u_{k,lj}^{(1)}+e_{kij}\phi_{,kj}^{(1)}+\rho f_i^{(1)}-\rho\ddot{u}_i^{(1)})+(c_{ijkl}u_{k,lj}^{(2)}+e_{kij}\phi_{,kj}^{(2)}+\rho f_i^{(2)}-\rho\ddot{u}_i^{(2)}) \\
&= 0+0 \\
&= 0,
\end{aligned} \tag{1.22}
$$

and

$$
\begin{aligned}
& e_{ikl}(u_k^{(1)}+u_k^{(2)})_{,li}-\varepsilon_{ij}(\phi^{(1)}+\phi^{(2)})_{,ij}-(q^{(1)}+q^{(2)}) \\
&= e_{ikl}u_{k,li}^{(1)}+e_{ikl}u_{k,li}^{(2)}-\varepsilon_{ij}\phi_{,ij}^{(1)}-\varepsilon_{ij}\phi_{,ij}^{(2)}-q^{(1)}-q^{(2)} \\
&= (e_{ikl}u_{k,li}^{(1)}-\varepsilon_{ij}\phi_{,ij}^{(1)}-q^{(1)})+(e_{ikl}u_{k,li}^{(2)}-\varepsilon_{ij}\phi_{,ij}^{(2)}-q^{(2)}) \\
&= 0+0 \\
&= 0.
\end{aligned} \tag{1.23}
$$

The principle of superposition can be generalized to include boundary loads.

1.4 Hamilton's Principle

The equations and boundary conditions of linear piezoelectricity can be derived from a variational principle [2]. Consider

$$
\begin{aligned}
\Pi(\mathbf{u},\phi) = & \int_{t_0}^{t_1} dt \int_V \left[\frac{1}{2}\rho\dot{u}_i\dot{u}_i - H(\mathbf{S},\mathbf{E}) + \rho f_i u_i - q\phi\right] dV \\
& + \int_{t_0}^{t_1} dt \int_{S_T} \bar{t}_i u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} \bar{\sigma}\phi \, dS,
\end{aligned} \tag{1.24}
$$

where $\mathbf{S}$ and $\mathbf{E}$ are considered as functions of the displacement and potential through

$$
S_{ij} = \frac{(u_{i,j}+u_{j,i})}{2}, \qquad E_i = -\phi_{,i}. \tag{1.25}
$$

$\mathbf{u}$ and ϕ are variationally admissible if they are smooth enough and satisfy

$$
\begin{aligned}
\delta u_i|_{t_0} = \delta u_i|_{t_1} = 0 \quad & \text{in} \quad V, \\
u_i = \bar{u}_i \quad & \text{on} \quad S_u, \quad t_0 < t < t_1, \\
\phi = \bar{\phi} \quad & \text{on} \quad S_\phi, \quad t_0 < t < t_1.
\end{aligned} \tag{1.26}
$$

The first variation of Π is

$$\delta\Pi = \int_{t_0}^{t_1} dt \int_V \left[(T_{ji,j} + \rho f_i - \rho\ddot{u}_i)\delta u_i + (D_{i,i} - q)\delta\phi\right] dV \\ - \int_{t_0}^{t_1} dt \int_{S_T} (T_{ji}n_j - \bar{t}_i)\delta u_i dS - \int_{t_0}^{t_1} dt \int_{S_D} (D_i n_i + \bar{\sigma})\delta\phi\, dS, \tag{1.27}$$

where we have denoted

$$\mathbf{T} = \frac{\partial H}{\partial \mathbf{S}}, \qquad \mathbf{D} = -\frac{\partial H}{\partial \mathbf{E}}. \tag{1.28}$$

Therefore, the stationary condition of Π is

$$T_{ji,j} + \rho f_i = \rho\ddot{u}_i \quad \text{in} \quad V, \quad t_0 < t < t_1, \quad D_{i,i} = q \quad \text{in} \quad V, \quad t_0 < t < t_1, \\ T_{ji}n_j = \bar{t}_i \quad \text{on} \quad S_T, \quad t_0 < t < t_1, \quad D_i n_i = -\bar{\sigma} \quad \text{on} \quad S_D, \quad t_0 < t < t_1. \tag{1.29}$$

Hamilton's principle can be stated as: among all the admissible $\{\mathbf{u}, \phi\}$, the one that also satisfies Equation (1.29) makes Π stationary.

1.5 Poynting's Theorem and Energy Integral

We begin with the rate of change of the total internal energy density, given as

$$\begin{aligned} \dot{U} &= \frac{\partial U}{\partial S_{ij}}\dot{S}_{ij} + \frac{\partial U}{\partial D_i}\dot{D}_i \\ &= T_{ij}\dot{S}_{ij} + E_i\dot{D}_i = T_{ij}\dot{u}_{i,j} - \phi_{,i}\dot{D}_i \\ &= (T_{ij}\dot{u}_i)_{,j} - T_{ij,j}\dot{u}_i - (\phi\dot{D}_i)_{,i} + \phi\dot{D}_{i,i} \\ &= (T_{ij}\dot{u}_i)_{,j} - (\rho\ddot{u}_i - \rho f_i)\dot{u}_i - (\phi\dot{D}_i)_{,i} + \phi\dot{q} \\ &= (T_{ji}\dot{u}_j)_{,i} - \frac{\partial}{\partial t}\Big(\frac{1}{2}\rho\dot{u}_i\dot{u}_i\Big) + \rho f_i\dot{u}_i - (\phi\dot{D}_i)_{,i} + \phi\dot{q}. \end{aligned} \tag{1.30}$$

Therefore,

$$\frac{\partial}{\partial t}(T + U) = \rho f_i\dot{u}_i + \phi\dot{q} - (\phi\dot{D}_i - T_{ji}\dot{u}_j)_{,i}, \tag{1.31}$$

where

$$T = \frac{1}{2}\rho\dot{u}_i\dot{u}_i \tag{1.32}$$

is the kinetic energy density, and $\phi\dot{D}_i$ is the quasi-static Poynting vector. Equation (1.31) is the Poynting theorem of piezoelectricity.

Integration of Equation (1.31) over V gives

$$\frac{\partial}{\partial t}\int_V (T+U)dV = \int_V \rho(f_i\dot{u}_i + \phi\dot{q})dV + \int_{S_u} T_{ji}n_j\dot{\bar{u}}_i dS + \int_{S_T} \bar{t}_i\dot{u}_i dS - \int_{S_\phi} \dot{D}_i n_i\bar{\phi}dS + \int_{S_D} \dot{\bar{\sigma}}\phi dS. \quad (1.33)$$

Integrating Equation (1.33) from t_0 to t, we obtain

$$\int_V (T+U)|_t dV = \int_V (T+U)|_{t_0} dV + \int_{t_0}^t dt \int_V \rho(f_i\dot{u}_i + \phi\dot{q})dV + \int_{t_0}^t dt \int_{S_u} T_{ji}n_j\dot{\bar{u}}_i dS + \int_{t_0}^t dt \int_{S_T} \bar{t}_i\dot{u}_i dS - \int_{t_0}^t dt \int_{S_\phi} \dot{D}_i n_i\bar{\phi}dS + \int_{t_0}^t dt \int_{S_D} \dot{\bar{\sigma}}\phi dS. \quad (1.34)$$

Equation (1.34) is called the energy integral which states that the energy at time t is the energy at time t_0 plus the work done to the body from t_0 to t.

1.6 Uniqueness

Consider two solutions to the following initial boundary value problem:

$$\begin{aligned} T_{ji,j} + \rho f_i &= \rho\ddot{u}_i \quad \text{in} \quad V, \quad t > t_0, \\ D_{i,i} &= q \quad \text{in} \quad V, \quad t > t_0, \\ T_{ij} &= c_{ijkl}S_{kl} - e_{kij}E_k \quad \text{in} \quad V, \quad t > t_0, \\ D_i &= e_{ijk}S_{jk} + \varepsilon_{ij}E_j \quad \text{in} \quad V, \quad t > t_0, \\ S_{ij} &= (u_{i,j} + u_{j,i})/2 \quad \text{in} \quad V, \quad t > t_0, \end{aligned} \quad (1.35)$$

and

$$\begin{aligned} &u_i = \bar{u}_i \quad \text{on} \quad S_u, \quad t > t_0, \qquad T_{ji}n_j = \bar{t}_i \quad \text{on} \quad S_T, \quad t > t_0, \\ &\phi = \bar{\phi} \quad \text{on} \quad S_\phi, \quad t > t_0, \qquad D_i n_i = -\bar{\sigma} \quad \text{on} \quad S_D, \quad t > t_0, \\ &u_i = u_i^0 \quad \text{in} \quad V, \quad t = t_0, \qquad \dot{u}_i = v_i^0 \quad \text{in} \quad V, \quad t = t_0, \\ &\phi = \phi^0 \quad \text{in} \quad V, \quad t = t_0. \end{aligned} \quad (1.36)$$

From the principle of superposition, the difference of the two solutions satisfies the homogeneous version of Equations (1.35) and (1.36). Let $\mathbf{u}^*, \phi^*$, $\mathbf{S}^*, \mathbf{T}^*$, $\mathbf{E}^*$, and $\mathbf{D}^*$ denote the differences of the corresponding fields and apply Equation (1.34) to them. The initial energy and the external work for the difference fields are zero. Then the energy integral implies that, for the difference fields, at any $t > t_0$,

$$\int_V (T^* + U^*)\big|_t \, dV = 0, \qquad t > t_0. \tag{1.37}$$

Because both T and U are nonnegative,

$$U^* = 0, \qquad T^* = 0 \quad \text{in} \quad V, \quad t > t_0. \tag{1.38}$$

From the positive definiteness of T and U,

$$\mathbf{S}^* = 0, \qquad \mathbf{E}^* = 0, \qquad \dot{\mathbf{u}}^* = 0 \quad \text{in} \quad V, \quad t > t_0. \tag{1.39}$$

Hence the two solutions are identical for $\mathbf{S}, \mathbf{E}, \mathbf{T}, \mathbf{D}$, and the velocity fields but may differ by a static rigid body displacement and a constant potential [2].

1.7 Four-Vector Formulation

Let us define the four-space coordinate system [6]

$$x_p = \{x_i, t\}, \tag{1.40}$$

and the four-vector

$$U_p = \{u_i, \phi\}, \tag{1.41}$$

where subscripts p, q, r, and s are assumed to run from 1 to 4. Also, define the second-rank four-tensor

$$\rho_{pq} = \begin{cases} \rho\delta_{pq}, & p, q = 1, 2, 3, \\ 0, & p, q = 4, \end{cases} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{1.42}$$

and the fourth-rank four-tensor M_{pqrs}, where

$$\begin{aligned} &M_{ijkl} = c_{ijkl}, \qquad M_{4jkl} = e_{jkl}, \qquad M_{ijk4} = e_{kij}, \\ &M_{4jk4} = -\varepsilon_{jk}, \qquad M_{p44s} = -\rho_{ps}, \end{aligned} \tag{1.43}$$

and all other components of $M_{pqrs} = 0$. Then

$$\begin{aligned} &(U_{p,q}M_{pqrl})_{,r} \\ &\quad = (U_{i,j}M_{ijrl} + U_{4,j}M_{4jrl} + U_{i,4}M_{i4rl} + U_{4,4}M_{44rl})_{,r} \\ &\quad = (U_{i,j}M_{ijkl} + U_{4,j}M_{4jkl} + U_{i,4}M_{i4kl} + U_{4,4}M_{44kl})_{,k} \\ &\qquad + (U_{i,j}M_{ij4l} + U_{4,j}M_{4j4l} + U_{i,4}M_{i44l} + U_{4,4}M_{444l})_{,4} \end{aligned}$$

$$
\begin{aligned}
&= (u_{i,j}c_{ijkl} + \phi_{,j}e_{jkl})_{,k} + (-\dot{u}_i\rho_{il})_{,4} \\
&= c_{ijkl}u_{i,jk} + e_{jkl}\phi_{,jk} - \rho\ddot{u}_l, \qquad (1.44)
\end{aligned}
$$

and

$$
\begin{aligned}
&(U_{p,q}M_{pqr4})_{,r} \\
&\quad = (U_{i,j}M_{ijr4} + U_{4,j}M_{4jr4} + U_{i,4}M_{i4r4} + U_{4,4}M_{44r4})_{,r} \\
&\quad = (U_{i,j}M_{ijk4} + U_{4,j}M_{4jk4} + U_{i,4}M_{i4k4} + U_{4,4}M_{44k4})_{,k} \\
&\qquad + (U_{i,j}M_{ij44} + U_{4,j}M_{4j44} + U_{i,4}M_{i444} + U_{4,4}M_{4444})_{,4} \\
&\quad = (u_{i,j}e_{kij} - \phi_{,j}\varepsilon_{jk})_{,k} \\
&\quad = u_{i,jk}e_{kij} - \phi_{,jk}\varepsilon_{jk}. \qquad (1.45)
\end{aligned}
$$

Therefore,

$$
(U_{p,q}M_{pqrs})_{,r} = 0 \qquad (1.46)
$$

yields the homogeneous equation of motion and the charge equation.

1.8 Cylindrical Coordinates

The cylindrical coordinates (r, θ, z) are defined by

$$
x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \qquad x_3 = z. \qquad (1.47)
$$

In cylindrical coordinates, we have the strain-displacement relation

$$
\begin{aligned}
&S_{rr} = u_{r,r}, \qquad S_{\theta\theta} = \frac{1}{r}u_{\theta,\theta} + \frac{u_r}{r}, \qquad S_{zz} = u_{z,z}, \\
&2S_{r\theta} = u_{\theta,r} + \frac{1}{r}u_{r,\theta} - \frac{u_\theta}{r}, \qquad 2S_{\theta z} = \frac{1}{r}u_{z,\theta} + u_{\theta,z}, \\
&2S_{zr} = u_{r,z} + u_{z,r}. \qquad (1.48)
\end{aligned}
$$

The electric field-potential relation is given by

$$
E_r = -\phi_{,r}, \qquad E_\theta = -\frac{1}{r}\phi_{,\theta}, \qquad E_z = -\phi_{,z}. \qquad (1.49)
$$

The equations of motion are

$$
\begin{aligned}
&\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{\partial T_{zr}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r} + \rho f_r = \rho\ddot{u}_r, \\
&\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{2}{r}T_{r\theta} + \rho f_\theta = \rho\ddot{u}_\theta, \\
&\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r}T_{rz} + \rho f_z = \rho\ddot{u}_z. \qquad (1.50)
\end{aligned}
$$

The electrostatic charge equation is

$$\frac{1}{r}(rD_r)_{,r} + \frac{1}{r}D_{\theta,\theta} + D_{z,z} = q. \tag{1.51}$$

1.9 Spherical Coordinates

The spherical coordinates (r, θ, φ) are defined by

$$x_1 = r\sin\theta\cos\phi, \qquad x_2 = r\sin\theta\sin\varphi, \qquad x_3 = r\cos\theta. \tag{1.52}$$

In spherical coordinates we have the strain-displacement relation

$$S_{rr} = \frac{\partial u_r}{\partial r}, \qquad S_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \qquad S_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta, \tag{1.53}$$

$$2S_{r\theta} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}, \qquad 2S_{\theta\varphi} = \frac{1}{r}\frac{\partial u_\varphi}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r}\cot\theta,$$
$$2S_{\varphi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}. \tag{1.54}$$

The electric field-potential relation is

$$E_r = -\frac{\partial \phi}{\partial r}, \qquad E_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}, \qquad E_\varphi = -\frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \varphi}. \tag{1.55}$$

The equations of motion are

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi r}}{\partial \varphi} + \frac{1}{r}(2T_{rr} - T_{\theta\theta} - T_{\varphi\varphi} + T_{\theta r}\cot\theta) + \rho f_r$$
$$= \rho\ddot{u}_r, \tag{1.56}$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi\theta}}{\partial \varphi} + \frac{1}{r}[3T_{r\theta} + (T_{\theta\theta} - T_{\varphi\varphi})\cot\theta] + \rho f_\theta$$
$$= \rho\ddot{u}_\theta, \tag{1.57}$$

$$\frac{\partial T_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\varphi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi\varphi}}{\partial \varphi} + \frac{1}{r}(3T_{r\varphi} + 2T_{\theta\varphi}\cot\theta) + \rho f_z$$
$$= \rho\ddot{u}_\varphi. \tag{1.58}$$

The electrostatic charge equation is

$$r^2\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(D_\theta \sin\varphi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}D_\varphi = q. \tag{1.59}$$

1.10 Compact Matrix Notation

We now introduce a compact matrix notation [1, 2]. This notation consists of replacing pairs of indices, ij or kl, by single indices, p or q, where i, j, k, and l take the values of 1, 2, and 3; and p and q take the values of 1, 2, 3, 4, 5, and 6 according to

$$\begin{array}{llllll} ij \text{ or } kl\text{:} & 11 & 22 & 33 & 23 \text{ or } 32 & 31 \text{ or } 13 & 12 \text{ or } 21 \\ p \text{ or } q\text{:} & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \tag{1.60}$$

Thus

$$c_{ijkl} \to c_{pq}, \quad e_{ikl} \to e_{ip}, \quad T_{ij} \to T_p. \tag{1.61}$$

For the strain tensor, we introduce S_p such that

$$\begin{aligned} &S_1 = S_{11}, \qquad S_2 = S_{22}, \qquad S_3 = S_{33}, \\ &S_4 = 2S_{23}, \qquad S_5 = 2S_{31}, \qquad S_6 = 2S_{12}. \end{aligned} \tag{1.62}$$

The constitutive relations can then be written as

$$T_p = c^E_{pq} S_q - e_{kp} E_k, \qquad D_i = e_{iq} S_q + \varepsilon^S_{ik} E_k. \tag{1.63}$$

In matrix form, Equation (1.63) becomes

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{pmatrix} c^E_{11} & c^E_{12} & c^E_{13} & c^E_{14} & c^E_{15} & c^E_{16} \\ c^E_{21} & c^E_{22} & c^E_{23} & c^E_{24} & c^E_{25} & c^E_{26} \\ c^E_{31} & c^E_{32} & c^E_{33} & c^E_{34} & c^E_{35} & c^E_{36} \\ c^E_{41} & c^E_{42} & c^E_{43} & c^E_{44} & c^E_{45} & c^E_{46} \\ c^E_{51} & c^E_{52} & c^E_{53} & c^E_{54} & c^E_{55} & c^E_{56} \\ c^E_{61} & c^E_{62} & c^E_{63} & c^E_{64} & c^E_{65} & c^E_{66} \end{pmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} - \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \\ e_{14} & e_{24} & e_{34} \\ e_{15} & e_{25} & e_{35} \\ e_{16} & e_{26} & e_{36} \end{pmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}, \tag{1.64}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} + \begin{pmatrix} \varepsilon^S_{11} & \varepsilon^S_{12} & \varepsilon^S_{13} \\ \varepsilon^S_{21} & \varepsilon^S_{22} & \varepsilon^S_{23} \\ \varepsilon^S_{31} & \varepsilon^S_{32} & \varepsilon^S_{33} \end{pmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}. \tag{1.65}$$

Similarly, Equations (1.10), (1.12), and (1.13) can also be written in matrix form. The matrices of the material constants in various expressions are related by

$$\begin{aligned} &c^E_{pr} s^E_{qr} = \delta_{pq}, \qquad c^D_{pr} s^D_{qr} = \delta_{pq}, \\ &\beta^S_{ik} \varepsilon^S_{jk} = \delta_{ij}, \qquad \beta^T_{ik} \varepsilon^T_{jk} = \delta_{ij}, \end{aligned} \tag{1.66}$$

$$
\begin{aligned}
& c_{pq}^{D} = c_{pq}^{E} + e_{kp} h_{kq}, \qquad s_{pq}^{D} = s_{pq}^{E} - d_{kp} g_{kq}, \\
& \varepsilon_{ij}^{T} = \varepsilon_{ij}^{S} + d_{iq} e_{jq}, \qquad \beta_{ij}^{T} = \beta_{ij}^{S} - g_{iq} h_{jq},
\end{aligned} \tag{1.67}
$$

$$
\begin{aligned}
& e_{ip} = d_{iq} c_{pq}^{E}, \qquad d_{ip} = \varepsilon_{ik}^{T} g_{kp}, \\
& g_{ip} = \beta_{ik}^{T} d_{kp}, \qquad h_{ip} = g_{ig} c_{qp}^{D}.
\end{aligned} \tag{1.68}
$$

References

[1] Meitzler AH, Tiersten HF, Warner AW et al. (1988) *IEEE Standard on Piezoelectricity*. IEEE, New York

[2] Tiersten HF (1969) *Linear Piezoelectric Plate Vibrations*. Plenum, New York

[3] Yang JS (2005) *An Introduction to the Theory of Piezoelectricity*. Springer, New York

[4] Tiersten HF (1971) On the nonlinear equations of thermoelectroelasticity. *Int J Engng Sci* 9: 587–604

[5] Lax M, Nelson DF (1971) Linear and nonlinear electrodynamics in elastic anisotropic dielectrics. *Phys Rev B* 4:3694–3731

[6] Holland R, EerNisse EP (1969) *Design of Resonant Piezoelectric Devices*. MIT Press, Cambridge, MA

Chapter 2
Green's Functions

Ernian Pan

2.1 Introduction

Coupling between mechanical and electric fields has stimulated interesting research related to the microelectromechanical system [1, 2]. The major applications are in sensor and actuator devices by which an electric voltage can induce an elastic deformation and vice versa. Because many novel materials, such as the nitride group semiconductors, are piezoelectric, study on quantum nanostructures is currently a cutting-edge topic with the strain energy band engineering in the center [3, 4]. Novel laminated composites (with adaptive and smart components) are continuously attracting great attention from mechanical, aerospace, and civil engineering branches [5]. In materials property study, the Eshelby-based micromechanics theory has been very popular [6]. In most of these exciting research topics, the fundamental solution of a given system under a unit concentrated force/charge or simply the Green's function solution is required. This motivates the writing of this chapter. In this chapter, however, only the static case with general anisotropic piezoelectricity is considered, even though a couple of closely related references on vibration and/or dynamics (time-harmonic) wave propagation are briefly reviewed. Furthermore, although emphasis is given to the generalized point and line forces, the Green's functions to the corresponding point and line dislocations, as well as point and line eigenstrain are also discussed or presented based on Betti's reciprocal theorem.

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J. Yang (ed.), *Special Topics in the Theory of Piezoelectricity*, DOI: 10.1007/978-0-387-89498-0_2,

2.2 Governing Equations

Consider a linear, anisotropic piezoelectric and heterogeneous solid occupying the domain V bounded by the boundary S. In discussing the Green's functions, the problem domain and the corresponding boundary conditions are clearly described later. We also assume that the deformation is static, and thus the field equations for such a solid consist of [7]:

(a) *Equilibrium equations (including Gauss equation):*

$$\sigma_{ji,j} + f_i = 0 \qquad D_{i,i} - q = 0, \tag{2.1}$$

where σ_{ij} and D_{i} are the stress and electric displacement, respectively; f_{i} and q are the body force and electric charge, respectively. In this and the following sections, summation from 1 to 3 (1 to 4) over repeated lowercase (uppercase) subscripts is implied. A subscript comma denotes the partial differentiation.

In the Cartesian coordinate system, the equilibrium equations are

$$\begin{aligned}
&\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x = 0 \\
&\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y = 0 \\
&\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0 \qquad (2.2a) \\
&\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} - q = 0 \,. \qquad (2.2b)
\end{aligned}$$

In the cylindrical coordinate system, the equilibrium equations are

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{r\theta}}{r\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = 0 \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta\theta}}{r\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + f_\theta = 0 \qquad (2.3a) \\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{\theta z}}{r\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = 0 \\
&\frac{\partial D_r}{\partial r} + \frac{\partial D_\theta}{r\partial \theta} + \frac{\partial D_z}{\partial_z} - q = 0 \,. \qquad (2.3b)
\end{aligned}$$

(b) *Constitutive relations:*

$$\sigma_{ij} = C_{ijlm}\gamma_{lm} - e_{kji}E_k \qquad D_i = e_{ijk}\gamma_{jk} + \varepsilon_{ij}E_j, \tag{2.4}$$

where γ_{ij} is the strain and E_{i} the electric field; C_{ijlm}, e_{ijk}, and $\varepsilon_{\mathrm{ij}}$ are the elastic moduli, piezoelectric coefficients, and dielectric constants, respectively. The uncoupled state (purely elastic and purely electric deformation) can be obtained by simply setting $e_{\mathrm{ijk}} = 0$. For transversely isotropic piezoelectric materials with the z-axis being the material symmetric (or the poling) axis, the constitutive relation in the Cartesian coordinate system is (using the reduced indices for C_{ijkl} and e_{ijk}, with the following correspondence between the one and two indices: 1 = 11, 2 = 22, 3 = 33, 4 = 23, 5 = 13, 6 = 12)

$$\begin{aligned}
\sigma_{xx} &= C_{11}\gamma_{xx} + C_{12}\gamma_{yy} + C_{13}\gamma_{zz} - e_{31}E_z \\
\sigma_{yy} &= C_{12}\gamma_{xx} + C_{11}\gamma_{yy} + C_{13}\gamma_{zz} - e_{31}E_z \\
\sigma_{zz} &= C_{13}\gamma_{xx} + C_{13}\gamma_{yy} + C_{33}\gamma_{zz} - e_{33}E_z \\
\sigma_{yz} &= 2C_{44}\gamma_{yz} - e_{15}E_y \\
\sigma_{xz} &= 2C_{44}\gamma_{xz} - e_{15}E_x \\
\sigma_{xy} &= 2C_{66}\gamma_{xy} && \text{(2.5a)} \\
D_x &= 2e_{15}\gamma_{xz} + \varepsilon_{11}E_x \\
D_y &= 2e_{15}\gamma_{yz} + \varepsilon_{11}E_y \\
D_z &= e_{31}(\gamma_{xx} + \gamma_{yy}) + e_{33}\gamma_{zz} + \varepsilon_{33}E_z\,, && \text{(2.5b)}
\end{aligned}$$

where $C_{66} = (C_{11} - C_{12})/2$.

Similarly, in the cylindrical coordinate system, the constitutive relation is

$$\begin{aligned}
\sigma_{rr} &= C_{11}\gamma_{rr} + C_{12}\gamma_{\theta\theta} + C_{13}\gamma_{zz} - e_{31}E_z \\
\sigma_{\theta\theta} &= C_{12}\gamma_{rr} + C_{11}\gamma_{\theta\theta} + C_{13}\gamma_{zz} - e_{31}E_z \\
\sigma_{zz} &= C_{13}\gamma_{rr} + C_{13}\gamma_{\theta\theta} + C_{33}\gamma_{zz} - e_{33}E_z \\
\sigma_{\theta z} &= 2C_{44}\gamma_{\theta z} - e_{15}E_\theta \\
\sigma_{rz} &= 2C_{44}\gamma_{rz} - e_{15}E_r \\
\sigma_{r\theta} &= 2C_{66}\gamma_{r\theta} && \text{(2.6a)} \\
D_r &= 2e_{15}\gamma_{rz} + \varepsilon_{11}E_r \\
D_\theta &= 2e_{15}\gamma_{\theta z} + \varepsilon_{11}E_\theta \\
D_z &= e_{31}(\gamma_{rr} + \gamma_{\theta\theta}) + e_{33}\gamma_{zz} + \varepsilon_{33}E_z\,. && \text{(2.6b)}
\end{aligned}$$

(c) *Elastic strain-displacement and electric field-potential relations:*

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad E_i = -\phi_{,i}\,, \tag{2.7}$$

where u_i and ϕ are the elastic displacement and electric potential, respectively.

In the Cartesian coordinate system, we have

$$\begin{aligned}
&\gamma_{xx} = \frac{\partial u_x}{\partial x}, \qquad \gamma_{yy} = \frac{\partial u_y}{\partial y}, \qquad \gamma_{zz} = \frac{\partial u_z}{\partial z} \qquad \gamma_{yz} = 0.5\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\
&\gamma_{xz} = 0.5\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \qquad \gamma_{xy} = 0.5\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) && \text{(2.8a)} \\
&E_x = -\frac{\partial \phi}{\partial x}, \qquad E_y = -\frac{\partial \phi}{\partial y}, \qquad E_z = -\frac{\partial \phi}{\partial z}\,, && \text{(2.8b)}
\end{aligned}$$

and in cylindrical coordinate system, we obtain

$$\gamma_{rr} = \frac{\partial u_r}{\partial r}, \qquad \gamma_{\theta\theta} = \frac{\partial u_\theta}{r\partial\theta} + \frac{u_r}{r}, \qquad \gamma_{zz} = \frac{\partial u_z}{\partial z}$$

$$\gamma_{\theta z} = 0.5\left(\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{r\partial\theta}\right) \qquad \gamma_{rz} = 0.5\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$$

$$\gamma_{r\theta} = 0.5\left(\frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \tag{2.9a}$$

$$E_r = -\frac{\partial\phi}{\partial r}, \qquad E_\theta = -\frac{\partial\phi}{r\partial\theta}, \qquad E_z = -\frac{\partial\phi}{\partial z}. \tag{2.9b}$$

The notation introduced by Barnett and Lothe [8] has been shown to be very convenient for the analysis of piezoelectric problems. With this notation, the elastic displacement and electric potential, the elastic strain and electric field, the stress and electric displacement, and the elastic and electric moduli (or coefficients) can be grouped together as [9]

$$u_I = \begin{cases} u_i & I = i = 1,2,3 \\ \phi & I = 4 \end{cases} \tag{2.10}$$

$$\gamma_{Ij} = \begin{cases} \gamma_{ij} & I = i = 1,2,3 \\ -E_j & I = 4 \end{cases} \tag{2.11}$$

$$\sigma_{iJ} = \begin{cases} \sigma_{ij} & J = j = 1,2,3 \\ D_i & J = 4 \end{cases} \tag{2.12a}$$

$$T_J = \sigma_{iJ} n_i = \begin{cases} \sigma_{ij} n_i & J = j = 1,2,3 \\ D_i n_i & J = 4 \end{cases} \tag{2.12b}$$

$$C_{iJKl} = \begin{cases} C_{ijkl} & J,K = j,k = 1,2,3 \\ e_{lij} & J = j = 1,2,3; K = 4 \\ e_{ikl} & J = 4; K = k = 1,2,3 \\ -\varepsilon_{il} & J,K = 4 \end{cases}. \tag{2.13}$$

It is noted that we have kept the original symbols instead of introducing new ones because they can be easily distinguished by the range of their subscript. In terms of this shorthand notation, the constitutive relations can be unified into a single equation as

$$\sigma_{iJ} = C_{iJKl}\gamma_{Kl}, \tag{2.14}$$

where the material property coefficients C_{iJKl} can be location-dependent in the region.

Similarly, the equilibrium equations in terms of the extended stresses can be recast into

$$\sigma_{iJ,i} + f_J = 0 \tag{2.15}$$

with the extended force f_J being defined as

$$f_J = \begin{cases} f_j & J = j = 1,2,3 \\ -q & J = 4 \end{cases} . \tag{2.16}$$

For the Green's function solutions, the body force and electric charge density are replaced by the following concentrated unit sources (k = 1, 2, 3),

$$f_I = \begin{cases} \delta_{ik}\delta(\boldsymbol{x} - \boldsymbol{x}_0), & I = i = 1,2,3 \\ \delta(\boldsymbol{x} - \boldsymbol{x}_0), & I = 4. \end{cases} . \tag{2.17}$$

It is observed that Equations (2.14) and (2.15) are exactly the same as their purely elastic counterparts. The only difference is the dimension of the index of the involved quantities. Therefore, the solution method developed for anisotropic elasticity can be directly applied to the piezoelectric case. For ease of reference, in this chapter, we still use *displacement* to stand for the elastic displacement and electric potential as defined in Equation (2.10), use *stress* for the stress and electric displacement as defined in Equation (2.12a), and use *traction* for the elastic traction and normal electric displacement as defined in Equation (2.12b).

2.3 Relations Among Different Sources and Their Responses

Relations among different concentrated sources and their responses can be studied via Betti's reciprocal theorem, which states that for two systems (1) and (2) belonging to the same material space, the following relation holds (i.e., [9])

$$\sigma_{iJ}^{(1)} u_{J,i}^{(2)} = \sigma_{iJ}^{(2)} u_{J,i}^{(1)}. \tag{2.18}$$

From (2.18), one can easily derive the following integral equation for these two systems

$$\int_S \sigma_{iJ}^{(1)} u_J^{(2)} n_i dS - \int_V \sigma_{iJ,i}^{(1)} u_J^{(2)} dV = \int_S \sigma_{iJ}^{(2)} u_J^{(1)} n_i dS - \int_V \sigma_{iJ,i}^{(2)} u_J^{(1)} dV. \tag{2.19}$$

We let system (1) be the real boundary value problem and (2) be the corresponding "point-force" Green's function problem; that is,

$$\sigma_{iJ,i} = -\delta_{JK}\delta(x_p^f - x_p^s), \tag{2.20}$$